Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 12 **(2011)**. Number 1. pp. 28 – 52

Free field realizations of certain modules for affine Lie algebra $\widehat{sl}(n,\mathbb{C})$

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Communicated by V. V. Kirichenko

Abstract. For the affine Lie algebra $\widehat{sl}(n,\mathbb{C})$ we study a realization in terms of infinite sums of partial differential operators of a family of representations introduced in [BBFK]. These representations generalize a construction of Imaginary Verma modules [F1]. The realization constructed in the paper extends the free field realization of Imaginary Verma modules constructed by B.Cox [C1].

1. Introduction

Representation theory of affine Lie algebras is a very rich subject with many applications in Mathematics and Physics. Representations of affine Lie algebras have some very distinct features which have no analogs in the finite dimensional case. For example, there exist modules for affine Lie algebras containing both finite and infinite-dimensional weight spaces. The simplest example of such modules is given by the so-called Imaginary Verma modules [F1]. These representations correspond to nonstandard partitions of the root system which are not equivalent under the Weyl group to the standard partition into positive and negative roots (for details see [DFG]). For affine Lie algebras, there are always only finitely many equivalence classes of such nonstandard partitions (see [F2]). These partitions give rise to *Verma-type modules*, which were first studied and classified by Jakobsen and Kac [JK], and by Futorny [F2, F3] (see also [C2, F1, FS, FK]).

In a recent paper [BBFK] different Borel-type subalgebras that do not correspond to partitions of the root system of affine Lie algebra $\mathfrak{g}$ were considered. They correspond to functions $\varphi : \mathbb{N} \to \{\pm\}$ on the set $\mathbb{N}$ of positive integers, and give rise to a class of modules called φ-*Imaginary*

2000 Mathematics Subject Classification: *17B67, 81R10.*

Verma modules. These modules can be viewed as induced from φ-highest weight modules over the Heisenberg subalgebra of $\mathfrak{g}$. If $\varphi(n) = +$ for all $n \in \mathbb{N}$, then *φ-Imaginary Verma modules* are just Imaginary Verma modules. It was shown in [BBFK] that *φ-Imaginary Verma module* is irreducible if and only if its central charge is nonzero.

In the case of Imaginary Verma modules for affine Lie algebra $\widehat{sl}(n, \mathbb{C})$ their free field realization was constructed by B.Cox in [C1]. This realization generalizes the classical Wakimoto construction in terms of infinite sums of partial differential operators. Later this has been generalized by B.Cox and V.Futorny [CF] for other Verma type modules.

The purpose of this paper is to extend the construction of [C1] to all φ-Imaginary Verma modules over the affine Lie algebra $\widehat{sl}(n, \mathbb{C})$.

The structure of the paper is as follows. In Section 3, we recall a construction of $\mathbb{J}$-Imaginary Verma modules for affine Lie algebras following [BBFK]. In Section 4, we consider the case $n = 2$ and construct a realization $\rho_{\mathbb{J}}$ of $\mathbb{J}$-Imaginary Verma modules for affine $sl(2)$. Finally, in Section 5, we construct a representation $\rho_{\mathbb{J}} : \widehat{sl}(n, \mathbb{C}) \to gl(\mathbb{C}[x, y])$, for all $\mathbb{J}$, using $\varphi_{\mathbb{J}}$ and two auxiliary anti-automorphisms $\rho_1 : \widehat{sl}(n, \mathbb{C}) \to \widehat{sl}(n, \mathbb{C})$ and $\rho_2 : gl(\mathbb{C}[x, y]) \to gl(\mathbb{C}[x, y])$. Note that generically (when the central charge of the module is not zero) $\mathbb{J}$-Imaginary Verma module is irreducible. Hence, our construction provides free field realization of a large family of irreducible modules for affine $\widehat{sl}(n, \mathbb{C})$.

2. Preliminaries

Let $sl(n, \mathbb{C})$ be a complex Lie algebra of $n{\times}n$-matrices with trace zero with the Killing form $(X|Y) = tr(XY)$. Denote by E_{ij} the matrix units, $i, j = 1, \ldots, n$ and set $H_i = E_{ii} - E_{i+1,i+1}$, $i = 1, \ldots, n-1$. Then E_{ij}, $i \neq j$ and H_i's form a basis of $sl(n, \mathbb{C})$.

The affine Lie algebra $\widehat{sl}(n, \mathbb{C})$ is the universal central extension with the 1-dimensional center $\mathbb{C}c$ of the loop algebra $sl(n, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. Set $X(m) := t^m \otimes X$, for all $X, Y \in sl(n, \mathbb{C})$ and $m \in \mathbb{Z}$. Then $\widehat{sl}(n, \mathbb{C})$ is generated by $E_i(m)$, $F_i(m)$ and $H_i(m)$, with $m \in \mathbb{Z}$ and $1 \leqslant i \leqslant n$, and the central element c.

Fix a symbol A and a sequence $\{A(m)\}_{m \in \mathbb{Z}}$. Define:

$$A^+(z) := \sum_{m \in \mathbb{Z}>0} A(m) z^{-m}, \tag{2.1}$$

$$A^-(z) := \sum_{m \in \mathbb{Z}\leqslant 0} A(m) z^{-m}, \tag{2.2}$$

$$A(z) := A^+(z) + A^-(z) = \sum_{m \in \mathbb{Z}} A(m) z^{-m}, \tag{2.3}$$

$$\dot{A}(z) := \dot{A}^{+}(z) + \dot{A}^{-}(z) = \sum_{m\in\mathbb{Z}} mA(m)z^{-m}, \tag{2.4}$$

where, for example, $\dot{H}_i^{+}(z) = \sum_{m\in\mathbb{Z}>0} mH_i(m)z^{-m}$.

Then we have the following defining relations in $\widehat{sl}(n,\mathbb{C})$:

$(R1)$ $[H_i(z), H_j(w)] = -(H_i|H_j)c\dot{\delta}(z-w)$,

$(R2)$ $[H_i(z), E_j(w)] = C_{ij}\sum_n E_j(z)z^n w^{-n} = C_{ij}E_j(z)\delta(z-w)$,

$(R3)$ $[H_i(z), F_j(w)] = -C_{ij}\sum_n F_j(z)z^n w^{-n} = -C_{ij}F_j(z)\delta(z-w)$,

$(R4)$ $[E_i(z), F_j(w)] = \delta_{ij}(H_i(z)\delta(z-w) - c\dot{\delta}(z-w))$,

$(R5)$ $[E_i(z), E_j(w)] = [F_i(z), F_j(w)] = 0$ if $C_{ij} \neq -1$,

$(R6)$ $[F_i(z_1), F_i(z_2), F_j(w)] = [E_i(z_1), E_i(z_2), E_j(w)] = 0$ if $C_{ij} = -1$,

where $C = (C_{ij})$ denotes the Cartan matrix of type A_n and $[X, Y, Z] := [X, [Y, Z]]$.

Elements $H_i, i = 1, \ldots, n-1$ and c span a Cartan subalgebra of $\widehat{sl}(n,\mathbb{C})$, while $H_i(m)$, $m \in \mathbb{Z}$, $1 \leqslant i \leqslant n$, and c span a Heisenberg subalgebra of $\widehat{sl}(n,\mathbb{C})$. The central element c acts a scalar on any irreducible module V. This scalar is called the *central charge* of V.

Now fix $\gamma \in \mathbb{C}^*$ and for all $1 \leqslant i \leqslant n$, fix $\lambda_i \in \mathbb{C}$. Let $2c = \gamma^2$. Then:

$$\mathbb{C}[x] := \mathbb{C}[x_{ij}(m) \mid i, j, m \in \mathbb{Z}, 1 \leqslant i, j \leqslant n] \tag{2.5}$$
$$\mathbb{C}[y] := \mathbb{C}[y_i(m) \mid i, m \in \mathbb{N}^*, 1 \leqslant i \leqslant n] \tag{2.6}$$

are the algebras generated over $\mathbb{C}$ by $x_{ij}(m)$ and $y_i(m)$, respectively. In $\mathbb{C}[x]$, define operators:

$$a_{ij} := -x_{ij}(m); \tag{2.7}$$
$$a^*_{ij}(m) := \frac{\partial}{\partial x_{ij}(-m)}. \tag{2.8}$$

where $[a_{ij}(m), a^*_{kl}(p)] = \delta_{ik}\delta_{jl}\delta_{m,-p}$. Fix an arbitrary $\mathbb{J} \subseteq \mathbb{N}^*$. In $\mathbb{C}[y]$

define operators $b_i(m)$, with $m \in \mathbb{Z}$ and $1 \leqslant i \leqslant n$ as follows:

$$b_i(m) := \begin{cases} -\gamma^{-1}\lambda_i & \text{if } m = 0; \\ -\gamma m \dfrac{\partial}{\partial y_i(m)} & \text{if } m \in \mathbb{N}^* \setminus \mathbb{J}; \\ -\gamma^{-1} y_i(m) & \text{if } m \in \mathbb{J}; \\ \gamma m \dfrac{\partial}{\partial y_i(-m)} & \text{if } -m \in \mathbb{J}; \\ -\gamma^{-1} y_i(-m) & \text{if } -m \in \mathbb{N}^* \setminus \mathbb{J}. \end{cases} \tag{2.9}$$

where we have $[b_i(m), b_j(p)] = m\delta_{ij}\delta_{m,-p}$. Let

$$\mathbb{C}[x,y] := \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y].$$

3. $\mathbb{J}$-Imaginary Verma modules for $\mathfrak{g}$

Fix $\mathbb{J} \subseteq \mathbb{N}^*$ and let $\psi_{\mathbb{J}} : \mathbb{Z} \to \{0,1\}$ be the function given by:

$$\psi_{\mathbb{J}}(m) := \begin{cases} 1 & \text{if } m \in \mathbb{J} \text{ or } -m \in \mathbb{N}^* \setminus \mathbb{J}; \\ 0 & \text{if } m \in \mathbb{N}^* \setminus \mathbb{J} \text{ or } -m \in \mathbb{J}. \end{cases} \tag{3.1}$$

