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H-supplemented modules with respect to a preradical

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Abstract. Let M be a right R-module and τ a preradical. We call M τ-H-supplemented if for every submodule A of M there exists a direct summand D of M such that $(A+D)/D \subseteq \tau(M/D)$ and $(A+D)/A \subseteq \tau(M/A)$. Let τ be a cohereditary preradical. Firstly, for a duo module $M = M_1 \oplus M_2$ we prove that M is τ-H-supplemented if and only if M_1 and M_2 are τ-H-supplemented. Secondly, let $M = \oplus_{i=1}^{n} M_i$ be a τ-supplemented module. Assume that M_i is τ-M_j-projective for all $j > i$. If each M_i is τ-H-supplemented, then M is τ-H-supplemented. We also investigate the relations between τ-H-supplemented modules and τ-($\oplus$-)supplemented modules.

Introduction

Throughout this paper, R denotes an associative ring with identity and modules are unital right R-modules. We use $N \leq M$ and $N \leq_d M$ to signify that N is a submodule and a direct summand of M, respectively.

A functor τ from the category of the right R-modules Mod − R to itself is called a *preradical* if it satisfies the following properties:

i) For any R-module M, $\tau(M)$ is a submodule of an R-module M,

ii) If $f : M' \to M$ is an R-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M')$.

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It is well known if K is a direct summand of M, then $\tau(K) = \tau(M) \cap K$ for a preradical τ. A preradical τ is said to be *cohereditary* if, for every $M \in \mathrm{Mod} - \mathrm{R}$ and every submodule N of M, $\tau(M/N) = (\tau(M) + N)/N$. We refer to [3] for details concerning radicals and preradicals. In this paper, τ will be a preradical unless otherwise stated. Recall that a module M has the *Summand Sum Property*, (SSP) if the sum of any two direct summands of M is again a direct summand (see [4]).

Let M be a module. A submodule X of M is called *fully invariant*, if for every $f \in End(M)$, $f(X) \subseteq X$. The module M is called a *duo module*, if every submodule of M is fully invariant. The submodule A of M is called *projection invariant* in M if $f(A) \subseteq A$, for any idempotent $f \in End(M)$. A submodule K of M is called *small* in M (denoted by $K \ll M$) if $N + K \neq M$ for any proper submodule N of M.

Lifting modules were defined and studied by many authors. H-supplemented modules were introduced in [11] as a generalization of lifting modules. According to [11], a module M is called *H-supplemented* if for every submodule A of M there exists a direct summand D of M such that $A + X = M$ if and only if $D + X = M$ for any submodule X of M. For more information about H-supplemented modules we refer the reader to [8], [10] and [11]. A module M is called *$\oplus$-supplemented* if for every submodule N of M there exists a direct summand D of M such that $M = N + D$ and $N \cap D \ll D$. According to [15], a module M is *semiperfect* if every factor module of M has a projective cover. By [15, 41.14 and 42.1], if P is projective, then P is semiperfect if and only if for every submodule K of P there exists a decomposition $K = A \oplus B$ such that A is a direct summand of P and $B \ll P$. By [5, Lemma 1.2] a projective module is $\oplus$-supplemented if and only if it is semiperfect.

In [2], for a radical τ, Al-Takhman, Lomp and Wisbauer defined and studied the concept of τ-lifting, τ-supplemented and τ-semiperfect modules. Following [2], a module M is called *τ-lifting* if every submodule N of M has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \subseteq \tau(M)$ and they call M *τ-supplemented* if for every submodule N of M there exists a submodule K of M such that $N + K = M$ and $N \cap K \subseteq \tau(K)$ (In this case K is called a *τ-supplement* of N in M). They call a module M *τ-semiperfect* if for every submodule N of M, M/N has a projective τ-cover. In this paper we define τ-H-supplemented modules and investigate the general properties of such modules.

In Section 1 we will define τ-H-supplemented modules and give an equivalent condition for such modules. Also we obtain some conditions which under the factor module of a τ-H-supplemented module will be τ-H-supplemented. Let M be a τ-H-supplemented module for a cohereditary preradical τ. Then

(1) If M is a distributive module, then M/X is τ-H-supplemented for every submodule X of M.
(2) Let $N \leq M$ such that for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Then M/N is τ-H-supplemented.
(3) Let X be a projection invariant submodule of M. Then M/X is τ-H-supplemented. In particular, for every fully invariant submodule A of M, M/A is τ-H-supplemented (Corollary 1).

In Section 2 we will study direct summands of τ-H-supplemented modules. We show that, if τ is a cohereditary preradical, every direct summand of a τ-H-supplemented module with SSP is τ-H-supplemented (Theorem 2).

In Section 3 we will study direct sums of τ-H-supplemented modules. Let τ be a cohereditary preradical. Let $M = M_1 \oplus M_2$ be a duo module. Then M is τ-H-supplemented if and only if M_1 and M_2 are τ-H-supplemented (Theorem 4). Let τ be a cohereditary preradical. Let $M = \oplus_{i=1}^{n} M_i$ be a τ-supplemented module. Assume that M_i is τ-M_j-projective for all $j > i$. If each M_i is τ-H-supplemented, then M is τ-H-supplemented (Corollary 4).

