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On primarily multiplication modules over pullback rings

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Abstract. The purpose of this paper is to present a new approach to the classification of indecomposable primarily multiplication modules with finite-dimensional top over pullback of two Dedekind domains. We extend the definition and results given in [10] to a more general primarily multiplication modules case.

1. Introduction

Modules arise when the representing subjects have an additive structure: representations as endomorphisms of abelian groups and vector spaces are the main cases. A representation of a group G over a field K is the same thing as a module over the corresponding group algebra $K[G]$. A module over a ring R is "really just" an abelian group M together with a ring morphism from to the endomorphism ring of M. It is an important feature of this strategy that any single representation will tell us only a certain amount about the original structure. So, for example, one looks at the set of all irreducible characters (simple modules) of a finite group, and even then for some purposes one has to look at more general modules. Thus arises the project of classifying all representations or, more realistically, all representations of a certain significant type. A commonly adopted strategy is to prove a decomposition theorem which says that every representation of the sort we are considering may be built up from certain simpler ones,

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and then to develop a classification and structure theory for these simpler building blocks.

An optimal structure theory for the blocks is one which provides us with a complete list and with representations of the members of the list, which are explicit enough to allow us to answer many questions about the blocks with relatively little effort. Of course it may be that, as in the case of finite groups, describing the ways in which the building blocks may be combined is problematic: fortunately with modules there are many structure theorems where the blocks are combined simply by forming direct sums. The structure theory for the building blocks is the theory of canonical forms for irreducible blocks. In fact many classification problems in linear algebra are most easily investigated by turning them into problems about modules. Also modules arise in various ways in analysis and topology, and classification may be relevant (see the discussion of Aronszajn and Fixman [2] in [26]). One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary rings R.

The reader is referred to [1], [27, Chapters 1 and 14], [28] and [29] for a detailed discussion of classification problems, their representation types (finite, tame, or wild), and useful computational reduction procedures.

Unfortunately, for the vast majority of rings, the classification of arbitrary module is infeasible. For example, the classification of all indecomposable pure-injective modules with infinite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M over a ring R we define its **top** as $M/\mathrm{rad}(R)M$) over the pullback ring formed by mapping two local Dedekind domains R_1 and R_2 onto a field $\bar{R}$ is at least as difficult as that problem. Why consider pure-injective modules? Pure-injective modules are model-theoretically typical: for example classification of the complete theories of R-modules reduces to classifying the (complete theories of) pure-injectives. Also, for some rings the "small" (finite-dimensional, finitely generated ...) modules are classified and in many cases this classification can be extended to give a classification of the (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules [5]. One point of this paper is to introduce a subclass of pure-injective modules.

Modules over pullback rings has been studied by several authors (see for example, [24],[14], [12], [11] and [13]). Notably, there is the monumental work of Levy [18], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. Common to all these classification is the reduction to a "‘matrix problem"’ over a division ring (see [28, Section 17.9] for background on matrix problems and their applications). In the present paper we introduce a new class of R-modules, called primarily multiplication modules (see Definition 3.1), and we study it in details from the classification problem point of view. We are mainly interested in case either R is a Dedekind domain or R is a pullback of two local Dedekind domains. Let R be the pullback of two local Dedekind domains over a common factor field. The purpose of this paper is to give a complete description of the indecomposable primarily multiplication modules over R. The classification is divided into two stages: the description of all indecomposable separated primarily multiplication R-modules and then, using this list of separated primarily multiplication modules we show that non-separated indecomposable primarily multiplication R-modules are factor modules of finite direct sums of separated primarily multiplication R-modules. Then we use the classification of separated primarily multiplication modules from Section 2, together with results of Levy [18], [19] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable primarily multiplication modules M (see Theorem 5.8). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable primarily multiplication modules (where infinite length primarily multiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated primarily multiplication modules.

