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RESEARCH ARTICLE

The influence of weakly s-permutably embedded subgroups on the p-nilpotency of finite groups

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Abstract. Suppose G is a finite group and H is a subgroup of G. H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G; H is called weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. We investigate the influence of weakly s-permutably embedded subgroups on the p-nilpotency of finite groups.

1. Introduction

Throughout the paper, all groups are finite. Recall that a subgroup H of a group G is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G [1]. From Ballester-Bolinches and Pedraza-Aguilera [2], we know H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Y. Wang [3] introduced the concept of c-normal subgroup. A subgroup H of a group G is said to be a c-normal if there exists a normal subgroup K of G

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such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In 2007, A. N. Skiba [4] introduced the concept of weakly s-permutable subgroup. A subgroup H of a group G is said to be weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. In [5], H. Wei introduced the concept of c^*-normal subgroup. A subgroup H of a group G is called c^*-normal in G if there is a normal subgroup K of G such that $G = HK$ and $H \cap K$ is s-permutably embedded in G. As a generalization of above series subgroups, Y. Li [6] introduced a new subgroup embedding property in a finite group called weakly s-permutably embedded subgroup.

Definition 1. *A subgroup H of a finite group G is said to be weakly s-permutably embedded in G if there are a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$.*

Y. Li studied the influence of weakly s-permutably embedded subgroups on the supersolvability of groups. If G has a normal Hall p'-subgroup, then we call that G is p-nilpotent. There are many results about the p-nilpotency. For example, if for an odd prime p, every subgroup of order p lies in center of G, then G is p-nilpotent ([13], IV, p.435). Recently, M. Asaad and A. A. Heliel proved that if every maximal subgroup of Sylow p-subgroup P of G is s-permutably embedded in G, where p is the smallest prime dividing $|G|$, then G is p-nilpotent ([15], Theorem 3.1). In the present paper, we continue to characterize p-nilpotency of finite groups with the assumption that some maximal subgroups or 2-maximal subgroups of Sylow subgroup of G are weakly s-permutably embedded.

2. Preliminaries

We use convention notions and notation, as in [11] and [13]. G always denotes a group, $|G|$ is the order of G, $O_p(G)$ is the maximal normal p-subgroup of G, $O^p(G) =< g \in G \mid p \nmid o(g) >$ and $\Phi(G)$ is the Frattini subgroup of G.

Lemma 1. ([3], Lemma 2.2.) *Let H be a weakly s-permutably embedded subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-permutably embedded in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-permutably embedded G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutably embedded in G/N.*

Lemma 2. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and P is abelian, then G is p-nilpotent.*

Proof. Since $N_G(P)$ is p-nilpotent, $N_G(P) = P \times H$, where H is the normal p-complement of $N_G(P)$. Since P is abelian and $[P, H] = 1$, we see that $C_G(P) = P \times H = N_G(P)$. By famous Burnside's Theorem, G is p-nilpotent. □

Lemma 3. ([7], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 4. ([8], Lemma 2.3.) *Suppose that H is s-permutable in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 5. ([14], Lemma A.) *If P is a s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

3. Main results

Theorem 1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Proof. It is easy to see that the theorem holds when $p = 2$ by [3, Theorem 3.1], so it suffices to prove the theorem for the case of odd prime. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is not a non-abelian simple group.

By Lemma 2, $p^3 \,||P|$ and so there exists a non-trivial maximal subgroup P_1 of P. By the hypothesis, P_1 is weakly s-permutably embedded in G. Then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)se$. Suppose G is simple, then $T = G$, and so $P_1 = (P_1)_{se}$ is s-permutable embedded in G. Therefore P_1 is a Sylow p-subgroup of some s-permutable subgroup K of G. Since G is simple, $K_G = 1$. By Lemma 4, P_1 is s-permutable in G. Thus $N_G(P_1) \geq O^p(G) = G$ by Lemma 5. It follows that $N_G(P_1) = G$, and so $P_1 \trianglelefteq G$, a contradiction.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. By Lemma 1, it is easy to see that every maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ is weakly s-permutably embedded in $G/O_{p'}(G)$. Since