In Section 4 we will obtain the relations between τ-H-supplemented modules and the other modules. Let τ be a cohereditary preradical. Let M be a projective module such that every τ-supplement submodule of M is a direct summand. The following are equivalent: (Theorem 6)
(1) M is τ-supplemented;
(2) M is τ-lifting;
(3) M is amply τ-supplemented;
(4) M is τ-H-supplemented and $\tau(M)$ is QSL in M;
(5) M is τ-$\oplus$-supplemented.

1. Factor modules of τ-H-supplemented modules

In this section we will define τ-H-supplemented modules and give an equivalent condition for a module to be τ-H-supplemented. Also we investigate some conditions for factor modules of a τ-H-supplemented module to be τ-H-supplemented.

Keskin Tütüncü, Nematollahi and Talebi give equivalent conditions for a module to be H-supplemented (see [8, Theorem 2.1]). Now we give the definition of a τ-H-supplemented module based on their definition.

Definition 1. Let M be a module. Then M is τ-H-*supplemented* in case for every $A \leq M$ there exists a direct summand D of M such that $(A + D)/A \subseteq \tau(M/A)$ and $(A + D)/D \subseteq \tau(M/D)$.

In this paper, τ-H-supplement will mean that a direct summand D of M exists with the stated inclusions in Definition 1. The definition shows that every τ-lifting module is τ-H-supplemented.

Next we give an equivalent condition for a module to be τ-H-supplemented.

Proposition 1. *Let M be a module. Then M is τ-H-supplemented if and only if for each $A \leq M$ there exists a direct summand D of M and a submodule X of M such that $A \subseteq X$, $D \subseteq X$, $X/A \subseteq \tau(M/A)$ and $X/D \subseteq \tau(M/D)$.*

Proof. ($\Rightarrow$) It is clear.
($\Leftarrow$) Let $A \leq M$. By assumption, there exist a direct summand D of M and $X \leq M$ such that $(A+D)/A \subseteq X/A \subseteq \tau(M/A)$ and $(A+D)/D \subseteq X/D \subseteq \tau(M/D)$. Hence M is τ-H-supplemented. □

A factor module of a τ-H-supplemented module need not be τ-H-supplemented in general. Before giving a counter example to the fact that a factor module of a τ-H-supplemented module need not be τ-H-supplemented in case $\tau = Rad$, we have to mention the following definitions:

A commutative ring R is a *valuation* ring if it satisfies one of the following three equivalent conditions:

(1) for any two elements a and b, either a divides b or b divides a.

(2) the ideals of R are linearly ordered by inclusion.

(3) R is a local ring and every finitely generated ideal is principal.

A module M is called *finitely presented* if $M \cong F/K$ for some finitely generated free module F and finitely generated submodule K of M.

Example 1. Let R be a commutative local ring which is not a valuation ring and let $n \geq 2$. By [16, Theorem 2], there exists a finitely presented indecomposable module $M = R^{(n)}/K$ which cannot be generated by fewer than n elements. By [5, Corollary 1.6], $R^{(n)}$ is $\oplus$-supplemented and hence H-supplemented by [9, Proposition 2.1]. Being finitely generated, $R^{(n)}$ is *Rad*-H-supplemented. Since M is not cyclic, it is not $\oplus$-supplemented, and hence not H-supplemented. Since M is finitely generated, it is not *Rad*-H-supplemented. (Note that since $R/JacR$ is semisimple, the preradical *Rad* is also cohereditary.)

In [8] and [10], the authors give some conditions for a factor module of an H-supplemented module to be H-supplemented. Now we give analogous of their conditions for a τ-H-supplemented module.

Theorem 1. *Let τ be a cohereditary preradical. Let M be a τ-H-supplemented module and $X \leq M$. If for every direct summand K of M, $(X+K)/X$ is a direct summand of M/X, then M/X is τ-H-supplemented.*

Proof. Let $N/X \leq M/X$. Since M is τ-H-supplemented, there exists a direct summand D of M such that $(N+D)/N \subseteq \tau(M/N)$ and $(N+D)/D \subseteq \tau(M/D)$. By assumption, $(D+X)/X$ is a direct summand of M/X. Since τ is a cohereditary preradical, it is easy to check that $\frac{N/X+(D+X)/X}{N/X} \subseteq \tau(\frac{M/X}{N/X})$ and $\frac{N/X+(D+X)/X}{(D+X)/X} \subseteq \tau(\frac{M/X}{(D+X)/X})$. Hence M/X is τ-H-supplemented. □

Let M be a module. Then M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N+(K\cap L) = (N+K)\cap(N+L)$ or $N\cap(K+L) = (N\cap K)+(N\cap L)$.

Corollary 1. *Let M be a τ-H-supplemented module for a cohereditary preradical τ.*
(1) If M is a distributive module, then M/X is τ-H-supplemented for every submodule X of M.
(2) Let $N \leq M$ such that for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Then M/N is τ-H-supplemented.
(3) Let X be a projection invariant submodule of M. Then M/X is τ-H-supplemented. In particular, for every fully invariant submodule A of M, M/A is τ-H-supplemented.

