Algebra and Discrete Mathematics SURVEY ARTICLE
Volume 11 **(2011)**. Number 1. pp. 75 – 108

Some related to pronormality subgroup families and the properties of a group

Vladimir V. Kirichenko, Leonid A. Kurdachenko and Igor Ya. Subbotin

Abstract. Some influential families of subgroups such as pronormal subgroups, contranormal subgroups, and abnormal subgroups, their generalizations, characterizations, interplays between them and the group, and their connections to other types of subgroups have been considered.

Introduction

Groups with certain prescribed properties of subgroups form one of the central subjects of research in group theory. Their investigation introduced many important notions such as the finiteness conditions, locally nilpotence, locally solubility, group ranks, and others. Choosing specific prescribed properties and concrete families of subgroups which posses these properties, we come to distinct classes of groups. Among many others, the following restrictive properties have been considered by numerous authors: the normality, generalized normality, to be abelian, nilpotency, complementability, transitivity, supersolubility, density, the minimal and maximal conditions, different restrictions on important characteristics of groups (in particular, on distinct ranks), other finiteness conditions. Topological and linear groups with the restrictions on their systems of subgroups have been also investigating.

The roots of these investigations lie in the famous classical paper due to R. Dedekind [19], in which he completely described finite non-abelian

2000 Mathematics Subject Classification: *20E15, 20E34, 20F19,20F22.*

Key words and phrases: *normal subgroups, subnormal subgroup, contranormal subgroup, abnormal subgroups, pronormal subgroups, transitivity of normality, permutable subgroups, soluble group, radical group, locally graded group.*

groups whose all subgroups are normal (the Hamiltonian groups). Recall that abelian and Hamiltonian groups together form the named in honor of R. Dedekind the class of Dedekind groups (i.e. groups with only normal subgroups). Later, R. Baer obtained a description of all infinite and finite Hamiltonian groups [3]. As it has been shown, such groups are direct products of a quaternion group, an elementary abelian 2-group, and an abelian periodic group with elements of only odd orders. In their famous paper [55], G. Miller and H. Moreno described the finite groups whose all proper subgroups are abelian. In this setting, we need to mention the remarkable article [68], in which O. Yu. Schmidt completely described finite groups whose all proper subgroups are nilpotent. O.Yu. Schmidt [69] continuing farther the Dedekind's research, described finite groups G having only one class of non-normal subgroups. Later, in the paper [70], he considered finite groups having only two classes of non-normal subgroups. As an evidence of nowadays actuality of these Schmidt's results, we can mention that these researches were continued in [76], generalized in [62], and quite recently they have been just repeated in [13] and [57]. The mentioned above Schmidt's works showed that normal subgroups and their generalizations quite strongly influence the structure of a finite group. Classical results of S.N. Chernikov concerning groups whose all infinite subgroups are normal, groups whose all non-abelian subgroups are normal [16, 17], and groups whose infinite abelian subgroups are normal [18] justified this point for infinite groups. The influence on the structure of a group such generalizations of normal subgroups as the subnormal subgroups, ascendant subgroups, permutable subgroups, almost normal subgroups, normal-by-finite subgroups and many others, became one of the central themes in infinite group theory. Numerous important results have been obtained in this area by many algebraists, containing S.N. Chernikov, R. Baer, P. Hall, B.H. Neumann, V.M. Glushkov, M.I. Kargapolov, S.E. Stonehewer, J. Wiegold, B. Hartley, D.J. Robinson, D.I. Zaitsev, M.J. Tomkinson, J.S. Wilson, J.C. Lennox, L.A. Kurdachenko, M.R. Dixon, H. Smith, C. Casolo, I.Ya. Subbotin, N.F. Kuzennyj, F.N. Liman.

Meanwhile, some other important types of subgroups having significant influence on the structure of a group have been introduced. Among them are such well-known subgroups as the abnormal, pronormal, contranormal, permutable, Carter subgroups, system normalizers, and so on.

A subgroup H of a group G is called *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element g of G. Abnormal subgroups have appeared in the paper [29] due to P. Hall, while the term "an abnormal subgroup" itself belongs to R. Carter [14].

Later, P. Hall has introduced the following generalization of abnormal subgroups. A subgroup H of a group G is said to be *pronormal* in G if for every $g \in G$ the subgroups H and H^g are conjugate in the subgroup $\langle H, H^g \rangle$. Such important subgroups of finite soluble groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal subgroups.