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent, $G/O_{p'}(G)$ satisfies all the hypotheses of our theorem. The minimality of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(3) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is clear to see $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is p-nilpotent. Applying Lemma 1, we immediately see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N. We will show G/N satisfies the hypothesis of the theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leqslant p^2$, then G/N is p-nilpotent by Lemma 2. So we suppose $|PN/N| \geqslant p^3$. Let M_1/N be a maximal subgroup of PN/N. Then $M_1 = N(M_1 \cap P)$. Let $P_1 = M_1 \cap P$. It follows that $P_1 \cap N = M_1 \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since

$$p = |PN/N : M_1/N| = |PN : (M_1 \cap P)N| = |P : M_1 \cap P| = |P : P_1|,$$

we have that P_1 is a maximal subgroup of P. By the hypothesis, P_1 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. So $G/N = M_1/N \cdot TN/N = P_1N/N \cdot TN/N$. Since $(|N : P_1 \cap N|, |N : T \cap N|) = 1$, $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap (P_1T)$. By Lemma 3, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that

$$(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_{se}N/N.$$

Since $(P_1)_{se}N/N$ is s-permutably embedded in G/N by [2, Lemma 2.1], M_1/N is weakly s-permutably embedded in G/N. Since $N_{G/N}(PN/N) = N_G(P)N/N$ is p-nilpotent, we have G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(5) $G = PQ$ is solvable, where Q is a Sylow q-subgroup of G with $p \neq q$.

Since G is not p-nilpotent, by a result of Thompson [9, Corollary], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. If $N_G(H) \neq G$, we must have $N_G(H)$ is p-nilpotent by Step (3), a contradiction. We obtain $N_G(H) = G$. This leads to $O_p(G) \neq 1$. By Step (4), $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that $G_1 = PQ$ is a subgroup of G [10, Theorem 6.3.5]. Invoking our claim (3) above, G_1 is p-nilpotent if $G_1 < G$. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ [11, Theorem 9.3.1], a contradiction. Thus, we have proved that $G = PQ$ is solvable.

(6) The final contradiction.

By Step (4), there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since N is elementary abelian p-group, $N \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$. By the uniqueness of N, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence $N = O_p(G) = C_G(N)$. Obviously $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we take a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By our hypotheses, P_1 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_1)_{se}$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{se}$. So there is an s-permutable subgroup K of G such that $(P_1)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_1)_{se} \leq P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 4, $(P_1)_{se}$ is s-permutable in G. From Lemma 5 we have $O^p(G) \leq N_G((P_1)_{se})$. Since $(P_1)_{se}$ is subnormal in G, $P_1 \cap T \leq (P_1)_{se} \leq O_p(G) = N$ by [12, Corollary 1.10.17]. Thus, $(P_1)_{se} \leq P_1 \cap N$ and $(P_1)_{se} \leq ((P_1)_{se})^G = ((P_1)_{se})^{O^p(G)P} = ((P_1)_{se})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $((P_1)_{se})^G = 1$ or $((P_1)_{se})^G = P_1 \cap N = N$. If $((P_1)_{se})^G = P_1 \cap N = N$, then $N \leq P_1$ and so $P = P_1$, a contradiction. So we may assume $((P_1)_{se})^G = 1$. Then $P_1 \cap T = 1$. Since $|G : T|$ is a power of p and $T \lhd \lhd G$, $O^p(G) \leq T$. From the fact that N is the unique minimal normal subgroup of G, we have $N \leq O^p(G) \leq T$. Hence $N \cap P_1 \leq T \cap P_1 = 1$. Since

$$|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p,$$

$P_1 \cap N$ is a maximal subgroup of N. Therefore $|N| = p$, and so $\mathrm{Aut}(N)$ is a cyclic group of order $p-1$. If $q > p$, then NQ is p-nilpotent and therefore $Q \leq C_G(N) = N$, a contradiction. On the other hand, if $q < p$, then, since $N = C_G(N)$, we see that $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and therefore M, and in particular Q, is cyclic. Since Q is a cyclic group and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction. □

Remark 1. In proving our Theorem 1, the assumption that $N_G(P)$ is p-nilpotent is essential. To illustrate the situation, we consider $G = A_5$ and $p = 5$. In this case, since every maximal subgroup of Sylow 5-subgroup of G is 1, we see that every maximal subgroup of Sylow 5-subgroup of G is weakly s-permutably embedded in G, but G is not 5-nilpotent.