Proof. (1) Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/X = [(D+X)/X]+[(D'+X)/X]$ and $X = X+(D\cap D') = (X+D)\cap(X+D')$. So $M/X = [(D+X)/X]\oplus[(D'+X)/X]$. By Theorem 1, M/X is τ-H-supplemented.
(2) Let $L/N \leq M/N$. Since M is τ-H-supplemented, there exists a direct summand D of M and a submodule X of M such that $X/L \subseteq \tau(M/L)$ and $X/D \subseteq \tau(M/D)$. Let $M = D \oplus D'$. Then by hypothesis, $N = (D \cap N) \oplus (D' \cap N) = (D+N) \cap (D'+N)$. So, $(D+N)/N \oplus (D'+N)/N = M/N$. Now we have $\frac{X/N}{L/N} \subseteq \tau(\frac{M/N}{L/N})$ and $\frac{X/N}{(D+N)/N} \subseteq \tau(\frac{M/N}{(D+N)/N})$ and hence M/N is τ-H-supplemented by Proposition 1.
(3) Clear by (2). □

Proposition 2. *Let M be a τ-H-supplemented module for a cohereditary preradical τ and $N \leq M$. If for each idempotent $e : M \to M$ there exists an idempotent $f : M/N \to M/N$ such that $\frac{(N+e(M))/N}{T/N} \subseteq \tau(\frac{M/N}{T/N})$ where $Imf = T/N$, then M/N is τ-H-supplemented.*

Proof. Let $Y/N \leq M/N$. Since M is τ-H-supplemented, there exists an idempotent $e : M \to M$ and a submodule X of M such that $X/e(M) \subseteq$

$\tau(M/e(M))$ and $X/Y \subseteq \tau(M/Y)$ by Proposition 1. By hypothesis, there exists an idempotent $f : M/N \to M/N$ with $Imf = T/N$ such that $(N + e(M))/T \subseteq \tau(M/T)$. Now, T/N is a direct summand of M/N and $T/N \subseteq X/N$. Clearly $\frac{X/N}{T/N} \subseteq \tau(\frac{M/N}{T/N})$ and $\frac{X/N}{Y/N} \subseteq \tau(\frac{M/N}{Y/N})$. □

Proposition 3. *Let τ be a cohereditary preradical and M_0 a direct summand of a module M such that for every decomposition $M = N \oplus K$ of M, there exist submodules N' of N and K' of K such that $M = M_0 \oplus N' \oplus K'$ with $\tau(K') = K'$. If M is τ-H-supplemented, then M/M_0 is τ-H-supplemented.*

Proof. Let $N/M_0 \leq M/M_0$. Since M is τ-H-supplemented, there exists a decomposition $M = K \oplus S$ such that $(N + K)/N \subseteq \tau(M/N)$ and $(N + K)/K \subseteq \tau(M/K)$. By hypothesis, $M = M_0 \oplus N' \oplus K'$ for $N' \leq K$ and $K' \leq S$ with $\tau(K') = K'$. Now it is easy to see that $(M_0 \oplus N')/M_0$ is a τ-H-supplement of N/M_0 in M/M_0. □

Let M be an R-module and τ a preradical. By $P_\tau(M)$ we denote the sum of all submodules N of M with $\tau(N) = N$. The following Lemma will be very useful for us to prove Corollary 2.

Lemma 1. *Let τ be any preradical and let M be any module. Then*
(1) $\tau(P_\tau(M)) = P_\tau(M)$.
(2) $P_\tau(M)$ is a fully invariant submodule of M.
(3) If $M = N \oplus K$, then $P_\tau(M) = P_\tau(N) \oplus P_\tau(K)$.

Corollary 2. *Let M be a τ-H-supplemented module for a cohereditary preradical τ. If $P_\tau(M)$ is a direct summand of M, then $P_\tau(M)$ and $M/P_\tau(M)$ are τ-H-supplemented.*

Proof. By Corollary 1(3) and Lemma 1(2), $M/P_\tau(M)$ is τ-H-supplemented. Let L be a submodule of M such that $M = P_\tau(M) \oplus L$. Let $M = N \oplus K$. Now, by Lemma 1(3), $M = P_\tau(N) \oplus P_\tau(K) \oplus L$. Therefore $M/L \cong P_\tau(M)$ is τ-H-supplemented by Proposition 3 and Lemma 1(1). □

2. Direct summands of τ-H-supplemented modules

In this section we will consider direct summands of τ-H-supplemented modules. We investigate some conditions for direct summands of a τ-H-supplemented module to be τ-H-supplemented. We call a module M *completely τ-H-supplemented* provided every direct summand of M is τ-H-supplemented. The following Theorem is an analogue of [10, Theorem 2.7].

Theorem 2. *(1) Every τ-lifting module is completely τ-H-supplemented.*

(2)Let M be a τ-H-supplemented module for a cohereditary preradical τ. If M has the SSP, then M is completely τ-H-supplemented.

Proof. (1) It is clear since by [2, 2.10] every direct summand of a τ-lifting module is again τ-lifting.

(2) Assume that M is τ-H-supplemented and M has the SSP. Let N be a direct summand of M. We will show that N is τ-H-supplemented. Let $M = N \oplus N'$ for some submodule N' of M. Suppose that A is a direct summand of M. Since M has the SSP, $A + N'$ is a direct summand of M. Let $M = (A + N') \oplus B$ for some $B \leq M$. Then $M/N' = (A + N')/N' \oplus (B + N')/N'$. Hence by Theorem 1, M/N' is τ-H-supplemented and so N is τ-H-supplemented. □

Proposition 4. *Let M be a duo module. Then M has the SSP.*

Proof. See [10, Page 969]. □

Corollary 3. *Let τ be a cohereditary preradical. Let M be a τ-H-supplemented duo module. Then M is completely τ-H-supplemented.*

The following is an example for Theorem 2(2) in case $\tau = Rad$.