Let $\mathfrak{g} = \widehat{sl}(n,\mathbb{C})$ and consider $\mathsf{S}_{\psi_{\mathbb{J}}} = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{N}, \varphi_{\mathbb{J}}(n) = 1\} \cup \{-m\delta \mid m \in \mathbb{N}, \varphi_{\mathbb{J}}(m) = 0\}$. Then the spaces $\mathfrak{g}_{\mathsf{S}_{\psi_{\mathbb{J}}}} = \bigoplus_{\alpha \in \mathsf{S}_{\psi_{\mathbb{J}}}} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\mathsf{S}_{\psi_{\mathbb{J}}}} = \bigoplus_{\alpha \in -\mathsf{S}_{\psi_{\mathbb{J}}}} \mathfrak{g}_\alpha$ are subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_{-\mathsf{S}_{\psi_{\mathbb{J}}}} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\mathsf{S}_{\psi_{\mathbb{J}}}}$ of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$ and suppose $\lambda(c) = a,\ a \in \mathbb{C}$. Let $\mathfrak{b}_{\psi_{\mathbb{J}}} = \mathfrak{h} \oplus \mathfrak{g}_{\mathsf{S}_{\psi_{\mathbb{J}}}}$ be the Borel subalgebra corresponding to $\mathsf{S}_{\psi_{\mathbb{J}}}$, and note that $\mathfrak{b}_{\psi_{\mathbb{J}}} \supset \mathbb{C}c \oplus \mathsf{L}^+_{\psi_{\mathbb{J}}}$. Let $\mathbb{C}v_\lambda$ be a module of dimension one for $\mathfrak{b}_{\psi_{\mathbb{J}}}$ such that for all $h \in \mathfrak{h}$, $\mathfrak{g}_{\mathsf{S}_{\psi_{\mathbb{J}}}} v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$.

Remark 3.1. We call the $\mathfrak{g}$-module

$$\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda) := \mathsf{M}_{\mathfrak{b}_{\psi_{\mathbb{J}}}}(\lambda) = \mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{b}_{\psi_{\mathbb{J}}})} \mathbb{C}v_\lambda$$

a $\mathbb{J}$-Imaginary Verma module. Identifying $1 \otimes v_\lambda$ with v_λ, the $\mathsf{U}(\mathsf{L})$-submodule of $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$ generated by v_λ is isomorphic to $\mathsf{M}_{\psi_{\mathbb{J}}}(a)$. If $\psi_{\mathbb{J}}(n) = 1$ for all n, then $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$ coincides with the imaginary Verma module $\mathsf{M}_{\mathsf{S}}(\lambda)$ where $\mathsf{S} = \Delta_+$.

Here we present some basic properties of $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$.

Proposition 3.2. [BBFK, Proposition 3.4] Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda(c) = a$. If $a \neq 0$, then the following statements are true for $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$.

- $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$ is a free $\mathsf{U}(\mathfrak{g}_{-\mathsf{S}_{\psi_{\mathbb{J}}}})$-module of rank 1.

- $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$ has a unique maximal submodule and hence a unique irreducible quotient.
- $\operatorname{supp}\big(\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)\big) = \bigcup_{\beta\in\dot{\mathsf{Q}}_+}\{\lambda-\beta+n\delta \mid n\in\mathbb{Z}\}$ where $\dot{\mathsf{Q}}_+$ is the free abelian monoid generated by all the simple roots in $\dot{\Delta}_+$.
- If $\psi_{\mathbb{J}}(k)\neq\psi_{\mathbb{J}}(\ell)$ for some $k,\ell\in\mathbb{N}$, then $\dim \mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)_\mu=\infty$ for any $\mu\in\operatorname{supp}\big(\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)\big)$.

We have the following irreducibility criterion for modules $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$.

Theorem 3.3. [BBFK, Theorem 3.5] Let $\lambda\in\mathfrak{h}^*$, $\lambda(c)=a$. Then $\mathsf{M}_{\psi_{\mathbb{J}}}(\lambda)$ is irreducible if and only if $a\neq 0$.

4. The case $n=2$

We denote $E(m)$ instead of $E_1(m)$, a_m instead of $a_{11}(m)$, etc.

Definition 4.1. Let $\varphi_{\mathbb{J}}:\ \widehat{sl}(2,\mathbb{C})\to gl(\mathbb{C}[x,y])$ be the function given by:

$\varphi_{\mathbb{J}}(E(m)) := -\dfrac{\partial}{\partial x_m};$

$\varphi_{\mathbb{J}}(F(m)) := \sum_{l,j} x_l x_{j-l-m}\dfrac{\partial}{\partial x_j} + \sum_{j\in\mathbb{J}} x_{j-m}\dfrac{\partial}{\partial y_j} + 2K\sum_{i\in\mathbb{J}} j y_j x_{-j-m} + Kmx_{-m} - 2Jx_{-m};$

$\varphi_{\mathbb{J}}(H(m)) := -2\sum_j x_{j-m}\dfrac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(m)\dfrac{\partial}{\partial y_m} + \psi_{\mathbb{J}}(-m)2mKy_{-m} + 2\delta_{m,0}J.$

Lemma 4.2. We have the following:

(*a*) $[\varphi_{\mathbb{J}}(H(m)),\varphi_{\mathbb{J}}(E(n))] = 2\varphi_{\mathbb{J}}(E(m+n));$

(*b*) $[\varphi_{\mathbb{J}}(H(m)),\varphi_{\mathbb{J}}(H(n))] = 2\delta_{m,-n}mK;$

(*c*) $[\varphi_{\mathbb{J}}(H(m)),\varphi_{\mathbb{J}}(F(n))] = -2\varphi_{\mathbb{J}}(F(m+n));$

(*d*) $[\varphi_{\mathbb{J}}(E(m)),\varphi_{\mathbb{J}}(F(n))] = \varphi_{\mathbb{J}}(H(m+n)) + m\delta_{m,-n}K.$

Proof. We have:

(*a*) $[\varphi_{\mathbb{J}}(H(m)),\varphi_{\mathbb{J}}(E(n))]$

$$= [-2\sum_j x_{j-m}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(m)\frac{\partial}{\partial y_m} + \psi_{\mathbb{J}}(-m)2mKy_{-m} + 2\delta_{m,0}J, -\frac{\partial}{\partial x_n}]$$

$$= [-2\sum_j x_{j-m}\frac{\partial}{\partial x_j}, -\frac{\partial}{\partial x_n}] = 2\frac{\partial}{\partial x_{m+n}} = 2\varphi_{\mathbb{J}}(E(m+n)).$$

(*b*) $[\varphi_{\mathbb{J}}(H(m)), \varphi_{\mathbb{J}}(H(n))]$

$$
\begin{aligned}
&= [-2\sum_j x_{j-m}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(m)\frac{\partial}{\partial y_m} + \psi_{\mathbb{J}}(-m)2mKy_{-m} + 2\delta_{m,0}J,\\
&\quad -2\sum_j x_{j-n}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(n)\frac{\partial}{\partial y_n} + \psi_{\mathbb{J}}(-n)2nKy_{-n} + 2\delta_{n,0}J]\\
&= 4\sum_j x_{j-m}\frac{\partial}{\partial x_{j+n}} - 4\sum_j x_{j-n}\frac{\partial}{\partial x_{j+m}} - \delta_{m,-n}\psi_{\mathbb{J}}(m)\psi_{\mathbb{J}}(-n)2nK\\
&\quad + 2\delta_{m,-n}\psi_{\mathbb{J}}(n)\psi_{\mathbb{J}}(-m)Km + 0 = 2\delta_{m,-n}mK(\psi_{\mathbb{J}}(m) + \psi_{\mathbb{J}}(-m))\\
&= 2\delta_{m,-n}mK.
\end{aligned}
$$

(*c*) $[\varphi_{\mathbb{J}}(H(m)), \varphi_{\mathbb{J}}(F(n))]$

$$
\begin{aligned}
&= \Big[-2\sum_j x_{j-m}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(m)\frac{\partial}{\partial y_m} + \psi_{\mathbb{J}}(-m)2mKy_{-m} + 2\delta_{m,0}J,\\
&\quad \sum_{l,i} x_l x_{i-l-n}\frac{\partial}{\partial x_i} + \sum_{i\in\mathbb{J}} x_{i-n}\frac{\partial}{\partial y_i}\\
&\quad + 2K\sum_{i\in\mathbb{J}} iy_i x_{-i-n} + Knx_{-n} - 2Jx_{-n}\Big]\\
&= -2\Big(\sum_j x_{j-m}\Big(\sum_i x_{i-j-n}\frac{\partial}{\partial x_i} + \sum_l x_l\frac{\partial}{\partial x_{j+l+n}}\Big)\\
&\quad -\sum_{i,l} x_l x_{i-l-n}\frac{\partial}{\partial x_{i+m}}\Big)\\
&\quad -2\sum_j x_{j-m}\psi_{\mathbb{J}}(j+n)\frac{\partial}{\partial y_{j+n}}\\
&\quad -4K\sum_j x_{j-m}\psi_{\mathbb{J}}(-j-n)(-j-n)y_{-j-n}\\
&\quad -2Knx_{-n-m} + 4JX_{-n-m} - 2\psi_{\mathbb{J}}(m)K\psi_{\mathbb{J}}(m)mx_{-m-n}\\
&\quad -\psi_{\mathbb{J}}(-m)2Km\psi_{\mathbb{J}}(-m)x_{-m-n}\\
&= -4\sum_{j,i} x_j x_{i-j-m-n}\frac{\partial}{\partial x_i} + 2\sum_{l,i} x_l x_{i-l-m-n}\frac{\partial}{\partial x_i}\\
&\quad -2\sum_j \psi_{\mathbb{J}}(j)x_{j-m-n}\frac{\partial}{\partial y_j}\\
&\quad -4K\sum_j \psi_{\mathbb{J}}(-j)(-j)y_{-j}x_{j-m-n} - 2Knx_{-m-n}\\
&\quad +4Jx_{-m-n} - 2Kmx_{-m-n}
\end{aligned}
$$

$$
\begin{aligned}
&= -2\sum_{j,i} x_j x_{i-j-m-n}\frac{\partial}{\partial x_i} - 2\sum_{j\in\mathbb{J}} x_{j-m-n}\frac{\partial}{\partial x_j}\\
&\quad - 4K\sum_{j\in\mathbb{J}} j y_j x_{-j-m-n} - 2K(m+n)x_{-m-n} + 4Jx_{-m-n}\\
&= -2\varphi_{\mathbb{J}}(F(m+n)).
\end{aligned}
$$

