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RESEARCH ARTICLE

On filters and upper sets in CI-algebras

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Abstract. CI-algebras are a generalization of BE-algebras and dual BCK/BCI/BCH-algebras. In this paper filters of CI-algebras are considered. Given a subset of a CI-algebra, the least filter containing it is constructed. An equivalent condition of the filters using the notion of upper sets is provided.

1. Introduction

In 1966, Y. Imai and K. Iséki [3] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras, such as BCI-algebras [4], BCH-algebras [2], BCC-algebras [8], BH-algebras [5], d-algebras [12], etc. In [6], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. They defined and studied the concept of a filter in BE-algebras. This concept was also investigated in [10] and [7]. As a generalization of BE-algebras, B. L. Meng [9] introduced the notion of CI-algebras and discussed its important properties.

In this paper, we consider filters in CI-algebras. Given a subset of a CI-algebra, we make the least filter containing it. We provide an equivalent condition of the filters using the notion of upper sets.

2. Preliminaries

Definition 2.1. ([9]) *A CI-algebra* is an algebra $(X; *, 1)$ of type $(2, 0)$ satisfying the following axioms:

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(CI-1) $x * x = 1$,
(CI-2) $1 * x = x$,
(CI-3) $x * (y * z) = y * (x * z)$.

A CI-algebra X is said to be a *BE-algebra* if for all $x \in X$
(BE) $x * 1 = 1$.

Throughout this paper X will denote a CI-algebra. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 1$.

Example 2.2. Let $X = \{1, a, b, c\}$ and $*$ be defined by the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	c	c	c	1

Then $(X, *, 1)$ is a CI-algebra, which is not a BE-algebra.

For any $x_1, \ldots, x_n, a \in X$, we define

$$\prod_{i=1}^{n} x_i * a = x_n * (\cdots * (x_1 * a) \cdots).$$

Proposition 2.3. ([9]) *For any $x, y \in X$ we have*
(a) $y * ((y * x) * x) = 1$,
(b) $1 \leqslant x \Rightarrow x = 1$.

Definition 2.4. ([11]) A CI-algebra X is said to be *transitive* if for all $x, y, z \in X$,

$$y * z \leqslant (x * y) * (x * z).$$

It is easily seen that the CI-algebra X of Example 2.2 is transitive. Consider the following example.

Example 2.5. Let $X = \{1, a, b, c, d\}$ and $*$ be defined by the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	1	d
c	1	a	c	1	d
d	d	d	d	d	1

Then $(X, *, 1)$ is a CI-algebra. Since $b * a = 1$ and $(c * b) * (c * a) = c * a = a$, X is not transitive.

Lemma 2.6. ([11]) *If a CI-algebra X is transitive, then for all $x, y, z \in X$, $x \leqslant y$ implies $z * x \leqslant z * y$.*

Lemma 2.7. *Let X be a transitive CI-algebra and let $x, y \in X$ such that $x * y = 1$. Then for all $a_1, \ldots, a_n \in X$, $\prod_{i=1}^n a_i * x = 1$ implies $\prod_{i=1}^n a_i * y = 1$.*

Proof. We have $x \leqslant y$ and from Lemma 2.6 we see that

$$1 = \prod_{i=1}^{n} a_i * x \leqslant \prod_{i=1}^{n} a_i * y.$$

Applying Proposition 2.3 (b) we conclude that $\prod_{i=1}^n a_i * y = 1$. □

3. Filters

Following [9], a *filter* of X is a subset F of X such that for all $x, y \in X$:

(F1) $1 \in F$,

(F2) if $x * y \in F$ and $x \in F$, then $y \in F$.

By Fil(X) we denote the set of all filters in X. It is obvious that $\{1\}, X \in$ Fil(X).

Example 3.1. Consider the CI-algebra X of Example 2.2. It is easy to check that Fil$(X) = \{\{1\}, \{1, a, b\}, X\}$.

Proposition 3.2. *If F_i $(i \in I)$ are filters of X, then $\bigcap_{i \in I} F_i$ is a filter of X.*

Proof. Straightforward. □

Proposition 3.3. *Let F be a subset of X containing* 1. *Then $F \in$ Fil(X) if and only if for any $a, b \in F$ and $x \in X$, $a * (b * x) = 1$ implies $x \in F$.*

Proof. ($\Leftarrow$) Since $1 \in F$, the condition (F1) holds. Suppose that $a * x \in F$ and $a \in F$. By Proposition 2.3 (a), $a * [(a * x) * x] = 1$. Then $x \in F$ and hence (F2) is true. Therefore F is a filter of X.