Example 2. Let F be a field and R the upper triangular matrix ring R $= \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Since $R/JacR$ is semisimple, the preradical Rad is cohereditary. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $M = A \oplus (R/B)$. Then M is H-supplemented by [6, Lemma 3]. Also M has the SSP. Therefore M is a completely τ-H-supplemented module by Theorem 2(2).

3. Direct sums of τ-H-supplemented modules

The following example shows that any (finite) direct sum of τ-H-supplemented modules need not be τ-H-supplemented for $\tau = Rad$. We will show that under some conditions it will be true.

Example 3. Let R be a commutative local ring and M a finitely generated R-module. Assume $M \cong \bigoplus_{i=1}^{n} R/I_i$. Since every I_i is fully invariant in R, every R/I_i is τ-H-supplemented by Corollary 1(3). By [11, Lemma A.4], M is τ-H-supplemented if $I_1 \leq I_2 \leq ... \leq I_n$. If we don't have the condition $I_1 \leq I_2 \leq ... \leq I_n$, M is not τ-H-supplemented. (Note that since M is finitely generated, M is H-supplemented if and only if it is τ-H-supplemented.)

We call a module M *τ-semilocal* provided that $M/\tau(M)$ is semisimple. Clearly τ-supplemented modules are τ-semilocal.

Lemma 2. *Let M be a τ-H-supplemented module for a cohereditary preradical τ. Then M is τ-semilocal.*

Proof. Let $N/\tau(M) \leq M/\tau(M)$. Since M is τ-H-supplemented, there exists a direct summand D of M such that $(N+D)/N \subseteq \tau(M/N)$ and $(N+D)/D \subseteq \tau(M/D)$. Since $D \leq_d M$, $M = D \oplus D'$ for some submodule D' of M. Then $M = D' + N$. It follows that $M/\tau(M) = N/\tau(M) + (D' + \tau(M))/\tau(M)$. Since $N \cap D' \subseteq \tau(D')$, $M/\tau(M) = N/\tau(M) \oplus (D' + \tau(M))/\tau(M)$. Hence $M/\tau(M)$ is semisimple. □

Proposition 5. *Let M be a module. Then the following are equivalent for a cohereditary preradical τ:*
(1) M is τ-H-supplemented;
(2) M is τ-semilocal and each submodule (direct summand) of $M/\tau(M)$ lifts to a direct summand of M.

Proof. (1) $\Rightarrow$ (2) By Lemma 2, we only prove the last statement. Let $N/\tau(M) \leq M/\tau(M)$. Since M is τ-H-supplemented, there exists $D \leq_d M$ such that $(N+D)/N \subseteq \tau(M/N)$ and $(N+D)/D \subseteq \tau(M/D)$. Then $D \subseteq N$. Hence $N/\tau(M) = (D + \tau(M))/\tau(M)$. This means $N/\tau(M)$ lifts to D.

(2) $\Rightarrow$ (1) Let $N \leq M$. Then by assumption, $(N + \tau(M))/\tau(M) = \overline{N}$ is a direct summand of $M/\tau(M) = \overline{M}$. Then by assumption $\overline{N} = \overline{L}$ such that $M = L \oplus K$. The rest is easy by taking L as a τ-H-supplement of N in M. □

Theorem 3. *Let τ be a cohereditary preradical. Let $M = \oplus_{i\in I} H_i$ be a direct sum of τ-H-supplemented modules H_i $(i \in I)$. Assume that each direct summand of $M/\tau(M)$ lifts to a direct summand of M. Then M is τ-H-supplemented.*

Proof. Clearly $M/\tau(M)$ is semisimple by Lemma 2. Now M is τ-H-supplemented by Proposition 5. □

Theorem 4. *Let τ be a cohereditary preradical. Let $M = M_1 \oplus M_2$ be a duo module. Then M is τ-H-supplemented if and only if M_1 and M_2 are τ-H-supplemented.*

Proof. Note that for $A \leq M$, we can write $A = (A \cap M_1) \oplus (A \cap M_2)$.
$(\Rightarrow)$ Assume that M is τ-H-supplemented. Since M_1 and M_2 are fully invariant submodules of M, M_1 and M_2 are τ-H-supplemented by Corollary 1(3).