Several problems stated in the paper remain open, and we hope that this paper encourages the researchers to study the following two problems:

(1) Describe the indecomposable primarily multiplication modules with finite-dimensional top over the pullback ring formed by mapping two local rings R_1 and R_2 onto a field $\bar{R}$.

(2) Describe the indecomposable primarily multiplication modules with finite-dimensional top over the pullback ring formed by mapping two local Dedekind domains R_1 and R_2 onto a semi-simple artinian ring $\bar{R}$.

2. Preliminaries

In this paper all rings are commutative with identity and all modules unitary. In order to make this paper easier to follow, we recall in this section various notions from module theory which will be in the sequel. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$

(so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [17]).

Definition 2.1. *An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$) [17].*

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [17, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [17, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi : S \to M$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [17, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [17, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [17, Theorem 2.8]. Moreover, R-homomorphisms lift to separated representation, preserving epimorphisms and monomorphisms [17, Theorem 2.6].

Definition 2.2. *(a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be* primary *submodule (resp.* prime *submodule) if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some n (resp. $m \in M$ or $rM \subseteq N$), so* $\mathrm{rad}(N : M) = P$ *(resp. $(N : M) = P'$) is a prime ideal of R, and N is said to be P-primary submodule (resp. P-prime submodule). The set of all primary submodules (resp. prime submodules) in an R-module M is denoted* $\mathrm{pSpec}(M)$ *(resp.* $\mathrm{Spec}(M)$*) [20, 22].*

(b) An R-module M is defined to be a multiplication *module if for each submodule N of M, $N = IM$, for some ideal I of R. In this case we can take $I = (N : M)$ [4].*

(c) An R-module M is defined to be a weak multiplication *module if* $\mathrm{Spec}(M) = \emptyset$ *or for every prime submodule N of M, $N = IM$, for some ideal I of R (note that we can take $I = (N : M)$) [3].*

(d) A submodule N of an R-module M is called pure *submodule if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called* relatively divisible *(or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$ [30].*

(e) A module M is pure-injective *if it has the injective property relative to all pure exact sequences [15, 30, 25].*

(f) A non-zero R-module M is said to be coprimary *if for each $r \in R$, the homothety $M \xrightarrow{r.} M$ is either injective or nilpotent. So* $\mathrm{rad}(0 : M) = J$*, the radical* $(0 : M)$*, is a prime ideal of R, and M is said to be J-coprimary [23].*

Remark 2.3. (i) An R-module is pure-injective if and only if it is algebraically compact [30, 25].

(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if N is an RD-submodule of M. Moreover, N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R [30, 25].

(iii) It is easy to see that an R-module M is coprimary if and only if whenever $rm = 0$ (for $r \in R$, $m \in M$), then either $m = 0$ or $r^n M = 0$ for some n. Moreover, it is clear that if N is a J-primary submodule of M, then M/N is a J-coprimary R-module.

3. Basic properties of primarily multiplication modules

The aim of this section is to classify primarily multiplication modules over a local Dedekind domain. First we collect some basic properties concerning primarily multiplication modules. We begin the key definition of this paper.

Definition 3.1. *An R-module M is defined to be a primarily multiplication module if* $\mathrm{pSpec}(M) = \emptyset$ *or for every primary submodule N of M, $N = IM$, for some ideal I of R*

One can easily show that if M is a primarily multiplication module, then $N = (N : M)M$ for every primary submodule N of M. We need the following lemma proved in [23, p. 160 Theorem 10, p. 101 Corollary and p. 99 Corollary 1].

Lemma 3.2. *(i) Let P be a prime ideal of R, let S be a multiplicatively closed set such that $P \cap S = \emptyset$ and let M be an R-module. Then there exists a one-to-one correspondence between the P-primary submodules of M and the P_S-primary submodules of M_S.*

(ii) Let $K \subseteq N$ be submodules of an R-module M. Then N is a primary submodule of M if and only if N/K is a primary submodule of M/K.