As we can see, in these definitions a group does not need to be finite. However, these subgroups first have been introduced and intensively studied in finite groups. Many interesting and important results on finite groups have been proven in connection with these concepts. In infinite groups, the study of pronormal and abnormal subgroups has begun much later. Z.I. Borevich was one of the initiators of this study. He came to the necessity of this investigation studying arrangements of subgroups in linear groups. In the survey [2], some new types of pronormal subgroups and their generalizations have been introduced, and some connections between them have been established. An initial program of investigations in this area has been also outlined there. Unfortunately, when Z.I. Borevich passed away, these researches were no longer continued in St Petersburg. At the same time, in the cycle of their articles N.F. Kuzennyi and I.Ya. Subbotin initiated consistent investigation of pronormal subgroups in infinite groups. Later, other mathematicians joined and actively contributed in this research, and this part of group theory became rich on many interesting results and new robust concepts. One of the main goals of our survey is to give a snap shot of the current stage of this theory.

1. Abnormal subgroups, their interplays and generalizations

By its meaning, the abnormality is an antagonist to the normality: a subgroup of a group is simultaneously normal and abnormal only if it coincides with the group. The maximal non-normal subgroups are trivial examples of abnormal subgroups. More interesting here is the well-known J. Tits example: the subgroup $\mathbf{T}(n, K)$ of all triangular matrices is abnormal in the general linear group $\mathbf{GL}(n, F)$ over a field F. Every *Carter subgroup* (that is a nilpotent self-normalizing subgroup) of a finite soluble group is abnormal (R.W. Carter [14]). In the mentioned paper, R.W. Carter also was able to obtain an important following characterization of abnormal subgroups. But first we need the following simple concept.

Let G be a group and G_0 be a subgroup of G. A subgroup H is called *intermediate* for G_0 if $G_0 \leq H \leq G$ [2]. Z.I. Borevich and his collaborators studied a variety of properties of the lattices of all intermediate subgroups for a fixed subgroup G_0 (see [10, 12, 11, 25, 2]).

Theorem 1.1 [14]. *Let G be a group and H a subgroup of G. Then H is abnormal in G if and only if the following two conditions hold:*

(i) If K is an intermediate subgroup for H, then K is self-normalizing.

(ii) If K, L are two intermediate subgroup for H such that $L = x^{-1}Kx$, then $K = L$.

In the case of soluble groups, the condition (ii) could be omitted. For finite soluble groups this fact is mentioned in the book of B. Huppert [33, p. 733, Theorem 11.17]. A very power generalization of this statement on infinite groups has been obtained by L.A. Kurdachenko and I.Ya. Subbotin in the paper [45]. This last result permits the following wide generalization.

A group G is called *an $\tilde{N}$-group* if G satisfies the following condition:

If M, L are subgroup of G such that M is maximal in L, then M is normal in L.

Remark that the property "to be an $\tilde{N}-$group" is local [49, § 8]. In particular, *every locally nilpotent group is an $\tilde{N}$-group*, but converse is not true [78]. We also observe that *a group G is an $\tilde{N}$-group if and only if every subgroup of G is a member of some Kurosh-Chernikov series of G* [49, § 8].

Let G be a group and S be a family subgroup of G. Then S is said to be a Kurosh-Chernikov series, if it satisfies the following conditions:

(KC 1) $\langle 1 \rangle, G \in S$;

(KC 2) for each pair A, B of subgroups from S either is $A \leq B$ or $B \leq A$;

(KC 3) for every subfamily L of S the intersection of all members of L belongs to S and the union of all member of L belongs to S; in particular, for each non-identity element $x \in G$ the union V_x of all members of S excluding the element x belongs to S, and the intersection Λ_x of all members of S excluding the element x belongs to S;

(KC 4) for each non-identity element $x \in G$ the subgroup V_x is normal in Λ_x.

The factor-groups Λ_x/V_x are called factors of the series S.

If every subgroup of S is normal in G, then S is called the normal Kurosh-Chernikov series.

These and other families have been introduced in the classical work of A.G. Kurosh and S.N. Chernikov [49]. In the paper [49], such series have been named *normal and invariant*.