($\Leftarrow$) Suppose that M_1 and M_2 are τ-H-supplemented. Let $A \leq M$. Then $A = (A \cap M_1) \oplus (A \cap M_2)$. By assumption, there exist direct summands D_1 of M_1 and D_2 of M_2 such that $((A \cap M_1) + D_1)/(A \cap M_1) \subseteq \tau(M_1/(A \cap M_1))$, $((A \cap M_1) + D_1)/D_1 \subseteq \tau(M_1/D_1)$ and $((A \cap M_2) + D_2)/(A \cap M_2) \subseteq \tau(M_2/(A \cap M_2))$, $((A \cap M_2) + D_2)/D_2 \subseteq \tau(M_2/D_2)$. It is not hard to see that $(A + (D_1 \oplus D_2))/A \subseteq \tau(M/A)$ and $(A + (D_1 \oplus D_2))/(D_1 \oplus D_2) \subseteq \tau(M/(D_1 \oplus D_2))$. Namely, $D_1 \oplus D_2$ is a τ-H-supplement of A in M. Hence M is τ-H-supplemented. □

Definition 2. Let M and N be two modules. Let τ be a preradical. Then N is called *τ-M-projective* if, for any $K \leq M$ and any homomorphism $f : N \longrightarrow M/K$ there exists a homomorphism $h : N \longrightarrow M$ such that $Im(f - \pi h) \subseteq \tau(M/K)$, where $\pi : M \longrightarrow M/K$ is the natural epimorphism.

Lemma 3. *Let $M = M_1 \oplus M_2$. Consider the following conditions:*

1. *M_1 is τ-M_2-projective;*
2. *For every $K \leq M$ with $K + M_2 = M$, there exists $M_3 \leq M$ such that $M = M_2 \oplus M_3$ and $(K + M_3)/K \subseteq \tau(M/K)$.*

Then $(1) \Rightarrow (2)$.

Proof. Let $K \leq M$ and $M = K + M_2$. Consider the epimorphism $\pi : M_2 \longrightarrow M/K$ with $m_2 \mapsto m_2 + K (m_2 \in M_2)$ and the homomorphism $h : M_1 \longrightarrow M/K$ with $m_1 \mapsto m_1 + K (m_1 \in M_1)$. Since M_1 is τ-M_2-projective, there exist a homomorphism $\overline{h} : M_1 \longrightarrow M_2$ and a submodule X of M with $K \subseteq X$ such that $Im(h - \pi\overline{h}) = X/K \subseteq \tau(M/K)$. Let $M_3 = \{a - \overline{h}(a) \mid a \in M_1\}$. Clearly $M = M_2 \oplus M_3$. Since $K + M_3 \subseteq X$, $(K + M_3)/K \subseteq X/K$. Hence $(K + M_3)/K \subseteq \tau(M/K)$. □

Lemma 4. *Let A and $\{M_i\}_{i=1}^{n}$ be modules. If each M_i is τ-A-projective, for $i = 1, 2, \ldots n$, then $\bigoplus_{i=1}^{n} M_i$ is τ-A-projective.*

Proof. The proof is straightforward. □

Theorem 5. *Let τ be a cohereditary preradical. Let $M = M_1 \oplus M_2$ be a τ-supplemented module. Assume M_1 is τ-M_2-projective (or M_2 is τ-M_1-projective). If M_1 and M_2 are τ-H-supplemented, then M is τ-H-supplemented.*

Proof. Let $Y \leq M$.
Case 1: Let $M = Y + M_2$. Then by Lemma 3, there exists $M_3 \leq M$ such that $M = M_3 \oplus M_2$ and $(Y + M_3)/Y \subseteq \tau(M/Y)$. Since M/M_3 is τ-H-supplemented, there exist $X/M_3 \leq M/M_3$ and a direct summand D/M_3

of M/M_3 such that $\frac{X/M_3}{(Y+M_3)/M_3} \subseteq \tau(\frac{M/M_3}{(Y+M_3)/M_3})$ and $\frac{X/M_3}{D/M_3} \subseteq \tau(\frac{M/M_3}{D/M_3})$ by Proposition 1. Clearly, D is a direct summand of M. It is easy to check that $X/D \subseteq \tau(M/D)$ and $X/Y \subseteq \tau(M/Y)$. Therefore M is τ-H-supplemented by Proposition 1.

Case 2: Let $Y + M_2 \neq M$. Since M is τ-supplemented, $M/\tau(M)$ is semisimple. Then there exists a submodule K of M containing $\tau(M)$ such that $M/\tau(M) = K/\tau(M) \oplus (Y+M_2+\tau(M))/\tau(M)$. So $M = (K+Y)+M_2$ and $\tau(M) = K \cap (Y + M_2 + \tau(M)) = \tau(M) + (K \cap (Y + M_2))$ and hence $K \cap (Y + M_2) \subseteq \tau(M)$. By Lemma 3, there exists $M_4 \leq M$ such that $M = M_2 \oplus M_4$ and $(K + Y + M_4)/(K + Y) \subseteq \tau(M/(K + Y))$. This implies that $K + Y + M_4 \subseteq \tau(M) + K + Y = K + Y$. Now M/M_2 and M/M_4 are τ-H-supplemented. Therefore there exist submodules X_1/M_2 of M/M_2 and X_2/M_4 of M/M_4 and direct summands D_1/M_2 of M/M_2 and D_2/M_4 of M/M_4 such that $\frac{X_1/M_2}{(Y+M_2)/M_2} \subseteq \tau(\frac{M/M_2}{(Y+M_2)/M_2})$, $\frac{X_1/M_2}{D_1/M_2} \subseteq \tau(\frac{M/M_2}{D_1/M_2})$, $\frac{X_2/M_4}{(Y+K+M_4)/M_4} \subseteq \tau(\frac{M/M_4}{(Y+K+M_4)/M_4})$ and $\frac{X_2/M_4}{D_2/M_4} \subseteq \tau(\frac{M/M_4}{D_2/M_4})$. Clearly, $D_1 \cap D_2$ is a direct summand of M. Let $M = (D_1 \cap D_2) \oplus L$ for some submodule L of M. Then by [7, Lemma 1.2], $M = D_2 \oplus (D_1 \cap L)$. Note that we have that $X_1 \subseteq \tau(M) + D_1$, $X_1 \subseteq \tau(M) + M_2 + Y$, $X_2 \subseteq \tau(M) + D_2$ and $X_2 \subseteq \tau(M) + Y + K + M_4 = K + Y$. Now,