(iii) Let N be a P-primary submodule of the R-module M and suppose that I is an ideal of R and K a submodule of M. If $IK \subseteq N$, Then either $I \subseteq P$ or $K \subseteq N$.

Lemma 3.3. *Let M be an R-module, N a P-primary submodule of M and I an ideal of R with $I \subseteq (0 : M)$. Then N is P/I-primary submodule of M as an R/I-module.*

Proof. By Remark 2.3, M/N is a P-coprimary R-module. Let $(a+I)m \in N$ for some $m \in M$ and $a + I \in R/I$, so $a(m + N) = 0$; hence either $m \in N$ or $a^s M \subseteq N$ for some s, as needed. □

Lemma 3.4. *Let I be an ideal of a ring R, M a primarily multiplication R-module and N a non-zero R-submodule of M. Then the following hold:*

(i) If $I \subseteq (N : M)$, then the R/I-module M/N is primarily multiplication. In particular, the R-module M/N is weakly multiplication.

(ii) Every direct summand of M is a primarily multiplication R-module.

Proof. (i) Let L be a primary submodule of M/N. Then by Lemma 3.2, there exists a primary submodule K of M such that $L = K/N$, so $K = (K :_R M)M$. An inspection show that $L = (L :_{R/I} M/N)(M/N)$. Finally, take $I = 0$. Moreover, (ii) follows from (i). □

Lemma 3.5. *Let R and R' be any rings, $f : R \to R'$ a surjective homomorphism and M an R'-module. Then the following hold:*

(i) If N is a primary R-submodule of M, then N is a primary R'-submodule of M.

(ii) If M is a primarily multiplication R'-module, then M is a primarily multiplication R-module.

Proof. (i) is clear. To see that (ii), assume that N is a primary R-submodule of M. Then N is primary R'-submodule of M, so $N = IM$ for some ideal I of R. Set $I' = f^{-1}(I)$. Then $f(I) = I'$ and $IM = f(I)M = N$, as required. □

Proposition 3.6. *Let M be a module over a ring R. Then the following hold:*

(i) Let S be a multiplicatively closed subset of the ring R. If N is a primary submodule of M, then $(N :_R M)_S = (N_S :_{R_S} M_S)$.

(ii) M is a primarily multiplication R-module if and only if the R_P-module M_P is a primarily multiplication for every prime (or maximal) ideal P of R.

Proof. (i) Let $S \cap \mathrm{rad}(N : M) = \emptyset$. Since the inclusion $(N :_R M)_S \subseteq (N_S :_{R_S} M_S)$ is clear, we will prove the reverse inclusion. Assume that $r/s \in (N_S :_{R_S} M_S)$ and let $m \in M$. Then $(r/s)(m/1) \in N_S$, so $utrm = sux \in N$ for some $t, u \in S$ and $x \in N$. Therefore, N primary gives $rm \in N$; hence $r/s \in (N :_R M)_S$, and so we have equality. So suppose that $S \cap \mathrm{rad}(N : M) \neq \emptyset$. Then obviously, $N_S = M_S$ so that $(N :_R M)_S = (N_S :_{R_S} M_S) = R_S$.

(ii) Let M be a primarily multiplication R-module and N a primary submodule of M_P, where P is a prime ideal of R. According to Lemma 3.2, we must have $N = K_P$ for some primary submodule K of M. So $K = IM$, therefore, $N = (IM)_P = I_P M_P$. Conversely, let N be a primary submodule of M. It suffices to show that $(N/(N : M)M)_P = 0$ for every maximal ideal P of R. If $\mathrm{rad}(N : M) \subseteq P$, then by Lemma 3.2 again, N_P is a primary submodule, so $N_P = (N_P :_{R_P} M_P) = (N :_R M)_P$ by (i); hence $(N/(N : M)M)_P = 0$. If $\mathrm{rad}(N : M) \nsubseteq P$, then clearly $N_P = M_P$ and $(N : M)_P = R_P$, so we have $(N/(N : M)M)_P = 0$, as required. □

Remark 3.7. (1) Let R be an integral domain which is not a field and $Q(R)$ the field of fractions of R. Since $\mathrm{pSpec}(Q(R)) = \{0\}$ by [11, Remark 2.7 (2)], we must have $Q(R)$ is primarily multiplication.