Let $\mathfrak{X}$ be a class of groups. Recall that *a group G is said to be a hyper-X-group if G has an ascending series of normal subgroups whose factors belong to the class $\mathfrak{X}$.*

Theorem 1.2. *Let G be a hyper-$\tilde{N}$-group and H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Proof. If H is abnormal subgroup of G and $K \geq H$, then $N_G(K) = K$ by Theorem 1.1. Let

$$\langle 1 \rangle = L_0 \leq L_1 \leq ...L_\alpha \leq L_{\alpha+1} \leq ...L_\gamma = G$$

be an ascending series of normal subgroups whose factors are $\tilde{N}$-group. We will prove that H is abnormal in $L_\alpha H$ for each $\alpha \leq \gamma$. Put $D = HL_1$. Choose an arbitrary element $g \in D$ and consider the subgroup $K = \langle H, H^g \rangle$. Without loss of generality we may assume that $g \in L_1$. Suppose the contrary, that is $g \notin K$. The inclusion $H \leq K$ implies that $K = H(K \cap L_1)$. Clearly $K \cap L_1$ is normal in K, in particular, $K \cap L_1$ is H-invariant. Since $g \notin K, g \notin K \cap L_1$. Put $V = \langle K \cap L_1, g \rangle$. We choose a subgroup M of $\langle K \cap L_1, g \rangle$, which is maximal relative to the properties $K \cap L_1 \leq M$ and $g \notin M$. By this choice, M is a maximal subgroup of V. Since V is an $\tilde{N}$-group, M is normal in V. In particular, $M^g = M$ and $(K \cap L_1)^g \leq M$. If $h \in H$, then $[h, g] = h^{-1}hg \in K$, that is $[h, g] \in K \cap L_1$. We have now $h^g = h[h, g] \in H(K \cap L_1) = K$ for each element $h \in H$.

Let $y \in M$, $h \in H$. Consider the element $g^{-1}(h^{-1}yh)g = g^{-1}h^{-1}yhg$. Since $ghg^{-1}h^{-1} = b \in K \cap L_1$, $g^{-1}h^{-1} = h^{-1}g^{-1}b$. Similarly $hg = agh$ for some element $a \in K \cap L_1$. Now we have

$$g^{-1}h^{-1}yhg = h^{-1}g^{-1}byagh = h^{-1}(g^{-1}byag)h.$$

The inclusions $K \cap L_1 \leq M$ and $(K \cap L_1)g \leq M$ imply $g^{-1}h^{-1}yhg \in h^{-1}Mh$, that is, $g^{-1}(h^{-1}Mh)g = h^{-1}Mh$. Let $C = \bigcap_{h \in H} M^h$. By proved above we have $C^g = C$. The inclusion $K \cap L_1 \leq M$ implies $K \cap L_1 = (K \cap L_1)^h \leq M^h$ for each $h \in H$, so that $K \cap L_1 \leq C$. Furthermore,

$$H^g \leq K = H(K \cap L_1) \leq HC.$$

It follows that $(HC)^g = H^g C^g \leq HC$. In other words, $g \in N_G(HC)$.

Since

$$HC \cap L_1 = (H \cap L_1)C \leq (K \cap L_1)C = C,$$

and $C \leq M$, $g \notin HC$. On the other hand, by our conditions, HC is self-normalizing. This contradiction proves that $g \in \langle H, H^g \rangle$. Hence H is abnormal in HL_1.

Suppose that we have already proved that H is abnormal in $L_\alpha H$ for all $\alpha < \gamma$. Choose an arbitrary element $x \in G$ and consider the subgroup $\langle H, H^x \rangle$. First suppose that γ is a limit ordinal. Then there is an ordinal $\alpha < \gamma$ such that $x \in L_\alpha$. By the induction hypothesis, H is abnormal in $L_\alpha H$, so that $x \in \langle H, H^x \rangle$. Assume now that γ is not a limit ordinal. Put $W = L_{\alpha-1}$. If $x \in WH$, then all is proved. So we must consider the case when $x \notin WH$. Put $X/W = \langle HW/W, xW \rangle$. Choose in X/W a subgroup Y/W, which is maximal relative to the properties $HW/W \leq Y/W$ and $xW \notin Y/W$. By such choice, Y/W is a maximal subgroup of X/W. Since X/W is an $\tilde{N}$-group, Y/W is normal in X/W. Then $X \leq N_G(Y)$. Since $x \notin Y, Y \neq N_G(Y)$. The inclusion $H \leq Y$ implies a contradiction. This contradiction proves the inclusion $x \in WH$. This case has been already considered. □

We obtain the following corollaries.