(*d*) $[\varphi_{\mathbb{J}}(E(m)), \varphi_{\mathbb{J}}(F(n))]$

$$
\begin{aligned}
&= \Big[-\frac{\partial}{\partial x_m}, \sum_{l,j} x_l x_{j-l-n}\frac{\partial}{\partial x_j} + \sum_{j\in\mathbb{J}} x_{j-n}\frac{\partial}{\partial y_j}\\
&\quad + 2K\sum_{j\in\mathbb{J}} j y_j x_{-j-n} + Knx_{-n} - 2Jx_{-n}\Big]\\
&= -\Big(\sum_j x_{j-m-n}\frac{\partial}{\partial x_j} + \sum_l x_l\frac{\partial}{\partial x_{l+m+n}} + \psi_{\mathbb{J}}(m+n)\frac{\partial}{\partial y_{m+n}}\\
&\quad - 2K\psi_{\mathbb{J}}(-m-n)(m+n)Y_{-m-n} - \delta_{m,-n}(Km+2J)\Big)\\
&= -2\sum_j X_{j-m-n}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(m+n)\frac{\partial}{\partial y_{m+n}}\\
&\quad + 2\psi_{\mathbb{J}}(-m-n)(m+n)Ky_{-m-n} + 2\delta_{m,-n}J + Km\delta_{m,-n}\\
&= \varphi_{\mathbb{J}}(H(m+n)) + m\delta_{m,-n}K. \qquad \square
\end{aligned}
$$

Now let $\rho_1 : \widehat{sl}(2,\mathbb{C}) \to \widehat{sl}(2,\mathbb{C})$ be the function given by

$$\rho_1(E(m)) = -F(-m),\ \rho_1(F(m)) = -E(-m),\ \rho_1(H(m)) = H(-m)$$

and $\rho_1(K) = K$.

Let $\rho_2 : gl(\mathbb{C}[x,y]) \to gl(\mathbb{C}[x,y])$ be the function given by

$$\rho_2(x_{-m}) = \frac{\partial}{\partial x_m},\ \rho_2(\frac{\partial}{\partial x_m}) = x_{-m},\ \rho_2(y_k) = -\frac{\partial}{\partial y_k} \text{ and } \rho_2(\frac{\partial}{\partial y_k}) = -y_k.$$

Note that ρ_1 and ρ_2 are anti-automorphisms of $\widehat{sl}(2,\mathbb{C})$ and $gl(\mathbb{C}[x,y])$, respectively.

Definition 4.3. Let $\rho_{\mathbb{J}} := \rho_2 \circ \varphi_{\mathbb{J}} \circ \rho_1$. We have:

(*a*) $\rho_{\mathbb{J}}(F(m)) = \rho_2 \circ \varphi_{\mathbb{J}} \circ \rho_1(F(m))$

$$= \rho_2 \circ \varphi_{\mathbb{J}}(-E(-m)) = \rho_2\Big(\frac{\partial}{\partial x_{-m}}\Big) = x_m.$$

(*b*) $\rho_{\mathbb{J}}(E(m)) = \rho_2 \circ \varphi_{\mathbb{J}} \circ \rho_1(E(m)) = \rho_2 \circ \varphi_{\mathbb{J}}(-F(-m))$

$$= \rho_2\Big(-\sum_{l,j} x_l x_{j-l+m}\frac{\partial}{\partial x_j} - \sum_{j\in\mathbb{J}} x_{j+m}\frac{\partial}{\partial y_j}$$

$$- 2K\sum_{j\in\mathbb{J}} j y_j x_{-j+m} - K(-m)x_m + 2Jx_m\Big)$$

$$= -\sum_{l,j} x_{-j}\frac{\partial}{\partial x_{-l}}\frac{\partial}{\partial x_{-j+l-m}} - \sum_{j\in\mathbb{J}}(-y_j)\frac{\partial}{\partial x_{-j-m}}$$

$$- 2K\sum_{j\in\mathbb{J}} j(-\frac{\partial}{\partial y_j})\frac{\partial}{\partial x_{j-m}} - K(-m)\frac{\partial}{\partial x_{-m}} + 2J\frac{\partial}{\partial x_{-m}}$$

$$= -\sum_{l,j} x_j\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_{j-l-m}} + \sum_{j\in\mathbb{J}} y_j\frac{\partial}{\partial x_{-j-m}}$$

$$+ 2K\sum_{j\in\mathbb{J}} j\frac{\partial}{\partial y_j}\frac{\partial}{\partial x_{j-m}} + (Km+2J)\frac{\partial}{\partial x_{-m}}$$

$$= -\sum_{l,j} x_{j+l+m}\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_j} + \sum_{j\in\mathbb{J}} y_j\frac{\partial}{\partial x_{-j-m}}$$

$$+ 2K\sum_{j\in\mathbb{J}} j\frac{\partial}{\partial y_j}\frac{\partial}{\partial x_{j-m}} + (Km+2J)\frac{\partial}{\partial x_{-m}}.$$

(c) $\rho_{\mathbb{J}}(H(m)) = \rho_2\circ\varphi_{\mathbb{J}}\circ\rho_1(H(m)) = \rho_2\circ\varphi_{\mathbb{J}}(H(-m))$

$$= \rho_2\Big(-2\sum_{j} x_{j+m}\frac{\partial}{\partial x_j} - \psi_{\mathbb{J}}(-m)\frac{\partial}{\partial y_{-m}}$$

$$+ \psi_{\mathbb{J}}(m)2(-m)Ky_m + 2\delta_{m,0}J\Big)$$

$$= -2\sum_{j} x_{-j}\frac{\partial}{\partial x_{-j-m}} + \psi_{\mathbb{J}}(-m)y_{-m} + \psi_{\mathbb{J}}(m)2mK\frac{\partial}{\partial y_m} + 2\delta_{m,0}J$$

$$= -2\sum_{j} x_{j+m}\frac{\partial}{\partial x_j} + \psi_{\mathbb{J}}(-m)y_{-m} + \psi_{\mathbb{J}}(m)2mK\frac{\partial}{\partial y_m} + 2\delta_{m,0}J.$$

The function $\rho_{\mathbb{J}}$ satisfy $(R1)$ - $(R6)$ because $\varphi_{\mathbb{J}}$ satisfy these relations and ρ_1 and ρ_2 are anti-automorphisms. Hence, it defines a representation of $\widehat{sl}(2,\mathbb{C})$. Now we can start to study a candidate $\rho_{\mathbb{J}}$ when n is arbitrary.

5. Free field realization of $\widehat{sl}(n,\mathbb{C})$

Let $\rho_{\mathbb{J}}$ be the function given by:

(a) $\rho_{\mathbb{J}}(F_r)(w) := -a_{r,r+1}(w) + \sum_{j=1}^{r-1} a_{j,r+1}(w)a^*_{jr}(w);$

$$(b)\ \rho_{\mathbb{J}}(H_r)(w) := 2a_{r,r+1}(w)a^*_{r,r+1}(w) + \sum_{i=1}^{r-1}\Big(a_{i,r+1}(w)a^*_{i,r+1}(w) - a_{ir}(w)a^*_{ir}(w)\Big) + \sum_{j=r+2}^{n}\Big(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\Big) - \gamma b_r(w) + \frac{\gamma}{2}\Big(b^+_{r-1}(w) + b^+_{r+1}(w)\Big);$$

$$(c)\ \rho_{\mathbb{J}}(E_r)(w) := a_{r,r+1}(w)a^*_{r,r+1}(w)^2 - \sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{r+1,j}(w) + \sum_{j=1}^{r-1} a_{jr}(w)a^*_{j,r+1}(w) + \sum_{j=r+2}^{n}\Big(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\Big)a^*_{r,r+1}(w) - \gamma a^*_{r,r+1}(w)b_r(w) + \frac{\gamma}{2}a^*_{r,r+1}(w)\Big(b^+_{r-1}(w) + b^+_{r+1}(w)\Big) - \frac{\gamma}{2}a^*_{r,r+1}(w).$$

Theorem 5.1. The function $\rho_{\mathbb{J}} : \widehat{sl}(n,\mathbb{C}) \to gl(\mathbb{C}[x,y])$ is a representation of $\widehat{sl}(n,\mathbb{C})$.

To prove that $\rho_{\mathbb{J}}$ defines a representation of $\widehat{sl}(n,\mathbb{C})$ we need only verify the relations $(R1) - (R6)$ but to do it, we need the following lemmas:

Lemma 5.2. We have the following:

$$[a_{ij}(z), a^*_{kl}(w)] = \delta_{ik}\delta_{jl}\delta(z-w); \tag{5.1}$$

$$[a_{ij}(z)a^*_{kl}(z), a_{mn}(w)a^*_{pq}(w)] = \delta_{pi}\delta_{qj}a_{mn}(w)a^*_{kl}(z)\delta(z-w) - \delta_{km}\delta_{ln}a_{ij}(z)a^*_{pq}(w)\delta(w-z); \tag{5.2}$$

$$[a_{ij}(z), \dot{a}^*_{kl}(w)] = \delta_{ik}\delta_{jl}\dot{\delta}(z-w); \tag{5.3}$$

$$\dot{a}^*_{ij}(z)\delta(w-z) = a^*_{ij}(z)\dot{\delta}(w-z) - a^*_{ij}(w)\dot{\delta}(w-z); \tag{5.4}$$

$$[b_i(z), b^{\pm}_j(w)] = [b^{\mp}_i(z), b^{\pm}_j(w)] = \delta_{ij}\dot{\delta}^{\mp}(w-z); \tag{5.5}$$

$$[b_i(z), b_j(w)] = [b^+_i(z), b^-_j(w)] + [b^-_i(z), b^+_j(w)] = \delta_{ij}\dot{\delta}(w-z); \tag{5.6}$$

$$[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)a^*_{r,r+1}(w)] = a_{r,r+1}(w)a^*_{r,r+1}(z)a^*_{r,r+1}(w)\delta(w-z); \tag{5.7}$$

$$[a_{rj}(z)a^*_{rj}(z), f(w)a^*_{kl}(w)] = f(w)\delta_{rk}\delta_{jl}a^*_{rj}(z)\delta(w-z), \tag{5.8}$$

where $f(w)$ comutes with $a_{rj}(z)$, $a^*_{rj}(z)$ and $a^*_{kl}(w)$.