(2) Let R be a local Dedekind domain with unique maximal ideal $P = Rp$. As $\mathrm{pSpec}(E(R/P)) = \emptyset$ (where $E = E(R/P)$ is the injective hull of R/P) by [11, Remark 2.7 (b)], E is primarily multiplication.

(3) It is easy to see that every primarily multiplication module over a ring R always is a weak multiplication module.

Theorem 3.8. *Let R be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the following is a complete list, up to isomorphism, of the indecomposable primarily multiplications modules:*

(i) R

(ii) R/P^n, $n \geq 1$, the indecomposable torsion modules;

(iv) $R_{P^\infty} = E(R/P)$, the P-Prüfer module;

(v) $Q(R)$, the field of fractions of R.

Proof. By [5, Proposition 1.3]) these modules are indecomposable. Clearly, R and R/P^n $(n \geq 1)$ are multiplication, so they are primarily multiplication. Moreover, $Q(R)$ and $E(R/P)$ are primarily multiplication by Remark 3.7. Now we show that there are no more indecomposable primarily multiplication R-modules.

Now let M be an indecomposable primarily multiplication and choose any non-zero element $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$ (so $h(a)$ is a nonnegative integer or ∞). Also let $(0 : a) = \{r \in R : ra = 0\}$: thus $(0 : a)$ is an ideal of the form P^m or 0. Because $(0 : a) = P^{m+1}$ implies that $p^m a \neq 0$ and $p.p^m a = 0$, we can choose a so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

Case 1. $\mathrm{pSpec}(M) = \emptyset$. Since $\mathrm{Spec}(M) \subseteq \mathrm{pSpec}(M)$, it follows from [11, Theorem 2.8 (Case 1)] $M \cong E(R/P)$. So we may assume that $\mathrm{pSpec}(M) \neq \emptyset$.

Case 2. $h(a) = n$, $(0 : a) = P$. Say $a = p^n b$. Then we have $Rb \cong R/P^{n+1}$ is primarily multiplication. We also have Rb is pure in M. To see this, let $r \in R$: then $r = p^k u$ for some integer $k \geq 0$ and unit $u \in R$. If we have $sb \in rM$, say $sb = p^k uc$, then write $s = p^l t$ for some $t \in R$. So $a = sp^{n-1}vb$ for some unit $v \in R$, that is $a = uvp^{n-l+k}c$. Therefore $k \leq l$ and so we obtain $sb = tu^{-1}p^{l-k}.p^k b \in rM$. Thus Rb is pure in M. Hence, since Rb is a pure submodule of bonded order of M, we obtain Rb is a direct summand of M by [16, Theorem 5]; hence $M = Rb \cong R/P^{n+1}$.

Case 3. $h(a) = n$, $(0 : a) = 0$. Say $a = p^n b$. Then $rb = 0$ implies $ra = 0$ and so $r = 0$. Thus, $Rb \cong R$. Furthermore Rb is pure in M by an argument like that in case (2). As M is a torsion-free R-module by [16, Theorem 10], Rb is prime (so primary) submodule of M (see [20, Result 2]); hence $R \cong Rb = P^0 M = RM = M$ (since $M \neq 0$), as needed.