Corollary 1.3 [43]. *Let G be a radical group and H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Corollary 1.4 [26]. *Let G be a hyperabelian group and H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Corollary 1.5. *Let G be a soluble group and H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

The following very natural question arises in this connection:

Does Theorem 1.2 valid for all groups?

Even for some simple finite groups this question has a negative answer. There is a corresponding counterexample in [2, § 7].

Let G be a group. A subgroup H is called *weakly abnormal in G* if $x \in H^{\langle x \rangle}$ for each element $x \in G$ [2].

Theorem 1.6 [2]. *Let G be a group and H be a subgroup of G. Then H is weakly abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Proof. Let H be a weakly abnormal subgroup of G and $K \geq H$. If $x \in N_G(K)$, then $K^x = K$, so that $K^{\langle x\rangle} = K$. The inclusion $x \in H^{\langle x\rangle}$ implies $x \in H^{\langle x\rangle} \leq K^{\langle x\rangle} = K$, and $N_G(K) = K$.

Suppose contrary. Let x be an arbitrary element of G. Put $L = H^{\langle x\rangle}$. We observe that $L = \langle x^{-n}Hxn \mid n \in \mathbb{N}\rangle$. The equation

$$x^{-1}(x^{-n}Hx^n)x = x^{-n-1}Hx^{n+1} \leq L$$

implies $x^{-1}Lx = L$. Since $N_G(L) = L$, $x \in L = H^{\langle x\rangle}$, and H is weakly abnormal. □

If H is weakly abnormal (respectively, abnormal) in G, and K is an intermediate subgroup for H, then H is weakly abnormal (respectively, abnormal) in K, and K is weakly abnormal (respectively, abnormal) in G. In particular, if $\mathfrak{S}$ be a family of weakly abnormal (respectively, abnormal) subgroup of G, then the subgroup, generated by all subgroups of the family $\mathfrak{S}$ is weakly abnormal (respectively abnormal) in G. But we cannot justify the same statement about a lattice of all weakly abnormal (respectively, abnormal) subgroups, because the intersection of two weakly abnormal (respectively, abnormal) subgroups is not necessary weakly abnormal (respectively, abnormal). Here is the following simple example.

Let $G = \mathbf{Sym}(4)$ be the symmetric group of degree 4. Consider the following subgroups of G:

$$H = \langle (12), (123)\rangle \text{ and } K = \langle (12), (1324)\rangle .$$

Clearly, $H = \mathbf{Sym}(3)$; in particular, H is maximal in G. Since H is not normal, H is abnormal in G. A subgroup K is a dihedral group of order 8, so it of index 3 in G, and hence K is a maximal subgroup of G. Clearly, K is not normal, so that K is abnormal in G. We have $D = H \cap K = \langle (12)\rangle$. Let $x = (13)(24)$, then $\mathrm{D}^x = \langle (34)\rangle$, and $L = D^{\langle x\rangle} = \langle D, D^x\rangle = \langle (12), (34)\rangle$. The subgroup D is normal in L, in particular, L is not self-normalizing. Theorem 1.1 shows that D is not abnormal in G, and Theorem 1.6 shows that D is not weakly pronormal in G.

Consider now the following chain of subgroups $D < H < G$. Clearly, D is maximal in H, but not normal, so that D is abnormal in H. A subgroup H is abnormal in G. However, D is not abnormal in G. Consequently, the property "to be an abnormal subgroup" is not transitive. However there exists the following form of transitivity of abnormality.

Proposition 1.7 (P. Hall). *Let G be a group and H be a normal subgroup of G. If a subgroup D is abnormal in DH and DH is abnormal in G,* *then D is abnormal in G.*

Proof. Let x be an arbitrary element of G. Since DH is an abnormal subgroup of G, $x \in \langle DH, (DH)^x \rangle = H \langle D, D^x \rangle$. Then $x = hy$ where $h \in H$, $y \in \langle D, D^x \rangle$. Since D is abnormal in DH, then $h \in H \le DH$ implies

$$h \in \left\langle D, D^h \right\rangle = \left\langle D, (xy^{-1})^{-1} D(xy^{-1}) \right\rangle \le \langle D, D^x, y \rangle = D, D^x.$$

It follows that $x = hy \in \langle D, D^x \rangle$, which means that D is abnormal in G. □

In connection with this, it is worth mentioning the following most general yet result on transitivity of abnormality.