Proof. We have:

(5.1) $[a_{ij}(z), a^*_{kl}(w)] = \sum_{m,n\in\mathbb{Z}} [a_{ij}(m), a^*_{kl}(n)] z^{-m} w^{-n}$
$= \sum_{m,n\in\mathbb{Z}} \delta_{ik}\delta_{jl}\delta_{m,-n} z^{-m} w^{-n} = \sum_{n\in\mathbb{Z}} \delta_{ik}\delta_{jl} z^n w^{-n}$
$= \delta_{ik}\delta_{jl}\delta(z-w).$

(5.2) $[a_{ij}(z)a^*_{kl}(z), a_{mn}(w)a^*_{pq}(w)]$
$= a_{ij}(z)[a^*_{kl}(z), a_{mn}(w)]a^*_{pq}(w) + a_{ij}(z)a_{mn}(w)a^*_{kl}(z)a^*_{pq}(w)$
$\quad - a_{mn}(w)[a^*_{pq}(w), a_{ij}(z)]a^*_{kl}(z) - a_{mn}(w)a_{ij}(z)a^*_{pq}(w)a^*_{kl}(z)$
$= \delta_{pi}\delta_{qj} a_{mn}(w)a^*_{kl}(z)\delta(z-w) - \delta_{km}\delta_{ln} a_{ij}(z)a^*_{pq}(w)\delta(w-z).$

(5.3) $[a_{ij}(z), \dot{a}^*_{kl}(w)] = \sum_{m,n\in\mathbb{Z}} [a_{ij}(m), n a^*_{kl}(n)] z^{-m} w^{-n}$
$= \sum_{m,n\in\mathbb{Z}} n\delta_{ik}\delta_{jl}\delta_{m,-n} z^{-m} w^{-n} = \sum_{n\in\mathbb{Z}} \delta_{ik}\delta_{jl} n z^n w^{-n}$
$= \delta_{ik}\delta_{jl}\dot{\delta}(z-w).$

(5.4) $\dot{a}^*_{ij}(z)\delta(w-z)$
$= (\sum_{m\in\mathbb{Z}} (-m)a^*_{ij}(m)z^{-m})(\sum_{n\in\mathbb{Z}} z^{-n}w^n)$
$= \sum_{m,n\in\mathbb{Z}} (-m)a^*_{ij}(m)z^{-m-n}w^n$
$= \sum_{m,n\in\mathbb{Z}} -(m+n)a^*_{ij}(m)z^{-m-n}w^{(m+n)-m}$
$+ \sum_{m,n\in\mathbb{Z}} n a^*_{ij}(m)z^{-m-n}w^n$
$= \sum_{m,n\in\mathbb{Z}} -n a^*_{ij}(m)z^{-n}w^{n-m} + \sum_{m,n\in\mathbb{Z}} n a^*_{ij}(m)z^{-m}w^n z^{-n}$
$= -(\sum_{m\in\mathbb{Z}} a^*_{ij}(m)w^{-m})\dot{\delta}(w-z) + \sum_{m\in\mathbb{Z}} a^*_{ij}(m)z^{-m}\dot{\delta}(w-z)$
$= a^*_{ij}(z)\dot{\delta}(w-z) - a^*_{ij}(w)\dot{\delta}(w-z).$

(5.5) $[b_i(z), b^-_j(w)]$
$= \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^*_-}} [b_i(m), b_j(n)] z^{-m} w^{-n} = \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^*_-}} m\delta_{ij}\delta_{m,-n} z^{-m} w^{-n}$
$= \sum_{\substack{m\in\mathbb{Z}^*_+\\ n\in\mathbb{Z}^*_-}} m\delta_{ij}\delta_{m,-n} z^{-m} w^{-n} = [b^+_i(z), b^-_j(w)]$

$$= \sum_{n\in\mathbb{Z}_-} (-n)\delta ij z^n w^{-n} = \delta_{ij}\dot{\delta}^+(w-z).$$

and similarly for the other case.

(5.6) $[b_i(z), b_j(w)] = [b_i^+(z), b_j^-(w)] + [b_i^-(z), b_j^+(w)]$

$$= \sum_{m,n\in\mathbb{Z}} m\delta_{ij}\delta_{m,-n} z^{-m}w^{-n} = \delta_{ij}\sum_{n\in\mathbb{Z}}(-n)z^n w^{-n}$$

$$= \delta_{ij}\dot{\delta}(w-z).$$

(5.7) $[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)a^*_{r,r+1}(w)]$

$$\begin{aligned}
&= a_{r,r+1}(z)[a^*_{r,r+1}(z), a_{r,r+1}(w)]a^*_{r,r+1}(w)a^*_{r,r+1}(w)\\
&\quad - a_{r,r+1}(w)a^*_{r,r+1}(w)[a^*_{r,r+1}(w), a_{r,r+1}(z)]a^*_{r,r+1}(z)\\
&\quad + a_{r,r+1}(z)a_{r,r+1}(w)a^*_{r,r+1}(z)a^*_{r,r+1}(w)a^*_{r,r+1}(w)\\
&\quad - a_{r,r+1}(w)a^*_{r,r+1}(w)a_{r,r+1}(z)a^*_{r,r+1}(w)a^*_{r,r+1}(z)\\
&= -a_{r,r+1}(z)\delta(w-z)a^*_{r,r+1}(w)a^*_{r,r+1}(w)\\
&\quad + a_{r,r+1}(w)a^*_{r,r+1}(w)\delta(w-z)a^*_{r,r+1}(z)\\
&\quad + a_{r,r+1}(w)[a_{r,r+1}(z), a^*_{r,r+1}(w)]a^*_{r,r+1}(z)a^*_{r,r+1}(w)\\
&= a_{r,r+1}(w)a^*_{r,r+1}(z)a^*_{r,r+1}(w)\delta(w-z).
\end{aligned}$$

(5.8) $[a_{rj}(z)a^*_{rj}(z), f(w)a^*_{kl}(w)]$

$$\begin{aligned}
&= f(w)(a_{rj}(z)a^*_{rj}(z)a^*_{kl}(w) - a^*_{kl}(w)a_{rj}(z)a^*_{rj}(z))\\
&= f(w)[a_{rj}(z), a^*_{kl}(w)]a^*_{rj}(z) = f(w)\delta_{rk}\delta_{jl}a^*_{rj}(z)\delta(w-z). \qquad \square
\end{aligned}$$

Lemma 5.3. We have the following:

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{ks}(w)a^*_{k,s+1}(w), a_{rj}(z)a^*_{rj}(z)]$$

$$= -\delta_{s,r+1}a_{r,r+1}(w)a^*_{r,r+2}(z)\delta(w-z); \tag{5.9}$$

$$\sum_{j=1}^{r-1}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w), a_{jr}(z)a^*_{j,r+1}(z)] = 0; \tag{5.10}$$

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{ks}(w)a^*_{k,s+1}(w), a_{r+1,j}(z)a^*_{r+1,j}(z)] = 0; \tag{5.11}$$

$$\sum_{j=r+2}^{n}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w), a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)]$$

$$= -2\delta_{rs}\delta(w-z)\sum_{j=r+2}^{n}a_{r+1,j}(w)a^*_{rj}(z)$$

$$+ \delta_{r,s+1}\delta(w-z) \sum_{j=r+2}^{n} a_{rj}(z)a^*_{r-1,j}(w)$$
$$+ \delta_{s,r+1}\delta(w-z) \sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{r+1,j}(z); \qquad (5.12)$$

$$\sum_{j=1}^{r-1}[a^*_{s,s+1}(w), a_{jr}(z)a^*_{j,r+1}(z)] = -\delta_{r,s+1}a^*_{r-1,r+1}(z)\delta(w-z); \qquad (5.13)$$

$$\sum_{j=r+2}^{n}[a^*_{s,s+1}(w), a_{r+1,j}(z)a^*_{rj}(z)] = -\delta_{r+1,s}a^*_{r,r+2}(z)\delta(w-z). \qquad (5.14)$$

Now we are able to verify (R_1) - (R_6).

Lemma 5.4. $(R1)$ $[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_s)(w)] = -(H_r|H_s)c\dot{\delta}(z-w)$.