Case 4. First we show that if $h(a) = \infty$, then $(0 : a) = 0$. Indeed, suppose not. Then $(0 : a) = P$. Since $h(a) = \infty$, there is an element a_1 of M such that $a = a_0 = pa_1$, with $a \neq a_1$ since $a \neq 0$ and $pa = 0$. If $h(a_1) < \infty$, then by case (2), M is a module of finite length, and this contradicts the height of a is ∞. So $a_1 = pa_2$ for some $a_2 \in M$. By this process, one can show that $M \cong E(R/P)$ (see [9, Proposition 2.7]); hence $\mathrm{pSpec}(M) = \emptyset$ by Remark 3.7 (2), which is a contradiction. So we may assume that $h(a) = \infty$, $(0 : a) = 0$. By the Case (3) we may assume that every non-zero element of M satisfies these conditions. So a uniquely divisible by every non-zero element of $Q(R)$ and so we can define a map, which is easily checked to be an R-homomorphism, from $Q(R)$ to M which takes q to the element qa which, we have just shown, is defined and unique. Thus we have a copy of the injective module $Q(R)$ embedded in M which must, therefore, be isomorphic to $Q(R)$ (see [9, Proposition 2.7]). □

Corollary 3.9. *Let $R \neq M$ be a primarily multiplication module over a local Dedekind domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$ and K is a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every primarily multiplication R-module not isomorphic with R is pure-injective.*

Proof. Let N_i denote the indecomposable summand of M. So by Lemma 3.4, N_i is an indecomposable primarily multiplication module. Now the assertion follows from Theorem 3.8 and [5, Proposition 1.3]. □

4. The separated case

Throughout this section we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two local Dedekind domain R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

In this section we determine the indecomposable primarily multiplication separated R-modules where R is the pullback of two local Dedekind domains.

Remark 4.1. Let R be the pullback ring as in (1), and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\}$$

$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}$$

Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi_1' = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus Ker(f_2) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\mathrm{Ker}(f_1) = P_1S_1$ and $\mathrm{Ker}(f_2) = P_2S_2$).

Theorem 4.2. *Let R be the pullback ring as in (1), and let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be a separated R-module. Then S is a primarily multiplication R-module if and only if each S_i is a primarily multiplication R_i-module ($i = 1, 2$).*

Proof. By [11, Remark 3.9], we may assume that $\mathrm{pSpec}(S) \neq \emptyset$. Let S be a primarily multiplication R-module. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S :_R S)$, Lemma 3.3 gives $S_1 \cong S/(0 \oplus P_2)S$ is a primarily multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a primarily multiplication R_2-module. Conversely, assume that each S_i is a primarily multiplication R_i-module. Let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a primary submodule of S. We split the proof into two cases for $\mathrm{rad}(T : S)$.

Case 1. $\mathrm{rad}(T : S) = P$. By [11, Proposition 3.3 (i)], we must have $T_2 = P_2^m S_2 \subseteq P_2 S_2$ for some m since S_2 is primarily multiplication. Similarly, $T_1 = P_1^k S_1 \subseteq P_1 S_1$ for some k. Let $n = \min\{m, k\}$. Therefore,

$$T \subseteq T_1 \oplus T_2 \subseteq P^{n-1}(P_1 S_1 \oplus P_2 S_2) = P^n S.$$

For the other containment, assume that $s = (p_1^n, p_2^n)(s_1, s_2) = (p_1^n s_1, p_2^n s_2) \in P^n S$. Then $s \in T$ since $p_1^n s_1 \in T_1$, $p_2^n s_2 \in T_2$ and $f_1(p_1^n s_1) = f_2(p_2^n s_2) = 0$ (note that $\mathrm{Ker}(f_1) = P_1 S_1$ and $\mathrm{Ker}(f_2) = P_2 S_2$). Thus $T = P^n S$.

Case 2. $\mathrm{rad}(T : S) = P_1 \oplus 0$. Suppose that T is a $(P_1 \oplus 0)$-primary submodule of S. Then T_2 is a 0-primary R_2-submodule of S_2 by [11, Proposition 3.3 (ii)]; hence $T_2 = 0$ since $(T_2 : S_2) \subseteq \mathrm{rad}(T_2 : S_2) = 0$. Therefore, $T = (P_1 \oplus 0)S$; hence S is a primarily multiplication R-module. Similarly, if $\mathrm{rad}(T : S) = 0 \oplus P_2$, we get S is a primarily multiplication R-module. $\square$