Corollary 1. *Let p be a prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $N_G(P)$ is p-nilpotent and there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Proof. It is clear that $N_H(P)$ is p-nilpotent and that every maximal subgroup of P is weakly s-permutably embedded in H. By Theorem 1, H is p-nilpotent. Now let $H_{p'}$ be the normal Hall p'-subgroup of H. Then $H_{p'}$ is normal in G. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemma 1. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. In this case, by our hypotheses, $N_G(P) = G$ is p-nilpotent. □

Theorem 2. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several Steps.

(1) G is not a non-abelian simple group.

(2) $O_{p'}(G) = 1$.

(3) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

(5) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (4) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, hence by G, the uniqueness of N yields $N = O_p(G)$. Since $P \cap M < P$, there is a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = P \cap MN = N(P \cap M) = NP_1$ and $P_1 = P_1 \cap P = P_1 \cap N(P \cap M) = (P \cap M)(P_1 \cap N)$. If $P \cap M = 1$, then $P = N$ and so $N_G(P) = G$ is p-nilpotent, a contradiction. Thus we may assume $P \cap M \neq 1$. Take a maximal subgroup P_0 of $P \cap M$. Let $P_2 = P_0(P_1 \cap N)$. Obviously P_2 is a maximal subgroup of P_1. Therefore P_2 is a 2-maximal subgroup of P. By our hypotheses, P_2 is weakly s-permutably embedded in G, then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. So there is an s-permutable subgroup K of G such that $(P_2)se$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$, and so $N \leq (P_2)_{se} \leq P_2 \leq P_1$. It follows that $P = NP_1 = P_1$, a contradiction. If $K_G = 1$, then $(P_2)_{se}$ is s-permutable in G by Lemma 4. Thus $(P_2)_{se} \lhd \lhd G$. From Lemma 5 we have $O^p(G) \leq N_G((P_2)_{se})$. By [12, Corollary 1.10.17], $P_2 \cap T \leq (P_2)_{se} \leq O_p(G) = N$ and so $(P_2)_{se} \leq P_2 \cap N \leq P_1 \cap N$. Then $(P_2)_{se} \leq ((P_2)_{se})^G = ((P_2)_{se})^{O^p(G)P} = ((P_2)_{se})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $((P_2)_{se})^G = 1$ or $((P_2)_{se})^G = P_1 \cap N = N$. If $((P_2)_{se})^G = P_1 \cap N = N$, then $N \leq P_1$, a contradiction. If $((P_2)_{se})^G = 1$, then $P_2 \cap T = 1$. Since $|G : T|$ is a number of p-power and $T \lhd \lhd G$, $O^p(G) \leq T$. From the fact that N is the unique minimal normal subgroup of G, we have $N \leq O^p(G) \leq T$. Thus $N \cap P_2 \leq T \cap P_2 = 1$. Since $N \cap P_2 = N \cap P_1$ and $N \cap P_1$ is a maximal subgroup of N, we have $|N| = p$. Since $M \cong G/N = G/C_G(N) \lesssim \mathrm{Aut}(N)$ is abelian, $P \cap M$ is normalized by M. Therefore $P = N(P \cap M) \trianglelefteq G$. It follows that $G = N_G(P)$ is p-nilpotent, a contradiction.

(6) The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by J.Tate's theorem ([13], IV, 4.7). Hence, by $N_{p'}$ *char* $N \lhd G$, $N_{p'} \leq O_{p'}(G) = 1$. It follows that N is a p-group. Then $N \leq O_p(G) = 1$, a contradiction. Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. We take a 2-maximal subgroup P_2 of P such that $P_2 < P_1$. By the hypothesis, P_2 is weakly s-permutably embedded in G. Then there are a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. So there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_2)_{se} \cap N$ is a Sylow p-subgroup of N. We know $(P_2)_{se} \cap N \leq P_2 \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $(P_2)_{se} \cap N = P_2 \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_2 \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 4, $(P_2)_{se}$ is s-permutable in G and so $(P_2)_{se} \lhd \lhd G$. Hence $P_2 \cap T \leq (P_2)_{se} \leq O_p(G) = 1$. It follows that $|P \cap T| \leqslant p^2$. It is easy to see that $|N \cap P| \leqslant p^2$. Thus $N \cap P$ is abelian. Since $P \leq N_G(P \cap N) < G$, we have $N_G(P \cap N)$, and so $N_N(P \cap N)$ is p-nilpotent by Step (3). By Lemma 2, N is p-nilpotent, a contradiction with Steps (2) and (3). □

Corollary 2. *Let p be a prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $N_G(P)$ is p-nilpotent and there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

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