Proof. Observe that if $|r-s|> 1$, then $[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_s)(w)] = 0$ because all summations are equal to zero. So, writing in terms of $\delta_{s,f(r)}$, we have the following:

$$[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_s)(w)] = \delta_{rs}([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_r)(w)])$$
$$+ \delta_{r,s+1}([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r-1})(w)])$$
$$+ \delta_{r,s-1}([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)])$$
$$= \delta_{rs}\Big([2a_{r,r+1}(z)a^*_{r,r+1}(z), \rho_{\mathbb{J}}(H_r)(w)]$$
$$+ \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{r,r+1}(z) - a_{ir}(z)a^*_{ir}(z),$$
$$\rho_{\mathbb{J}}(H_r)(w)] + \sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho_{\mathbb{J}}(H_r)(w)]$$
$$+ [-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)), \rho_{\mathbb{J}}(H_r)(w)]\Big)$$
$$+ \delta_{r,s+1}\Big([2a_{r,r+1}(z)a^*_{r,r+1}(z), \rho_{\mathbb{J}}(H_{r-1})(w)]$$
$$+ \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho_{\mathbb{J}}(H_{r-1})(w)]$$
$$+ \sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho_{\mathbb{J}}(H_{r-1})(w)]$$
$$+ [-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)), \rho_{\mathbb{J}}(H_{r-1})(w)]\Big)$$
$$+ \delta_{r,s-1}\Big([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)]\Big)$$

$$
\begin{aligned}
&= \delta_{rs}\Big(4[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)] \\
&\quad + \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{r,r+1}(z), a_{i,r+1}(w)a^*_{i,r+1}(w)] \\
&\quad + \sum_{i=1}^{r-1}[a_{ir}(z)a^*_{ir}(z), a_{ir}(w)a^*_{ir}(w)] + 0 + \gamma^2[b_r(z), b_r(w)] \\
&\quad + \frac{\gamma^2}{4}[b^+_{r-1}(z), b^+_{r-1}(w)] + \frac{\gamma^2}{4}[b^+_{r+1}(z), b^+_{r+1}(w)]\Big) \\
&+ \delta_{r,s+1}\Big([2a_{r,r+1}(z)a^*_{r,r+1}(z), \sum_{j=r+1}^{n} -a_{(r-1)+1,j}(w)a^*_{(r-1)+1,j}(w)] \\
&+ \sum_{i=1}^{r-1}[-a_{ir}(z)a^*_{ir}(z), \sum_{j=1}^{r-2} -a_{jr}(w)a^*_{jr}(w)] \\
&\quad + \sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z), \sum_{i=r+1}^{n} -a_{ri}(w)a^*_{ri}(w)] \\
&+ [-\gamma b_r(z), \frac{\gamma}{2}b^+_{(r-1)+1}(w)] + [\frac{\gamma}{2}b^+_{r-1}(z), -\gamma b_{r-1}(w)]\Big) \\
&+ \delta_{r,s-1}\Big([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)]\Big) \\
&= \delta_{rs}(0 + 0 + 0 + 0 + 2c\dot{\delta}(w-z) + 0 + 0) \\
&+ \delta_{r,s+1}(-2[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)] \\
&- \sum_{i=1}^{r-2}[a_{ir}(z)a^*_{ir}(z), a_{ir}(w)a^*_{ir}(w)] \\
&+ \sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z), -a_{rj}(w)a^*_{rj}(w)] \\
&- c\dot{\delta}^-(w-z) + c\dot{\delta}^-(z-w)) + \delta_{r,s-1}([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)]) \\
&+ \delta_{rs}(2c\dot{\delta}(w-z)) + \delta_{r,s+1}(0 + 0 + 0 + -c\dot{\delta}^-(w-z) - c\dot{\delta}^+(w-z)) \\
&+ \delta_{r,s-1}([\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)]) \\
&= \delta_{rs}(2c\dot{\delta}(w-z)) + \delta_{r,s+1}(-c\dot{\delta}(w-z)) + \delta_{r,s-1}(-c\dot{\delta}(w-z)),
\end{aligned}
$$

where in the last equality we have:

$$
\begin{aligned}
&[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(H_{r+1})(w)] = \\
&\qquad - [\rho_{\mathbb{J}}(H_{r+1})(w), \rho_{\mathbb{J}}(H_r)(z)] = c\dot{\delta}(z-w) = -c\dot{\delta}(w-z).
\end{aligned}
$$

Then the relation $(R1)$ is verified. □

Lemma 5.5. $(R2)[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(E_s)(w)] = C_{rs}\rho_{\mathbb{J}}(E_s)(z)\delta(z-w)$.

Proof. We have the following:

$$(a)\ 2[a_{r,r+1}(z)a^*_{r,r+1}(z), \rho_{\mathbb{J}}(E_s)(w)]$$
$$= \delta_{sr}(2[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)a^*_{r,r+1}(w)]$$
$$+ 2[a_{r,r+1}(z)a^*_{r,r+1}(z),$$
$$(\sum_{j=r+2}^{n} a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w))a^*_{r,r+1}(w)]$$
$$+ 2[a_{r,r+1}(z)a^*_{r,r+1}(z),$$
$$- \gamma a^*_{r,r+1}(w)(\frac{1}{2} + b_r(w) - \frac{1}{2}b^+_{r-1}(w) - \frac{1}{2}b^+_{r+1}(w))])$$
$$+ \delta_{s,r+1}(2[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+2}(w)])$$
$$+ \delta_{s,r-1}(-2a^*_{r-1,r+1}(z)a_{r,r+1}(w)\delta(w-z))$$
$$\overset{(5.8)}{\underset{(5.7)}{=}} \delta_{sr}(2a_{r,r+1}(w)a^*_{r,r+1}(z)a^*_{r,r+1}(w)\delta(w-z)$$
$$+ 2\sum_{j=r+2}^{n}(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w))a^*_{r,r+1}(z)\delta(w-z)$$
$$- 2\gamma a^*_{r,r+1}b_r(w)\delta(w-z) + \gamma a^*_{r,r+1}(z)(b^+_{r-1}(w) + b^+_{r+1}(w))\delta(w-z)$$
$$- \gamma^2 a^*_{r,r+1}(z)\dot{\delta}(w-z)) + \delta_{s,r+1}(2a_{r,r+1}(z)a^*_{r,r+2}(w)\delta(w-z))$$
$$+ \delta_{s,r-1}(-2a^*_{r-1,r+1}(z)a_{r,r+1}(w)\delta(w-z)).$$

$$(b)\ \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho_{\mathbb{J}}(E_s)(w)]$$
$$= \delta_{sr}(\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \sum_{j=1}^{r-1} a_{jr}(w)a^*_{j,r+1}(w)])$$
$$+ \delta_{s,r+1}(\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z), a_{i,r+1}(w)a^*_{i,r+2}(w)])$$
$$+ \delta_{s,r-1}(-(a_{r-1,r}(w)a^*_{r-1,r}(z)a^*_{r-1,r}(w) + \sum_{j=r+1}^{n}(a_{r-1,j}(w)a^*_{r-1,j}(w)$$
$$- a_{rj}(w)a^*_{rj}(w))a^*_{r-1,r}(z) + \sum_{j=1}^{r-2} a_{j,r-1}(w)a^*_{jr}(z)$$
$$- a_{r,r+1}(w)a^*_{r-1,r+1}(z) - \gamma a^*_{r-1}(w)b_{r-1}(w)$$
$$+ \frac{\gamma}{2}(a^*_{r-1,r}(w)b^+_{r-2}(w) - a^*_{r-1,r}(w)b^+_{r}(w))\delta(w-z)$$

$$
\begin{aligned}
&+\frac{\gamma^2}{2}a^*_{r-1,r}(z)\dot{\delta}(w-z))\\
&\overset{(5.2)}{=}\delta_{sr}(2\sum_{i=1}^{r-1}a_{ir}(w)a^*_{i,r+1}(w)\delta(w-z))\\
&\quad+\delta_{s,r+1}(-\sum_{i=1}^{r-1}a_{i,r+1}(z)a^*_{i,r+2}(w)\delta(w-z))\\
&\quad+\delta_{s,r-1}(-(a_{r-1,r}(w)a^*_{r-1,r}(z)a^*_{r-1,r}(w)+\sum_{j=r+1}^{n}(a_{r-1,j}(w)a^*_{r-1,j}(w)\\
&\quad-a_{rj}(w)a^*_{rj}(w))a^*_{r-1,r}(z)+\sum_{j=1}^{r-2}a_{j,r-1}(w)a^*_{jr}(z)\\
&\quad-a_{r,r+1}(w)a^*_{r-1,r+1}(z)-\gamma a^*_{r-1}(w)b_{r-1}(w)\\
&\quad+\frac{\gamma}{2}(a^*_{r-1,r}(w)b^+_{r-2}(w)-a^*_{r-1,r}(w)b^+_{r}(w))\delta(w-z)\\
&\quad+\frac{\gamma^2}{2}a^*_{r-1,r}(z)\dot{\delta}(w-z)).
\end{aligned}
$$

$$
\begin{aligned}
&(c)\quad\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z)-a_{r+1,j}(z)a^*_{r+1,j}(z),\rho_{\mathbb{J}}(E_s)(w)]\\
&=\delta_{sr}(\sum_{j=r+2}^{n}[-a_{r+1,j}(z)a^*_{r+1,j}(z),-a_{r+1,j}(w)a^*_{r+1,j}(w)]\\
&\quad+\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z),a_{rj}(w)a^*_{rj}(w)]\\
&\quad+\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z)-a_{r+1,j}(z)a^*_{r+1,j}(z),\\
&\qquad\qquad(a_{rj}(w)a^*_{rj}(w)-a_{r+1,j}(w)a^*_{r+1,j}(w))a^*_{r,r+1}(w)])\\
&\quad+\delta_{s,r+1}(-(a_{r+1,r+2}(w)a^*_{r+1,r+2}(z)a^*_{r+1,r+2}(w)\\
&\quad+\sum_{j=r+3}^{n}(a_{r+1,j}(w)a^*_{r+1,j}(w)\\
&\quad-a_{r+2,j}(w)a^*_{r+2,j}(w))a^*_{r+1,r+2}(z)-a_{r,r+1}(w)a^*_{r,r+2}(z)\\
&\quad-\sum_{j=r+3}^{n}a_{r+2,j}(w)a^*_{r+1,j}(w)\\
&\quad-\gamma a^*_{r+1,r+2}(w)b_{r+1}(w)+\frac{\gamma}{2}a^*_{r+1,r+2}(w)(b^+_{r}(w)+b^+_{r+2}(w)))\delta(w-z)
\end{aligned}
$$

$$+\frac{\gamma^2}{2}a^*_{r+1,r+2}(z)\dot{\delta}(w-z)) + \delta_{s,r-1}(\sum_{j=r+2}^{n} a_{rj}(w)a^*_{r-1,j}(w)\delta(w-z))$$

$$\overset{(5.2)}{\underset{(5.7)}{=}} \delta_{sr}(0+0+2\sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{rj}(z)\delta(w-z))$$
$$+\delta_{s,r+1}(-(a_{r+1,r+2}(w)a^*_{r+1,r+2}(z)a^*_{r+1,r+2}(w)$$
$$+\sum_{j=r+3}^{n}(a_{r+1,j}(w)a^*_{r+1,j}(w)$$
$$-a_{r+2,j}(w)a^*_{r+2,j}(w))a^*_{r+1,r+2}(z) - a_{r,r+1}(w)a^*_{r,r+2}(z)$$
$$-\sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{r+1,j}(w)$$
$$-\gamma a^*_{r+1,r+2}(w)b_{r+1}(w) + \frac{\gamma}{2}a^*_{r+1,r+2}(w)(b^+_r(w)+b^+_{r+2}(w)))\delta(w-z)$$
$$+\frac{\gamma^2}{2}a^*_{r+1,r+2}(z)\dot{\delta}(w-z)) + \delta_{s,r-1}(\sum_{j=r+2}^{n} a_{rj}(w)a^*_{r-1,j}(w)\delta(w-z)).$$