Lemma 4.3. *Let R be the pullback ring as in (1), and let $R \neq S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be a separated R-module. If either S_1 or S_2 is torsion-free, then $\bar{S} = 0$.*

Proof. Assume that $\bar{S} \neq 0$ and let S_1 be a torsion-free R_1-module. Then $P_1 S_1 \neq S_1$; hence $P_1 S_1$ is a prime submodule of S_1. By [20, Result 2], $P_1^{n+1} S_1 = P_1^n (P_1 S_1) = P_1 S_1 \cap P_1^n S_1 = P_1^n S_1$. Let $s_1 \in S_1$. Then there exists $t_1 \in S_1$ such that $p_1^n (s_1 - p_1 t_1) = 0$; thus $s_1 = p_1 t_1 \in P_1 S_1$. It follows that $S_1 = P_1 S_1$, which is a contradiction. Thus $\bar{S} = 0$. $\square$

Lemma 4.4. *Let R be the pullback ring as in (1). The following separated R-modules are indecomposable and primarily multiplication:*

(1) $R = (R_1 \to \bar{R} \leftarrow R_2)$

(2) $S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2)$ where $E(R_i/P_i$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;

(3) $S = (Q(R_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow Q(R_2)$ where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;

and, for all positive integers n, m,

(4) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$.

Proof. By [5, Lemma 2.8] and [19, 1.9], these modules are indecomposable. Primarily multiplicativity follows from Theorem 3.8 and Theorem 4.2. $\square$

We refer to modules of type (2) in Lemma 4.4 as P_1-Prüfer and P_2-Prüfer, respectively.

Theorem 4.5. *Let R be the pullback ring as in (1), and let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2$ be an indecomposable separated primarily multiplication R-module. Then S is isomorphic to one of the modules listed in Lemma 4.4.*

Proof. We may assume that $S \neq R$. If $\mathrm{pSpec}(S) = \emptyset$, then $\mathrm{pSpec}(S_i) = \emptyset$ by [11, Remark 3.9], so $S_i = P_i S_i$ for each $i = 1, 2$ by Theorem 3.8; hence $S = PS = P_1 S_1 \oplus P_2 S_2 = S_1 \oplus S_2$. Therefore, $S = S_1$ or S_2 and so S is of type (2) in the list of Lemma 4.4 by Theorem 3.8. So we may assume that $\mathrm{pSpec}(S) \neq \emptyset$. Next suppose that $PS = S$. Then by [5, Lemma 2.7 (i)], $S = S_1$ or S_2 and so S is an indecomposable primarily multiplication R_i-modules for some i and, since $PS = S$, is type (3) in the list of Lemma 4.4 by Theorem 3.8. So we may assume that $S/PS \neq 0$.

By Theorem 4.2, S_i is a primarily multiplication R_i-module, for each $i = 1, 2$ (note that for each i, S_i is torsion and it not divisible R_i-module by Theorem 1. Hence, by the structure of primarily multiplication modules over a local Dedekind domain (see Theorem 3.9), $S_i = M_i \oplus N_i$ where N_i is a direct sum of copies of R_i/P_i^n $(n \geq 1)$ and M_i is a direct sum of copies of $E(R_i/P_i)$ and $Q(R_i)$. Then we have $S = (N_1 \to \bar{S} \leftarrow N_2) \oplus (M_1 \to 0 \leftarrow 0) \oplus (0 \to 0 \leftarrow M_2)$. Since S is indecomposable and $S/PS \neq 0$ it follows that $S = (N_1 \to \bar{S} \leftarrow N_2)$. We will see that each S_i $(= N_i)$ is indecomposable.