Recall that a group G is called an *N-group or a group with the normalizer condition* if $H \neq N_G(H)$ for each subgroup H.

Theorem 1.8. (L.A. Kurdachenko, I.Ya. Subbotin [45]). *Let G be a group and H be a normal subgroup of G. Suppose that G/H has no proper abnormal subgroups and H satisfies the normalizer condition. Then abnormality is transitive in G.*

Following J.S.Rose [67], a subgroup H of a group G is called *contranormal* if $H^G = G$.

Abnormal subgroups are contranormal. However not every contranormal subgroup is abnormal. The following example justifies this.

Let P be a quasicyclic 2-group, that is

$$P = \left\langle a_n \mid a_1^2 = 1, a_{n+1}^2 = a_n, n \in \mathbb{N} \right\rangle .$$

Being abelian, P has an automorphism α such that $\alpha(a) = a^{-1}$ for all $a \in P$. Clearly, $|\alpha| = 2$. Consider a semi-direct product $G = P \leftthreetimes \langle d \rangle$ where $|d| = 2$ and $a^d = \alpha(a)$ for all $a \in P$. Then the series

$$\langle 1 \rangle < \langle a_1 \rangle > \langle a_2 \rangle < ... \langle a_n \rangle < ...P < G$$

is the upper central series of G. Being hypercentral, G satisfies the normalizer condition. Hence G has no proper abnormal subgroups. We have $d^{-1}a_n d = a_n^{-1}$, and this implies that $a_n^{-1}d^{-1}a_n d = a_n^{-2} = a_{n-1}^{-1}$. This equation shows that $\langle d \rangle^G = G$, so that $\langle d \rangle$ is a contranormal subgroup of G.

If H is an abnormal subgroup of a group G and K is a subgroup containing H, then H is abnormal in K, in particular, H is contranormal in K.

A subgroup H of a group G is said to be *nearly abnormal*, if H is contranormal in K for every subgroup K containing H.

From Theorem 1.2 we obtain

Corollary 1.9. *Let G be a hyper-$\tilde{N}$-group and H be a subgroup of G. Then H is nearly abnormal in G if and only if H is abnormal in G.*

Proof. If H is an abnormal subgroup of G, then,as we saw above, H is nearly abnormal. Suppose that H is nearly abnormal in G. Let K be an arbitrary subgroup containing H. Suppose that $N_G(K) = L \neq K$. Then K is normal in L. The inclusion $H \leq K$ implies that $H^L \leq K$, so that H is not contranormal in L. This contradiction shows that $N_G(K) = K$. Theorem 1.2 proves now that H is abnormal in G. $\square$

The Carter subgroups are an important subclass of abnormal subgroups. These subgroups have been introduced by R. Carter [14] as the self-normalizing nilpotent subgroups of a finite group. Some attempts of extending the definition of a Carter subgroup to infinite groups were made by S.E. Stonehewer [72, 73], A.D. Gardiner, B. Hartley and M.J. Tomkinson [23], and M.R. Dixon [20]. In [45], this concept have been extended to the class of nilpotent-by-hypercentral (not necessary periodic) groups.

We may define a Carter subgroup of a finite metanilpotent group as *a minimal abnormal subgroup.* The first logical step here is to consider the groups whose locally nilpotent residual is nilpotent.

Let G be a group, A be a normal subgroup of G. We say that A *satisfies the condition $Max - G$ (respectively, $Min - G$)*, if A satisfies the maximal (respectively, the minimal) condition for G-invariant subgroups.

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an *artinian-by-$\mathfrak{X}$-group* if G has a normal subgroup H such that $G/H \in \mathfrak{X}$ and H satisfies $Min - G$.

Let $\mathfrak{X}$ be a class of groups, G be a group and

$$R(\mathfrak{X}) = \{H \mid H \text{ is a normal subgroup of } G \text{ such that } G/H \in \mathfrak{X}\}.$$

Then the intersection $G_{\mathfrak{X}}$ of all normal subgroups of the family $R(\mathfrak{X})$ is called the $\mathfrak{X}$-*residual* of the group G.

We will deal with artinian-by-hypercentral groups whose locally nilpotent residuals are nilpotent. This is a natural first step. Since these groups are generalizations of finite metanilpotent groups, for the definition of the Carter subgroups in this class we may use some characterizations of these subgroups that are valid for finite metanilpotent groups.