Finally, we have:

$$(d)\ [-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z)+b^+_{r+1}(z)), \rho_{\mathbb{J}}(E_s)(w)]$$
$$= \delta_{sr}(\gamma^2[b_r(z), a^*_{r,r+1}(w)b_r(w)])$$
$$+\delta_{s,r+1}(-\frac{\gamma^2}{2}a^*_{r+1,r+2}(w)([b_r(z), b^+_r(w)] + [b^+_{r+1}(z), b_{r+1}(w)]))$$
$$+\delta_{s,r-1}(\frac{\gamma^2}{2}a^*_{r-1,r}(w)([b_r(z), b^+_r(w)] + [b^+_{r-1}(z), b_{r-1}(w)]))$$
$$= \delta_{sr}(\gamma^2 a^*_{r,r+1}(w)\dot{\delta}(w-z)) + \delta_{s,r+1}(-\frac{\gamma^2}{2}a^*_{r+1,r+2}(w)\dot{\delta}(w-z))$$
$$+\delta_{s,r-1}(\frac{\gamma^2}{2}a^*_{r-1,r}(w)\dot{\delta}(w-z)).$$

Adding the four equations up we get:

$$[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(E_s)(w)]$$
$$= \delta_{sr}2\rho_{\mathbb{J}}(E_r)(z)\delta(w-z) + \delta_{s,r+1}(-\rho_{\mathbb{J}}(E_{r+1})(z)\delta(w-z))$$
$$+\delta_{s,r-1}(-\rho_{\mathbb{J}}(E_{r-1})(z)\delta(w-z)),$$

proving (R_2). □

Lemma 5.6. $(R3)[\rho_{\mathbb{J}}(H_r)(z), \rho_{\mathbb{J}}(F_s)(w)] = -C_{rs}\rho_{\mathbb{J}}(F_s)(z)\delta(z-w)$.

Proof. We have:

$$(a)\ 2[a_{r,r+1}(z)a^*_{r,r+1}(z),\rho_{\mathbb{J}}(F_s)(w)]$$
$$=\delta_{sr}(2a_{r,r+2}(z)\delta(z-w))$$
$$+\delta_{s,r+1}(2[a_{r,r+1}(z)a^*_{r,r+1}(z),a_{r,r+2}(w)a^*_{r,r+1}(w)])+\delta_{s,r-1}0$$
$$=\delta_{sr}(2a_{r,r+2}(z)\delta(z-w))+\delta_{s,r+1}(2a_{r,r+2}(w)a^*_{r,r+1}(z)\delta(w-z)).$$

$$(b)\ \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z)-a_{ir}(z)a^*_{ir}(z),\rho_{\mathbb{J}}(F_s)(w)]$$
$$=\delta_{sr}(\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z),a_{i,r+1}(w)a^*_{ir}(w)]$$
$$-\sum_{i=1}^{r-1}[a_{ir}(z)a^*_{ir}(z),a_{i,r+1}(w)a^*_{ir}(w)])$$
$$+\delta_{s,r+1}(\sum_{j=1}^{r}[a_{i,r+1}(z)a^*_{i,r+1}(z),a_{j,r+2}(w)a^*_{j,r+1}(w)])$$
$$+\delta_{s,r-1}([a_{r-1,r}(z)a^*_{r-1,r}(z),a_{r-1,r}(w)]$$
$$-\sum_{i=1}^{r-2}a_{ir}(z)[a^*_{ir}(z),a_{ir}(w)]a^*_{i,r-1}(w))$$
$$=\delta_{sr}(-2\sum_{i=1}^{r-1}a_{i,r+1}(z)a^*_{ir}(w)\delta(z-w))$$
$$+\delta_{s,r+1}(\sum_{i=1}^{r-1}a_{i,r+2}(w)a^*_{i,r+1}(z)\delta(z-w))$$
$$+\delta_{s,r-1}((-a_{r-1,r}(z)+\sum_{i=1}^{r-2}a_{ir}(z)a^*_{i,r-1}(w))\delta(z-w)).$$

$$(c)\ \sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z)-a_{r+1,j}(z)a^*_{r+1,j}(z),\rho_{\mathbb{J}}(E_s)(w)]$$
$$=\delta_{sr}0+\delta_{s,r+1}(\sum_{j=r+2}^{n}\sum_{k=1}^{r}[a_{rj}(z)a^*_{rj}(z),a_{k,r+2}(w)a^*_{k,r+1}(w)]$$
$$+\sum_{j=r+2}^{n}[a_{r+1,j}(z)a^*_{r+1,j}(z),a_{r+1,r+2}(w)])+\delta_{s,r-1}0$$
$$=\delta_{s,r+1}(-a_{r,r+2}(z)a^*_{r,r+1}(w)\delta(z-w)-a_{r+1,r+2}(z)\delta(z-w)).$$

$$(d)\ [-\gamma b_r(z)+\frac{\gamma}{2}(b^+_{r-1}(z)+b^+_{r+1}(z)),\rho_{\mathbb{J}}(F_s)(w)]=0.$$

Adding the four equations up, we have the result. □

Lemma 5.7. ($R4$) $[\rho_{\mathbb{J}}(E_r)(z), \rho_{\mathbb{J}}(F_s)(w)] = \delta_{sr}(\rho_{\mathbb{J}}(H_r)(z)\delta(z-w) - c\dot{\delta}(z-w))$.

Proof. Consider the case $|s-r|>1$. We have $[\rho(E_r)(z), -a_{r,r+1}(w)] = 0$ and $\sum_{j=1}^{r-1}\sum_{k=1}^{s-1}[a_{kr}(z)a^*_{k,r+1}(z), a_{j,s+1}(w)a^*_{js}(w)] = 0$, because $r \neq s$. Finally:

$$\sum_{j=1}^{s-1}[\sum_{k=r+2}^{n}(a_{rk}(z)a^*_{rk}(z) - a_{r+1,k}(z)a^*_{r+1,k}(z))a^*_{r,r+1}(z)$$
$$+\sum_{k=r+2}^{n} a_{r+1,k}(z)a^*_{rk}, a_{j,s+1}(w)a^*_{js}(w)]$$
$$=\sum_{j=1}^{s-1}\sum_{k=r+2}^{n}[a_{rk}(z)a^*_{rk}(z) - a_{r+1,k}(z)a^*_{r+1,k}(z), a_{j,s+1}(w)a^*_{js}(w)]a^*_{r,r+1}(z)$$
$$-\sum_{j=1}^{s-1}\sum_{k=r+2}^{n}[a_{r+1,k}(z)a^*_{rk}(z), a_{j,s+1}(w)a^*_{js}(w)]$$
$$=\sum_{j=1}^{s-1}\sum_{k=r+2}^{n}(a_{j,s+1}(w)a^*_{rk}(z)\delta_{sk}\delta_{rj} - a_{rk}(z)a^*_{js}(w)\delta_{s+1,k}\delta_{jr}$$
$$- a_{j,s+1}(w)a^*_{r+1,k}(z)\delta_{r+1,j}\delta_{ks}$$
$$+ a_{r+1,k}(z)a^*_{js}(w)\delta_{r+1,j}\delta_{k,s+1})a^*_{r,r+1}(z)\delta(z-w)$$
$$+\sum_{j=1}^{s-1}\sum_{k=r+2}^{n}(a_{r+1,k}(z)a^*_{js}(w)\delta_{rj}\delta_{k,s+1}$$
$$- a_{j,s+1}(w)a^*_{rk}(z)\delta_{r+1,j}\delta_{ks})\delta(z-w) = \circledast$$

If $s+1>n$ then $a_{j,s+1}(w) = 0 = \delta_{k,s+1}$ and $\circledast = 0$. Now suppose $s+1 \leqslant n$. If $s-1<r$ then $j \leqslant s-1 < r$, $\delta_{jr} = \delta_{j,r+1} = 0$ and $\circledast = 0$. If $s-1 \geqslant r$ we have $s-1>r$ because $|s-r|>1$. Then $s \geqslant r+2$ and for this reason the terms $j = r, r+1$ and $k = s, s+1$ appears in the expression. Then:

$$\circledast =(a_{r,s+1}(w)a^*_{rs}(z) - a_{r,s+1}(z)a^*_{rs}(w) - a_{r+1,s+1}(w)a^*_{r+1,s}(z)$$
$$+ a_{r+1,s+1}(z)a^*_{r+1,s}(w))a^*_{r,r+1}(z)\delta(z-w)$$
$$+ (a_{r+1,s+1}(z)a^*_{rs}(w) - a_{r+1,s+1}(w)a^*_{rs}(z))\delta(z-w) = 0.$$