There exist positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^k S_2 = 0$ and $P^n S = 0$. Now choose $t \in S_1 \cup S_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is maximal. There is a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in S_i for $i = 1, 2$ (see [5, Theorem 2.9]). Therefore, $R_1 t_1 \cong R_1/P_1^m$ (resp. $R_2 t_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i t_i$ is pure-injective [5]. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 t_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2 t_2)$. Then M is an R-submodule of S which is primarily multiplication by Lemma 4.4, and is a direct summand of S; this implies that $S = M$, and S is an in (4) (see [5, Theorem 2.9]). □

Corollary 4.6. *Let R be the pullback ring as in (1), and let $S \neq R$ be a separated primarily multiplication R-module. Then S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in (3), N is a direct sum of copies of the modules as in (1)-(2) of the Lemma 4.4. In particular, every separated primarily multiplication R-module not isomorphic with R is pure-injective.*

Proof. Apply Theorem 4.5 and [5, Theorem 2.9]. □

5. The non-separated case

We continue to use the notation already established, so R is a pullback ring as in (1). In this section we find the indecomposable non-separated primarily multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable primarily multiplication modules.

Remark 5.1. Assume that R is the pullback ring as in (1) and let $F = E(R/P)$, the injective hull of R/P. Then $\mathrm{pSpec}(F) = \emptyset$ by [11, Proposition 4.1]; hence $E(R/P)$ is a non-separated primarily multiplication R-module (also see [5, p. 4053]).

Proposition 5.2. *Let R be the pullback ring as in (1) and let M be any primarily multiplication R-module. Then the following hold:*

(i) If M has a $P_1 \oplus 0$-primary submodule N, then M is separated.

(ii) If M has a $0 \oplus P_2$-primary submodule N, then M is separated.

Proof. (i) First, we show that the $(P_1 \oplus 0)$-coprimary R-module M/N is separated. It suffices to show that $(P_1 \oplus 0)(M/N) = 0$. As $(0, p_2)(p_1, 0)(m+N) = 0$ $(m \in M)$, we must have $(p_1, 0)m = 0$. Thus M/N is a separated R-module. By assumption, $(0 \oplus P_2)N = (0 \oplus P_2)(P_1^n \oplus 0)M = 0$ for some n; hence $P_1 N \cap P_2 N = 0$. Therefore, N is separated. Now we show that M is separated. It suffices to show that $p_1 M \cap p_2 M = 0$. Let $x = p_1 a = p_2 b$ for some $a, b \in M$. Then $p_1(a+N) = p_2(b+N) \in (P_1(M/N)) \cap (P_2(M/N)) = 0$, so $p_1 a = p_2 b \in N$. Then N primary gives $b \in N$; hence $x = 0$, and the proof is complete. The proof of (ii) is similar. $\square$

Theorem 5.3. *Let R be the pullback ring as in (1) and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is primarily multiplication if and only if M is primarily multiplication.*

Proof. By [11, Proposition 4.4], we may assume that $\mathrm{pSpec}(S) \neq \emptyset$. If S is primarily multiplication, then $M \cong S/K$ is primarily multiplication, by Lemma 3.2. Conversely, assume that M is a primarily multiplication R-module and let N be a primary submodule of S. First suppose that $\mathrm{rad}(N : S) = P$. Then by [11, Lemma 4.3], $K \subseteq N$, and N/K is a primary submodule of $S/K \cong M$ by Lemma 3.2, so $P^n(S/K) = N/K$ for some n since S/K is primarily multiplication. As $K \subseteq P^n S$ (see [11, Lemma 4.3]), we must have $N = P^n S$. If $\mathrm{rad}(N : S) = P_1 \oplus 0$ (resp. $\mathrm{rad}(N : S) = 0 \oplus P_2$), then N/K is a $(P_1 \oplus 0)$-primary (resp. $(0 \oplus P_2)$-primary) submodule of M, which is a contradiction by Proposition 2. Thus S is primarily multiplication. $\square$

Proposition 5.4. *Let R be the pullback ring as in (1) and let M be an indecomposable primarily multiplication non-separated R-module with finite-dimensional over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [5, Proposition 2.6 (i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Corollary 4.6 and Theorem 5.3. □