$$\begin{aligned} X_1 \cap X_2 &\subseteq (\tau(M) + M_2 + Y) \cap (Y + K) \\ &= (\tau(M) + Y) + (M_2 \cap (Y + K)) \\ &\subseteq \tau(M) + Y + [K \cap (Y + M_2)] + [Y \cap (K + M_2)] \\ &= \tau(M) + Y \end{aligned}$$

and

$$\begin{aligned} X_1 \cap X_2 &\subseteq (\tau(M) + D_1) \cap (\tau(M) + D_2) \\ &= (\tau(D_2) + D_1) \cap (\tau(D_1 \cap L) + D_2) \\ &= \tau(D_2) + [(D_2 + \tau(D_1 \cap L)) \cap D_1] \\ &= \tau(D_2) + \tau(D_1 \cap L) + (D_1 \cap D_2) \\ &\subseteq \tau(M) + (D_1 \cap D_2). \end{aligned}$$

Therefore $(X_1 \cap X_2)/Y \subseteq \tau(M/Y)$ and $(X_1 \cap X_2)/(D_1 \cap D_2) \subseteq \tau(M/(D_1 \cap D_2))$. Thus M is τ-H-supplemented by Proposition 1. □

Corollary 4. *Let τ be a cohereditary preradical. Let $M = \oplus_{i=1}^{n} M_i$ be a τ-supplemented module. Assume that M_i is τ-M_j-projective for all $j > i$. If each M_i is τ-H-supplemented, then M is τ-H-supplemented.*

Proof. By Lemma 4 and Theorem 5. □

4. Relations between τ-H-supplemented modules and the others

A module M is called *τ-$\oplus$-supplemented* if for every $A \leq M$, there exists a $B \leq_d M$ such that $A + B = M$ and $A \cap B \subseteq \tau(B)$. Clearly every τ-lifting module is τ-$\oplus$-supplemented and every τ-$\oplus$-supplemented module is τ-supplemented.

Next we will show that under some conditions every τ-$\oplus$-supplemented module is τ-H-supplemented.

Proposition 6. *Let τ be any preradical. Assume M is τ-$\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective. Then M is τ-H-supplemented.*

Proof. Let $N \leq M$. Since M is τ-$\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \subseteq \tau(M_2)$ for submodules M_1, M_2 of M. By hypothesis, M_1 is M_2-projective. By [11, Lemma 4.47], we obtain $M = A \oplus M_2$ for some submodule A of M such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. It is easy to see that $(N + A)/A \subseteq \tau(M/A)$ and $(N + A)/N \subseteq \tau(M/N)$. Thus M is τ-H-supplemented. □

Corollary 5. *Let τ be any preradical. Let M be a τ-$\oplus$-supplemented module. If M is projective, then M is τ-H-supplemented.*

Let $e = e^2 \in R$. Then e is called a *left (right) semicentral idempotent* if $xe = exe$ $(ex = exe)$, for all $x \in R$. The set of all left (right) semicentral idempotents is denoted by $S_l(R)$ $(S_r(R))$. A ring R is called *Abelian* if every idempotent is central.

Proposition 7. *Let τ be a preradical and M an R-module such that $End(M)$ is Abelian and $X \leq M$ implies $X = \sum_{i \in I} h_i(M)$ where $h_i \in End(M)$. If M is τ-$\oplus$-supplemented, then M is τ-H-supplemented and satisfies the (D_3)-condition.*

Proof. Let $X \leq M$, $X = \sum_{i \in I} h_i(M)$ with $h_i \in End(M)$. Since M is τ-$\oplus$-supplemented, there exists a direct summand eM such that $X + eM = M$ and $(X \cap eM) \subseteq \tau(eM)$ for some $e^2 = e \in End(M)$. Since $End(M)$ is Abelian, $(1-e)X = (1-e)M = (1-e)\sum_{i \in I} h_i(M) = \sum_{i \in I} h_i(1-e)(M) \subseteq X$. Therefore $X = (1 - e)M \oplus (X \cap eM)$. Then $(1 - e)M$ is a τ-H-supplement of X. If $eM + fM = M$ for $e^2 = e$, $f^2 = f \in End(M)$, then $eM \cap fM = efM$ with $(ef)^2 = ef$. So M satisfies the (D_3)-condition. □

Recall that for a commutative ring R, an R-module M is said to be a *multiplication module* if for each $X \leq M$, $X = MA$ for some ideal A of R.

Corollary 6. *Let τ be a preradical and M a τ-$\oplus$-supplemented module. If M satisfies one of the following conditions, then M is τ-H-supplemented.*
(1) M is a multiplication module and R is commutative.
(2) M is cyclic and R is commutative.