So we only have nontrivial expressions when $r=s$, $r=s-1$ or $r=s+1$. Then:

$$[\rho_{\mathbb{J}}(E_r)(z), \rho_{\mathbb{J}}(F_s)(w)] = \delta_{rs}([\rho_{\mathbb{J}}(E_r(z)), -a_{r,r+1}(w)]$$

$$
\begin{aligned}
&+ [\rho_{\mathbb{J}}(E_r(z)), \sum_{j=1}^{r-1}(a_{j,r+1}(w)a^*_{jr}(w)])\\
&+ \delta_{r,s-1}([\rho_{\mathbb{J}}(E_{s-1}(z)), -a_{s,s+1}(w)]\\
&+ [\rho_{\mathbb{J}}(E_{s-1}(z)), \sum_{j=1}^{s-1}(a_{j,s+1}(w)a^*_{js}(w)])\\
&+ \delta_{r,s+1}([\rho_{\mathbb{J}}(E_{s+1})(z), -a_{s,s+1}(w)]\\
&+ [\rho_{\mathbb{J}}(E_{s+1})(z), \sum_{j=1}^{s-1} a_{j,s+1}(w)a^*_{js}(w)])\\
= \delta_{rs}\Big(&(2a_{r,r+1}(z)a^*_{r,r+1}(z)\\
&+ \sum_{j=r+2}^{n} (a^*_{rj}(z)a_{rj}(z) - a^*_{r+1,j}(z)a_{r+1,j}(z)) - \gamma b_r(z)\\
&+ \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)))\delta(z-w) - \frac{\gamma^2}{2}\dot{\delta}(z-w)\\
&+ \sum_{j=1}^{r-1}[a_{jr}(z)a^*_{j,r+1}(z), a_{j,r+1}(w)a^*_{jr}(w)]\Big) + \delta_{r,s+1}(0+0)\\
&+ \delta_{r,s-1}(-\sum_{j=s+1}^{n}[a_{s,s+1}(w), a_{sj}(z)a^*_{s-1,s}(z)a^*_{sj}(z)]\\
&+ a_{s,s+1}(z)a^*_{s-1,s}(z)\delta(w-z))\\
= \delta_{rs}(&(2a_{r,r+1}(z)a^*_{r,r+1}(z)\\
&+ \sum_{j=r+2}^{n} (a^*_{rj}(z)a_{rj}(z) - a^*_{r+1,j}(z)a_{r+1,j}(z)) - \gamma b_r(z)\\
&+ \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)))\delta(z-w)\\
&- \frac{\gamma^2}{2}\dot{\delta}(z-w) + \sum_{j=1}^{r-1}(a_{j,r+1}(z)a^*_{j,r+1}(z)\\
&- a_{jr}(z)a^*_{jr}(z))\delta(z-w)) + \delta_{r,s-1}(-a_{s,s+1}(z)a^*_{s-1,s}(z)\delta(w-z)\\
&+ a_{s,s+1}(z)a^*_{s-1,s}(z)\delta(w-z))\\
= \delta_{sr}(&\rho_{\mathbb{J}}(H_r)(z)\delta(z-w) - c\dot{\delta}(z-w)).
\end{aligned}
$$

Then the lemma is verified. □

Lemma 5.8. We have the following:

$(R5|R6)$ $[\rho_{\mathbb{J}}(F_r)(z), \rho_{\mathbb{J}}(F_s)(w)] = [\rho_{\mathbb{J}}(E_r)(z), \rho_{\mathbb{J}}(E_s)(w)] = 0$ if $C_{rs} \neq -1$;

$$[\rho_{\mathbb{J}}(F_r)(z_1), \rho_{\mathbb{J}}(F_r)(z_2), \rho_{\mathbb{J}}(F_s)(w)]$$
$$= [\rho_{\mathbb{J}}(E_r)(z_1), \rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_s)(w)] = 0 \text{ if } C_{rs} = -1.$$

Proof. We have:

$$\begin{aligned}
&[\rho_{\mathbb{J}}(F_r)(z), \rho_{\mathbb{J}}(F_s)(w)]\\
&= [-a_{r,r+1}(z), \rho_{\mathbb{J}}(F_s)(w)] + [\sum_{j=1}^{r-1} a_{j,r+1}(z)a^*_{jr}(z), \rho_{\mathbb{J}}(F_s)(w)]\\
&= \delta_{s,r+1}a_{r,r+2}(w)\delta(w-z) + \delta_{r,s+1}a_{r-1,r+1}(w)\delta(w-z)\\
&- \delta_{r,s+1}\sum_{j=1}^{s-1} a_{j,r+1}(w)a^*_{js}(w)\delta(w-z)\\
&+ \delta_{s,r+1}\sum_{j=1}^{r-1} a_{j,r+2}(w)a^*_{jr}(w)\delta(w-z)\\
&= (\delta_{s,r+1}(a_{r,r+2}(w) + \sum_{j=1}^{r-1} a_{j,r+2}(w)a^*_{jr}(w)) + \delta_{r,s+1}(a_{r-1,r+1}(w)\\
&\quad - \sum_{j=1}^{r-2} a_{j,r+1}(w)a^*_{j,r-1}(w)))\delta(w-z).
\end{aligned}$$

Then $[\rho_{\mathbb{J}}(F_r)(z), \rho_{\mathbb{J}}(F_s)(w)] = 0$ if $|r-s| \neq 1$ (or equivalently, if $C_{rs} \neq -1$). Now:

$$\begin{aligned}
&[\rho_{\mathbb{J}}(E_r)(z), \rho_{\mathbb{J}}(E_s)(w)] = [a_{r,r+1}(z)a^*_{r,r+1}(z)^2, \rho_{\mathbb{J}}(E_s)(w)]\\
&\quad + [\sum_{j=r+2}^{n} (a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z))a^*_{r,r+1}(z), \rho_{\mathbb{J}}(E_s)(w)]\\
&\quad + [\sum_{j=1}^{r-1} a_{jr}(z)a^*_{j,r+1}(z) - \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{rj}(z), \rho_{\mathbb{J}}(E_s)(w)]\\
&\quad + [-\gamma a^*_{r,r+1}(z)(b_r(z) - \frac{1}{2}a^*_{r,r+1}(z)(b^+_{r-1}(z) + b^+_{r+1}(z)))\\
&\quad - \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(z), \rho_{\mathbb{J}}(E_s)(w)]\\
&= (-\delta_{rs}\sum_{j=r+2}^{n} (a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w))a^*_{r,r+1}(z)^2\delta(w-z)\\
&\quad - \delta_{r,s+1}a_{r,r+1}(w)a^*_{r-1,r}(w)a^*_{r,r+1}(z)^2\delta(w-z)\\
&\quad + 2\delta_{r,s-1}a_{r,r+1}(z)a^*_{r,r+1}(z)a^*_{r,r+2}(w)\delta(w-z)\\
&\quad - 2\delta_{r,s+1}a_{r,r+1}(z)a^*_{r,r+1}(z)a^*_{r-1,r+1}(w)\delta(w-z)
\end{aligned}$$

$$
\begin{aligned}
&- \delta_{rs}\gamma a^*_{r,r+1}(z)^2(b_r(w) - \frac{1}{2}(b^+_{r-1}(w) + b^+_{r+1}(w)))\delta(w-z)\\
&- \delta_{rs}\frac{\gamma^2}{2}a^*_{r,r+1}(z)^2\dot{\delta}(w-z)) + (\delta_{rs}\sum_{j=r+2}^{n}(a_{rj}(w)a^*_{rj}(w)\\
&- a_{r+1,j}(w)a^*_{r+1,j}(w))a^*_{r,r+1}(z)^2\delta(w-z)\\
&+ \delta_{s,r+1}a_{s,s+1}(w)a^*_{s,s+1}(z)^2a^*_{s,s+1}(w)\delta(w-z)\\
&+ \delta_{s,r+1}\sum_{k=r+3}^{n}(a_{r+1,k}(w)a^*_{r+1,k}(w)\\
&- a_{r+2,k}(w)a^*_{r+2,j}(w))a^*_{r,r+1}(z)a^*_{r+1,r+2}(z)\delta(w-z)\\
&- \delta_{r,s+1}\sum_{j=r+2}^{n}(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z))\\
&\qquad a^*_{r-1,r}(w)a^*_{r,r+1}(w)\delta(w-z)\\
&+ \sum_{j=r+2}^{n}(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z))\\
&\qquad (\delta_{r,s+1}a^*_{r,r+2}(w) - \delta_{r,s+1}a^*_{r-1,r+1}(w))\delta(w-z)\\
&- \delta_{s,r+1}(a_{r,r+1}(w)a^*_{r,r+2}(z)\\
&+ \sum_{j=r+3}^{n}a_{r+2,k}(w)a^*_{r+1,j}(z))a^*_{r,r+1}(z)\delta(w-z)\\
&+ 2\delta_{rs}\sum_{j=r+2}^{n}a_{r+1,j}(w)a^*_{rj}(z)a^*_{r,r+1}(z)\delta(w-z)\\
&- \delta_{r,s+1}\sum_{j=r+2}^{n}a_{rj}(z)a^*_{r-1,j}(w)a^*_{r,r+1}(z)\delta(w-z)\\
&+ a^*_{r,r+1}(z)a^*_{r+1,r+2}(z)\delta_{s,r+1}(\gamma b_s(w)\delta(w-z)\\
&- \frac{\gamma}{2}(b^+_{s-1}(w) + b^+_{s+1}(w))\delta(w-z) + \frac{\gamma^2}{2}\dot{\delta}(w-z)))\\
&+ ((-2\delta_{s+1,r}a_{s,s+1}(z)a^*_{s,s+1}(z)a^*_{s,s+2}(w)\\
&+ 2\delta_{r+1,s}a_{s,s+1}(z)a^*_{s,s+1}(z)a^*_{s-1,s+1}(w)\\
&- \delta_{r,s+1}(\sum_{j=s+2}^{n}(a_{sj}(z)a^*_{sj}(z) - a_{s+1,j}(z)a^*_{s+1,j}(z))a^*_{s,s+2}(w)\\
&+ a_{s,s+1}(w)a^*_{s,s+1}(z)a^*_{s,s+2}(z)) - \sum_{j=s+3}^{n}a_{s+2,k}(w)a^*_{s+1,j}(z)a^*_{s,s+1}(z)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \delta_{s,r+1} \sum_{j=s+2}^{n} a_{sj}(z) a^*_{s-1,j}(w) a^*_{s,s+1}(z) \\
&\quad + \delta_{s,r+1} \sum_{j=s+2}^{n} (a_{sj}(z) a^*_{sj}(z) - a_{s+1,j}(z) a^*_{s+1,j}(z)) a^*_{s-1,s+1}(w) \\
&+ \delta_{s+1,r}(-\sum_{j=1}^{r-2} a_{j,r-1}(w) a^*_{j,r+1}(z) + \sum_{j=r+2}^{n} a_{r+1,j}(z) a^*_{r-1,j}(w)) \\
&+ \delta_{s,r+1}(\sum_{j=1}^{r-1} a_{jr}(z) a^*_{j,r+2}(w) - \sum_{j=r+3}^{n} a_{r+2,j}(w) a^*_{rj}(z)))\delta(w-z) \\
&+ (\delta_{r+1,s} a^*_{s-1,s+1}(z) - \delta_{r,s+1} a^*_{r-1,r+1}(z))((\gamma b_s(z) \\
&- \frac{\gamma}{2}(b^+_{s-1}(z) + b^+_{s+1}(z)))\delta(w-z) \\
&+ \frac{\gamma}{2}\dot{\delta}(w-z)) - \frac{\gamma^2}{2}\delta_{r,s+1}\dot{a}_{r-1,r+1}(z)\delta(w-z)) + (\delta_{sr}\gamma a^*_{s,s+1}(z)^2(b_s(w) \\
&- \frac{1}{2}(b^+_{s-1}(w) + b^+_{s+1}(w)))\delta(w-z) + \delta_{sr}\frac{\gamma^2}{2} a^*_{s,s+1}(z)^2\dot{\delta}(w-z) \\
&- a^*_{s,s+1}(w) a^*_{s+1,s+2}(w)\delta_{r,s+1}(\gamma b_r(z)\delta(z-w) \\
&- \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z))\delta(z-w) + \frac{\gamma^2}{2}\dot{\delta}(z-w)) \\
&+ (\delta_{s,r+1} a^*_{s-1,s+1}(z) - \delta_{s+1,r} a^*_{r-1,r+1}(z))((\gamma b_r(z) - \frac{\gamma}{2}(b^+_{r-1}(z) \\
&+ b^+_{r+1}(z)))\delta(w-z) + \frac{\gamma^2}{2}\dot{\delta}(w-z)) - \frac{\gamma^2}{2}\delta_{r,s+1}\dot{a}_{r-1,r+1}(z)\delta(w-z) \\
&+ \gamma^2 a^*_{r,r+1}(z) a^*_{s,s+1}(w)(\delta_{rs} - \frac{1}{2}(\delta_{r,s+1} + \delta_{s,r+1}))\dot{\delta}(w-z)).
\end{aligned}
$$