($\Rightarrow$) Let $F \in$ Fil(X). Assume $a, b \in F$ and $x \in X$ such that $a*(b*x) = 1$. From (F1) we obtain $a * (b * x) \in F$. Applying (F2) twice we have $x \in F$. □

By induction we easily obtain

Corollary 3.4. *Let F be a subset of X containing* 1. *Then $F \in$ Fil(X) if and only if for any $a_1, \ldots, a_n \in F$ and $x \in X$, $\prod_{i=1}^n a_i * x = 1$ implies $x \in F$.*

Definition 3.5. For every subset $A \subseteq X$, the smallest filter of X which contains A, that is, the intersection of all filters $F \supseteq A$, is said to be the *filter generated by* A, and will be denoted $[A)$. Obviously, $[\emptyset) = \{1\}$.

Theorem 3.6. *Let A be a nonvoid subset of a transitive CI-algebra X. Then*

$[A) = \{x \in X : x = 1$ *or* $\prod_{i=1}^{n} a_i * x = 1$ *for some* $a_1, \ldots, a_n \in A\}$.

Proof. Set $F = \{x \in X : x = 1$ or $\prod_{i=1}^{n} a_i * x = 1$ for some $a_1, \ldots, a_n \in A\}$. Since $a * a = 1$ for all $a \in A$, we obtain $A \subseteq F$. Obviously, $1 \in F$. Let $x * y \in F$ and $x \in F$. To prove that $y \in F$, we will consider three cases.

Case 1: $x = 1$.

Then $y = 1 * y \in F$.

Case 2: $x * y = 1$ and $x \neq 1$.

Since $x \in F$ and $x \neq 1$, we conclude that $\prod_{i=1}^{n} a_i * x = 1$ for some $a_1, \ldots, a_n \in A$. From Lemma 2.7 it follows that $\prod_{i=1}^{n} a_i * y = 1$. Therefore $y \in F$.

Case 3: $x * y \neq 1$ and $x \neq 1$.

Then there are $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that $\prod_{i=1}^{n} a_i * (x * y) = 1$ and $\prod_{j=1}^{m} b_j * x = 1$. Applying (CI-3) we deduce that $x \leq \prod_{i=1}^{n} a_i * y$. From Lemma 2.6 we see that

$$1 = \prod_{j=1}^{m} b_j * x \leqslant \prod_{j=1}^{m} b_j * \left(\prod_{i=1}^{n} a_i * y \right).$$

By Proposition 2.3 (b), $\prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i * y) = 1$. Hence $y \in F$, and so F is a filter of X.

Suppose now that U is any filter of X containing A. Let $x \in F$. If $x = 1$, then obviously $x \in U$. Assume that $x \neq 1$. Then there are $a_1, \ldots, a_n \in A$ such that $\prod_{i=1}^{n} a_i * x = 1$. Since $A \subseteq U$, it follows that $a_1, \ldots, a_n \in U$. Therefore $x \in U$ by Corollary 3.4. Thus $F \subseteq U$ and hence $F = [A)$. □

Let $F_1, F_2 \in \mathrm{Fil}(X)$. We define the meet of F_1 and F_2 (denoted by $F_1 \wedge F_2$) by $F_1 \wedge F_2 = F_1 \cap F_2$ and the join of F_1 and F_2 (denoted by $F_1 \vee F_2$) by $F_1 \vee F_2 = [F_1 \cup F_2)$. We note that $(\mathrm{Fil}(X); \wedge, \vee)$ is a lattice. Moreover, by Proposition 3.2 we have

Theorem 3.7. $(\mathrm{Fil}(X); \wedge, \vee)$ *is a complete lattice.*

4. Upper sets

For any $x, y \in X$, we define

$$A(x, y) = \{z \in X : z = 1 \text{ or } x * (y * z) = 1\}$$

and

$$A(x) = \{z \in X : z = 1 \text{ or } x * z = 1\}.$$

Applying (CI-2) we conclude that $A(x) = A(1, x)$.

The set $A(x)$ (resp. $A(x, y)$) is called an *upper set* of x (resp. of x and y). We say that a subset A of X is an upper set of X if $A = A(x, y)$ for some $x, y \in X$. By US(X) we denote the set of all upper sets in X.

Remark 4.1. By (CI-3), $A(x, y) = A(y, x)$ for all $x, y \in X$.