Proof. (1) Assume M is a multiplication module. Let $X \leq M$. Then $X = MA$ for some ideal A of R. For each $a \in A$, define $h_a : M \to M$ by $h_a(m) = ma$ for all $m \in M$. Then h_a is an R-homomorphism and $X = MA = \sum_{a\in A} h_a(M)$. Since every multiplication module is a duo module, thus if $e^2 = e \in S = End(M)$, then $e, 1-e \in S_l(S)$. Therefore e is central. So $End(M)$ is Abelian. By Proposition 7, M is τ-H-supplemented.
(2) Clear by (1) since every cyclic module over a commutative ring is a multiplication module. □

Now we investigate the relations between τ-H-supplemented modules and the others. A module M is called *amply τ-supplemented* if for any submodules K and V of M such that $M = K + V$, there is a submodule U of V such that $K + U = M$ and $K \cap U \subseteq \tau(U)$.

Lemma 5. *Let τ be any preradical and let M be a projective module. The following are equivalent:*
(1) M is τ-supplemented;
(2) M is amply τ-supplemented.

Proof. Clearly an amply τ-supplemented module is τ-supplemented. For the converse: Let $M = U + V$ and X be a τ-supplement of U in M. For an $f \in End(M)$ with $Im(f) \subseteq V$ and $Im(I - f) \subseteq U$ we have $f(U) \subseteq U$, $M = U + f(X)$ and $f(U \cap X) = U \cap f(X)$ (from $u = f(x)$ we derive $x - u = (I - f)(x) \in U$ and $x \in U$). Since $U \cap X \subseteq \tau(X)$, we also have $U \cap f(X) \subseteq \tau(f(X))$, i.e. $f(X)$ is a τ-supplement of U with $f(X) \subseteq V$. Hence M is amply τ-supplemented. □

Let M be any module. A submodule U of M is called *quasi strongly lifting* (QSL) in M if whenever $(A + U)/U$ is a direct summand of M/U, there exists a direct summand P of M such that $P \leq A$ and $P+U = A+U$ (see [1]).

Lemma 6. *Let τ be a cohereditary preradical and let M be any module. The following are equivalent:*
(1) M is τ-lifting;
(2) M is τ-H-supplemented and $\tau(M)$ is QSL in M.

Proof. By Lemma 2 and [1, Lemma 3.5 and Proposition 3.6]. □

Lemma 7. *Let τ be any preradical and let M be a projective module such that every τ-supplement submodule of M is a direct summand of M. The following are equivalent:*
(1) M is τ-supplemented;
(2) M is amply τ-supplemented;
(3) M is τ-lifting;
(4) M is τ-$\oplus$-supplemented.

Proof. (1) $\Leftrightarrow$ (2) By Lemma 5.

(1) $\Rightarrow$ (3) By [1, Lemma 3.2].

(3) $\Rightarrow$ (1) and (1) $\Leftrightarrow$ (4) are clear by definitions and the assumption that every τ-supplement submodule of M is a direct summand of M. □

Now we have the following Theorem:

Theorem 6. *Let τ be a cohereditary preradical. Let M be a projective module such that every τ-supplement submodule of M is a direct summand. The following are equivalent:*
(1) M is τ-supplemented;
(2) M is τ-lifting;
(3) M is amply τ-supplemented;
(4) M is τ-H-supplemented and $\tau(M)$ is QSL in M;
(5) M is τ-$\oplus$-supplemented.

As we see in Example 3 a finite direct sum of τ-H-supplemented modules need not be τ-H-supplemented. We will show that a finite direct sum of τ-$\oplus$-supplemented modules is τ-$\oplus$-supplemented.

Lemma 8. *Let $N, L \leq M$ such that $N + L$ has a τ-supplement H in M and $N \cap (H + L)$ has a τ-supplement G in N. Then $H + G$ is a τ-supplement of L in M.*

Proof. Let H be a τ-supplement of $N+L$ in M and G be a τ-supplement of $N \cap (H+L)$ in N. Then $M = (N+L)+H$ such that $(N+L)\cap H \subseteq \tau(H)$ and $N = [N \cap (H + L)] + G$ such that $(H + L) \cap G \subseteq \tau(G)$. Since $(H+G)\cap L \subseteq [(G+L)\cap H]+[(H+L)\cap G] \subseteq \tau(H)+\tau(G) \subseteq \tau(H+G)$, $H + G$ is a τ-supplement of L in M. □

Theorem 7. *For a ring R, any finite direct sum of τ-$\oplus$-supplemented R-modules is τ-$\oplus$-supplemented.*

Proof. Let $M = M_1 \oplus ... \oplus M_n$ and M_i be a τ-$\oplus$-supplemented module for each $1 \leq i \leq n$. To prove that M is τ-$\oplus$-supplemented it is sufficient to assume $n = 2$.