After doing the calculations above, the term in δ_{rs} become equal to zero. So we have only terms in $\delta_{r,s+1}$ and $\delta_{r,s-1}$. Then:

$$[\rho_{\mathbb{J}}(E_r)(z), \rho_{\mathbb{J}}(E_s)(w)] = 0 \text{ if } |r-s| \neq 1,$$

and $(R5)$ is verified. To finish we have:

$$
\begin{aligned}
&[\rho_{\mathbb{J}}(F_r)(z_1), \rho_{\mathbb{J}}(F_r)(z_2), \rho_{\mathbb{J}}(F_s)(w)] := [\rho_{\mathbb{J}}(F_r)(z_1), [\rho_{\mathbb{J}}(F_r)(z_2), \rho_{\mathbb{J}}(F_s)(w)]] \\
&= [-a_{r,r+1}(z_1), (a_{r-1,r+1}(w) - \sum_{j=1}^{r-2} a_{j,r+1}(w) a^*_{j,r-1}(w))\delta(w-z_2)] \\
&\quad + [\sum_{j=1}^{r-1} a_{j,r+1}(z_1) a^*_{jr}(z_1), a_{r-1,r+1}(w)
\end{aligned}
$$

$$- \sum_{l=1}^{r-2} a_{l,r+1}(w)a^*_{l,r-1}(w))\delta(w-z_2)] = 0+0 = 0.$$

Similarly for $s = r+1$. If $s = r-1$ we have:

$$\begin{aligned}
(a)\ &[a_{r,r+1}(z_1)a^*_{r,r+1}(z_1)^2, [\rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r-1})(w)]] \\
&= -a_{r,r+1}(w)a^*_{r,r+1}(z_1)^2 a^*_{r-1,r+1}(w)\dot{\delta}(z_1-z_2)\delta(w-z_2) \\
&+ \sum_{j=r+2}^{n} (a_{rj}(z_2)a^*_{rj}(z_2) - a_{r+1,j}(z_2)a^*_{r+1,j}(z_2)) \\
&\times a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\delta(z_1-z_2)\delta(w-z_2) \\
&+ \sum_{j=r+2}^{n} a_{rj}(z_2)a^*_{r-1,j}(w)a^*_{r,r+1}(z_1)^2\delta(z_1-z_2)\delta(w-z_2) \\
&+ a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\delta(z_1-z_2)\delta(w-z_2)(\gamma b_r(z_2) \\
&- \frac{\gamma}{2}(b^*_{r-1}(z_2) + b^+_{r+1}(z_2))) \\
&- \frac{\gamma^2}{2} a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\dot{\delta}(w-z_1)\delta(z_2-w).
\end{aligned}$$

$$\begin{aligned}
(b)\ &[\sum_{j=r+2}^{n} (a_{rj}(z_1)a^*_{rj}(z_1) - a_{r+1,j}(z_1)a^*_{r+1,j}(z_1))a^*_{r,r+1}(z_1), \\
&[\rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r-1})(w)]] \\
&= -\sum_{j=r+2}^{n} (a_{rj}(z_1)a^*_{rj}(z_1) - a_{r+1,j}(z_1)a^*_{r+1,j}(z_1)) \\
&\times a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\delta(z_1-z_2)\delta(w-z_2) \\
&- \sum_{j=r+2}^{n} (a_{rj}(z_1)a^*_{rj}(z_1) - a_{r+1,j}(z_1)a^*_{r+1,j}(z_1)) \\
&\times a^*_{r,r+1}(z_1)a^*_{r-1,r+1}(w)\delta(z_1-z_2)\delta(w-z_2) \\
&- 2\sum_{j=r+2}^{n} a_{r+1,j}(z_1)a^*_{rj}(w)a^*_{r,r+1}(z_1)a^*_{r-1,r}(w)\delta(z_1-z_2)\delta(w-z_2) \\
&- \sum_{j=r+2}^{n} a_{rj}(z_2)a^*_{r-1,j}(w)a^*_{r,r+1}(z_1)^2\delta(z_1-z_2)\delta(w-z_2) \\
&- \sum_{j=r+2}^{n} a_{r+1,j}(z_1)a^*_{r-1,j}(w)a^*_{r,r+1}(z_1)\delta(w-z_2)\delta(z_1-z_2).
\end{aligned}$$

(c) $[\sum_{j=1}^{r-1} a_{jr}(z_1)a^*_{j,r+1}(z_1)$

$- \sum_{j=r+2}^{n} a_{r+1,j}(z_1)a^*_{rj}(z_1), [\rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r-1})(w)]]$

$= (a^*_{r-1,r+1}(z_1)a^*_{r,r+1}(w) \sum_{j=r+2}^{n} (a^*_{rj}(z_2)a_{rj}(z_2) - a^*_{r+1,j}(z_2)a_{r+1,j}(z_2))$

$- a^*_{r-1,r+1}(z_1)a^*_{r,r+1}(w)(\gamma b_r(z_1) - \frac{\gamma}{2}(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)))$

$+ \frac{\gamma^2}{2} a^*_{r-1,r+1}(z_1)\dot{a}^*_{r,r+1}(w)$

$+ 2 \sum_{j=r+2}^{n} a_{r+1,j}(z_1)a^*_{rj}(z_1)a^*_{r-1,r}(w)a^*_{r,r+1}(w)$

$+ \sum_{j=r+2}^{n} a_{r+1,j}(z_1)a^*_{r-1,j}(w)a^*_{r,r+1}(z_2))\delta(z_1 - z_2)\delta(w - z_2).$

(d) $[-\gamma a^*_{r,r+1}(z_1)(b_r(z_1) - \frac{1}{2}a^*_{r,r+1}(z_1)(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)))$

$- \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(z_1), [\rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r-1})(w)]]$

$= -a^*_{r,r+1}(z_2)^2 a^*_{r-1,r}(w)\delta(z_1 - z_2)\delta(w - z_2)(\gamma b_r(z_1)$

$- \frac{\gamma}{2}(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)))$

$- \frac{\gamma^2}{2} a^*_{r,r+1}(z_2)^2 a^*_{r-1,r}(w)\dot{\delta}(z_1 - w)\delta(w - z_2)$

$+ a^*_{r,r+1}(z_2)a^*_{r-1,r+1}(w)\delta(z_1 - z_2)\delta(w - z_2)(\gamma b_r(z_1)$

$- \frac{\gamma}{2}(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)))$

$- \frac{\gamma^2}{2} a^*_{r,r+1}(z_2)a^*_{r-1,r+1}(w)\dot{\delta}(z_1 - z_2)\delta(w - z_1)$

$- \gamma^2 a^*_{r,r+1}(z_1)a^*_{r-1,r}(w)a^*_{r,r+1}(w)\delta(z_2 - w)\dot{\delta}(z_2 - z_1)$

$+ \frac{\gamma^2}{2} a^*_{r,r+1}(z_1)a^*_{r-1,r+1}(z_2)\dot{\delta}(z_1 - z_2)\delta(w - z_2).$

Adding the four summations up we have:

$$[\rho_{\mathbb{J}}(E_r)(z_1), \rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r-1})(w)] = 0$$

Similarly: $[\rho_{\mathbb{J}}(E_r)(z_1), \rho_{\mathbb{J}}(E_r)(z_2), \rho_{\mathbb{J}}(E_{r+1})(w)] = 0$ and then $(R6)$ is proved. □

6. Acknowledgment

This work is part of the Ph.D. Thesis of the author, who was supported by FAPESP (Process number: 2008/06860-3). The author is grateful to his supervisor V. Futorny and to B. Cox for stimulating discussions.

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Received by the editors: 15.07.2011
and in final form 15.07.2011.