Let R be the pullback ring as in (1) and let M be an indecomposable primarily multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. First we claim that R (the module (1) on the list in Lemma 4.4) is not a direct summand of S. For otherwise we have $S = S' \oplus R$ and then, since $\mathrm{Soc}(R) = 0$, $K \subseteq S'$. Therefore $M \cong S'/K \oplus R$, a contradiction since M is indecomposable and non-separated. Moreover, By Proposition 5.4, S is pure-injective. So in the proofs of [5, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [5, Proposition 2.6 (ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable primarily multiplication non-separated R-module": because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following results:

Corollary 5.5. *Let R be the pullback ring as in (1) and let M be an indecomposable primarily multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summand of S.*

Corollary 5.6. *Let R be the pullback ring as in (1) and let M be an indecomposable primarily multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is a direct sum of finitely many indecomposable primarily multiplication modules.*

Corollary 5.7. *Let R be the pullback ring as in (1) and let M be an indecomposable primarily multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then at most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Before we state the main theorem of this section let us explain the idea of proof. Let M be an indecomposable primarily multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then by Corollary 5.6, S is a direct sum of just finitely many indecomposable separated primarily multiplication modules and these are known by Theorem 4.3. In any separated representation $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ the kernel of the map φ to M is annihilated by P, hence is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socle of the various direct summands of S. This explains Corollary 5.5: the modules $Q(R_1)$ and $Q(R_2)$ have zero socle and so cannot occur in a separated (hence "minimal") representation. So the questions are: does this provide any further condition on the possible direct summands of S? How can these summands be amalgamated in order to form M?

In [19], Levy shows that the indecomposable finitely generated R-modules are of two non-overlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type which are most relevant to us. Such a module is obtained from a direct sum, S, of indecomposable separated modules by amalgamating the direct summands of S in pairs to form a chain but leaving the two ends unamalgamated [19], see also [18, Section 11].

Since every "block cycle" type R-module is a quotient of a primarily multiplication R-module, so it is primarily multiplication by Lemma 3.2. So by Corollary 5.7, the infinite length non-separated indecomposable primarily multiplication modules are obtained in just the same way as the deleted cycle type indecomposable are except that at least one of the two "end" modules must be a separated indecomposable primarily multiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer are direct summands of S then we will describe these modules as **doubly infinite**. Those where S has just one infinite length summand we will call **singly infinite** (see [5, Section 3]). It remains to show that the modules obtained by these amalgamation are, indeed, indecomposable primarily multiplication modules.

Theorem 5.8. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the* *indecomposable non-separated primarily multiplication modules with finite-dimensional top are the following:*

(i) the indecomposable modules of finite length (apart from R/P which is separated);

(ii) the doubly infinite primarily multiplication modules as described above;

(iii) the singly infinite primarily multiplication modules as described above, apart from the two Prüfer modules (2) in Lemma 4.4.

Proof. We know already that every indecomposable primarily multiplication non-separated module with finite-dimensional top has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable primarily multiplication modules. Let M be an indecomposable non-separated primarily multiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Since M is a quotient of a primarily multiplication R-module, so it is primarily multiplication by Lemma 3.2. The indecomposability follows from [19, 1.9].

(ii) and (iii) (involving one or two Prüfer modules) Since a quotient of any primarily multiplication R-module is primarily multiplication, M is primarily multiplication and the indecomposability follows from [5, Theorem 3.5]. □

Corollary 5.9. *Let R be the pullback ring as described in Theorem 5.8. Then every indecomposable non-separated primarily multiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [5, Theorem 3.5] and Theorem 5.8. □

Remark 5.10. Given a field k, the infinite-dimensional k-algebra $T = k[x, y : xy = 0]_{(x,y)}$ is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of the local Dedekind domains $k[x]_{(x)}, k[y]_{(y)}$. This paper includes the classification of those indecomposable primarily multiplication modules over T which have finite-dimensional top.

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