Let $L \leq M$. Then $M = M_1 + M_2 + L$ so that $M_1 + M_2 + L$ has a τ-supplement 0 in M. Let H be a τ-supplement of $M_2 \cap (M_1 + L)$ in M_2 such that $H \leq_d M_2$. By Lemma 8, H is a τ-supplement of $M_1 + L$ in M. Let K be a τ-supplement of $M_1 \cap (L + H)$ in M_1 such that $K \leq_d M_1$. Again by applying Lemma 8, we get that $H + K$ is a τ-supplement of L in M. Since $H \leq_d M_2$ and $K \leq_d M_1$, it follows that $H + K = H \oplus K \leq_d M$. Thus $M = M_1 \oplus M_2$ is τ-$\oplus$-supplemented. □

Note that by the same proof as the proof of Theorem 7, any finite sum of τ-supplemented modules is τ-supplemented.

Theorem 8. *Let τ be a cohereditary preradical. Let R be a τ-$\oplus$-supplemented ring (i.e. R_R is τ-$\oplus$-supplemented) such that every finite direct sum of the copies of R is distributive. Then the following are equivalent:*
(1) R is τ-H-supplemented;
(2) Every finitely generated free R-module is τ-H-supplemented;
(3) Every finitely generated projective R-module is τ-H-supplemented;
(4) If F is a finitely generated free R-module and N a fully invariant submodule, then F/N is τ-H-supplemented.

Proof. (1) ⇒ (3) Let M be a finitely generated projective R-module. Then M is isomorphic to a direct summand of a finitely generated free module F. By Corollary 4, F is τ-H-supplemented. Thus M is τ-H-supplemented by Corollary 1(1).

(3) ⇒ (2) ⇒ (1) and (4) ⇒ (1) are clear.

(2) ⇒ (4) By (2), F is τ-H-supplemented. The result follows from Corollary 1(3). □

We next consider the preradical $\overline{Z}$.

Let M be a module and $\mathcal{S}$ denote the class of all small modules. Talebi and Vanaja defined $\overline{Z}(M)$ in [13] as follows:

$\overline{Z}(M) = \bigcap\{ker g \mid g \in \mathrm{Hom}(M, L), L \in \mathcal{S}\}$. The module M is called *cosingular* (*non-cosingular*) if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Clearly every non-cosingular module is $\overline{Z}$-H-supplemented. Also if R is a non-cosingular ring, then every R-module is $\overline{Z}$-H-supplemented by [13, Proposition 2.5 and Corollary 2.6].

Let M be a module and τ_M a preradical on $\sigma[M]$. In [12], the authors call a module $N \in \sigma[M]$ τ_M-*semiperfect* if it satisfies one of the following conditions (see [12, Proposition 2.1 and Definition 2.2]):

(1) For every submodule K of N there exists a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$;

(2) For every submodule K of N, there exists a decomposition $N = A \oplus B$ such that A is projective in $\sigma[M]$, $A \leq K$ and $K \cap B \subseteq \tau_M(N)$.

If $\sigma[M] = \text{Mod} - \text{R}$, then they call N *τ-semiperfect*.

By the above definition, every τ-semiperfect module is τ-lifting and hence τ-H-supplemented. Also if M is projective we have the following:

τ-semiperfect $\Leftrightarrow$ τ-lifting $\Leftrightarrow$ τ-$\oplus$-supplemented $\Rightarrow$ τ-H-supplemented

In [12, Theorem 2.23], the authors showed that their τ-semiperfect module definition agrees with the definition of τ-semiperfect module in the sense of [2] for a projective module and for the preradical *Soc*. In [14], Tribak and Keskin Tütüncü studied $\overline{Z}$-lifting modules and $\overline{Z}$-semiperfect modules in the sense of [12]. They also investigate some conditions for the preradical $\overline{Z}$ for two definitions of τ-semiperfect modules to be agreed (see [14, Proposition 5.8 and Proposition 5.11]).

A τ-H-supplemented module need not be H-supplemented. Of course if $\tau(M) \ll M$ and τ is cohereditary, then every τ-H-supplemented module is H-supplemented.

Example 4. Let K be a field and let $R = \prod_{n \geq 1} K_n$ with $K_n = K$. By [14, Example 4.1(1)] R is not semiperfect. Since R is projective, R is not $\oplus$-supplemented by [5, Lemma 1.2]. Hence R is not H-supplemented. Again by [14, Example 4.1(1)], the module R is $\overline{Z}$-semiperfect in the sense of [12] and so it is $\overline{Z}$-H-supplemented.

If R is a DVR (Discrete Valuation Ring), then the R-module R is semiperfect and hence H-supplemented.

Now we give an equivalent condition for a module to be $\overline{Z}$-$\oplus$-supplemented module under some assumptions.

Proposition 8. *Let R be a commutative ring, P a projective module with $Rad(P) \ll P$ and assume P to have finite hollow dimension. Then the following are equivalent:*

(1) P is $\overline{Z}$-$\oplus$-supplemented;

(2) $P = P_1 \oplus P_2 \oplus P_3$ with P_1 $\oplus$-supplemented and $Rad(P_1) = \overline{Z}(P_1)$, P_2 semisimple and $\overline{Z}(P_3) = P_3$.

Proof. (1) $\Rightarrow$ (2) See the proof of [14, Corollary 4.3] and [5, Lemma 2.1].

(2) $\Rightarrow$ (1) By [14, Corollary 4.3] all P_1, P_2 and P_3 are $\overline{Z}$-semiperfect in the sense of [12] and hence $\overline{Z}$-$\oplus$-supplemented. By Theorem 7, P is $\overline{Z}$-$\oplus$-supplemented. □

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