Example 4.2. Let $X = \{1, a, b\}$ and $*$ be defined by the following table:

$*$	1	a	b
1	1	a	b
a	a	1	1
b	a	1	1

Then $(X, *, 1)$ is a CI-algebra. For $x, y \in X$, we have

$$A(x,y) = \begin{cases} X & \text{if } x \neq y \text{ and } (x = 1 \text{ or } y = 1) \\ \{1\} & \text{otherwise.} \end{cases}$$

Since Fil(X) $= \{\{1\}, X\}$, we see that Fil(X) $=$ US(X).

In general, not every filter is an upper set and not every upper set is a filter. Indeed, we consider the following example.

Example 4.3. Let X be the CI-algebra of Example 2.2. We have (see Example 3.1) Fil(X) $= \{\{1\}, \{1, a, b\}, X\}$. It is easy to check that US(X) $=$ $\{\{1\}, \{1, a, b\}, \{1, c\}\}$. Therefore X is not an upper set of X and $\{1, c\}$ is not a filter in X. □

Lemma 4.4. *For every $x, y \in X$,*

(a) $x \in A(x)$,
(b) $1 \in A(x, y)$ *and* $1 \in A(x)$,
(c) *if* $y * 1 = 1$, *then* $A(x) \subseteq A(x, y)$,
(d) *if* $y * 1 \neq 1$, *then* $A(x) - \{1\} \subseteq X - A(x, y)$,
(e) *if* $A(x)$ *is a filter of* X *and* $y \in A(x)$, *then* $A(x, y) \subseteq A(x)$.

Proof. (a) Let $x \in X$. Since $x * x = 1$, we have $x \in A(x)$.

(b) By the definition of upper sets.

(c) Let $y * 1 = 1$ and let $z \in A(x)$. If $z = 1$, then obviously $z \in A(x, y)$. Suppose that $x * z = 1$. Hence $y * (x * z) = y * 1 = 1$ and therefore $z \in A(y, x) = A(x, y)$. Consequently, $A(x) \subseteq A(x, y)$.

(d) Let $y * 1 \neq 1$ and $z \in A(x) - \{1\}$. Then $x * z = 1$ and applying (CI-3) we get $x * (y * z) = y * (x * z) = y * 1 \neq 1$. Thus $z \notin A(x, y)$ and we conclude that $A(x) - \{1\} \subseteq X - A(x, y)$.

(e) Let $A(x)$ be a filter of X and $y \in A(x)$. If $z \in A(x,y)$, then $z = 1$ or $x * (y * z) = 1$. In the first case $z = 1 \in A(x)$, in the second one $x * (y * z) \in A(x)$. Since $A(x)$ is a filter and $x, y \in A(x)$, we obtain $z \in A(x)$. □

Theorem 4.5. *Let F be a nonvoid subset of a CI-algebra X. Then F is a filter of X if and only if $A(x,y) \subseteq F$ for all $x, y \in F$.*

Proof. Suppose that F is a fillter of X. Let $x, y \in F$ and $z \in A(x,y)$. Then $z = 1$ or $x * (y * z) = 1$. Obviously $z = 1 \in F$. If $x * (y * z) = 1$, then applying twice (F2) we obtain $z \in F$. Hence $A(x,y) \subseteq F$.

Now let $A(x,y) \subseteq F$ for all $x, y \in F$. Since $F \neq \emptyset$, there exists $z \in F$. By definition, $1 \in A(z,z) \subseteq F$ and therefore (F1) holds. Let $x * y \in F$ and $x \in F$. By (CI-1), $(x * y) * (x * y) = 1$ and hence $y \in A(x * y, x) \subseteq F$. Thus (F2) also holds and consequently, F is a filter of X. □

Proposition 4.6. *If F is a filter of X, then $F = \bigcup_{x,y \in F} A(x,y)$.*

Proof. Let F be a filter. From Theorem 4.5 it follows that $A(x,y) \subseteq F$ for all $x, y \in F$. Hence $\bigcup_{x,y \in F} A(x,y) \subseteq F$.

Now let $z \in F$. By Lemma 4.4 (a),

$$z \in A(z) = A(1,z) \subseteq \bigcup_{x,y \in F} A(x,y).$$

Then $F \subseteq \bigcup_{x,y \in F} A(x,y)$. □

Proposition 4.7. *If F is a filter of X, then $F = \bigcup_{x \in F} A(x)$.*

Proof. Let F be a filter and let $z \in F$. By Lemma 4.4 (a), $z \in A(z) \subseteq \bigcup_{x \in F} A(x)$. Therefore $F \subseteq \bigcup_{x \in F} A(x)$. From Theorem 4.5 we conclude that $A(x) = A(1,x) \subseteq F$ for all $x \in F$. Hence $\bigcup_{x \in F} A(x) \subseteq F$ and consequently, $F = \bigcup_{x \in F} A(x)$. □

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