Theorem 1.10 *(L.A. Kurdachenko, I.Ya. Subbotin [43]). Let G be an artinian-by-hypercentral group and suppose that its locally nilpotent residual K is nilpotent. Then G contains a minimal abnormal subgroup L. Moreover, L is a maximal hypercentral subgroup and it contains the upper hypercenter of G. In particular, $G = KL$. If H is another minimal abnormal subgroup, then H conjugates with L.*

Corollary 1.11 [43]. *Let G be an artinian-by-hypercentral group and suppose that its locally nilpotent residual K is nilpotent. Then G contains a hypercentral abnormal subgroup L. Moreover, L is a maximal hypercentral subgroup and it contains the upper hypercenter of G. In particular, $G = KL$. If H is another hypercentral abnormal subgroup, then H conjugates with L.*

Let G be an artinian-by-hypercentral group with a nilpotent hypercentral residual. A subgroup L is called *a Carter subgroup of a group G* if H is a hypercentral abnormal subgroup of G (or, equivalently, H is a minimal abnormal subgroup of G).

A Carter subgroup in finite soluble group can be defined as *a covering subgroup for the formation of nilpotent groups*. As we shall see, this characterization can be extended on the groups under consideration.

Recall that a subgroup H of a group G is said to be a $\mathfrak{LN}$-*covering subgroup* if H is locally nilpotent and if $S = HS_{\mathfrak{LN}}$ for every subgroup S that contains H. (Here $S_{\mathfrak{LN}}$ is the locally nilpotent residual of the subgroup S).

Theorem 1.12 (L.A. Kurdachenko, I.Ya. Subbotin [43]). L*et G be an artinian-by-hypercentral group and suppose that its locally nilpotent residual K is nilpotent. If L is a Carter subgroup of G, then L is a $\mathfrak{LN}$-covering subgroup of G. Conversely, if H is a $\mathfrak{LN}$-covering subgroup of G, then H is a Carter subgroup of G.*

In a finite soluble group, the $\mathfrak{N}$-covering subgroups are exactly $\mathfrak{N}$-projectors. Therefore a Carter subgroup of a finite soluble group can be defined as an $\mathfrak{N}$-projector. This characterization can be also extended on artinian-by-hypercentral groups.

Recall that a subgroup L of a group G is said to be *a locally nilpotent projector*, if LH/H is a maximal locally nilpotent subgroup of G/H for each normal subgroup H of a group G.

Theorem 1.13 (L.A. Kurdachenko, I.Ya. Subbotin [43]). *Let G be an artinian-by-hypercentral group and suppose that its locally nilpotent residual K is nilpotent. If L is a Carter subgroup of G, then L is a locally nilpotent* *projector of G. Conversely, if H is a locally nilpotent projector of G, then H is a Carter subgroup of G.*

For some restricted classes of infinite groups, the Carter subgroups could be defined more traditionally.

Let G be a group and C be a normal subgroup of G. Then C is said to be a *G-minimax* if C has a finite series of G-invariant subgroups whose infinite factors are abelian and either satisfy $Min - G$ or $Max - G$.

A group G is said to be *generalized minimax*, if G is G-minimax.

Every soluble minimax group is obviously generalized minimax. However, the class of generalized minimax groups is significantly wider than the class of soluble minimax groups.

Theorem 1.14 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [35]). *Let G be a periodic generalized minimax group and suppose that its locally nilpotent residual K is nilpotent. If L is a self-normalizing locally nilpotent subgroup of G, then L is a $\mathfrak{LN}$-covering subgroup of G. In other words, L is a Carter subgroup of G.*

Following [42] we shall call normal and abnormal subgroups **U**-normal (from "union" and "U-turn"). We observe that finite groups with only **U**-normal subgroups have been considered in [22]. The locally soluble (in the periodic case, locally graded) infinite groups with **U**-subgroups have been studied in [75]. In [42], the groups with all **U**-normal subgroups and the groups with transitivity of **U**-normality, have been completely described.

Next natural question is regarding the structure of the groups whose **U**-normal subgroups form a lattice. These groups are denoted as #**U**-groups [47]. It is easy to see that the groups with no abnormal subgroups are #**U**-groups.

Observe that a union of any two **U**-normal subgroups is **U**-normal. However, the similar assertion is obviously false for intersections.

Note that in a soluble group, an abnormal subgroup R is exactly a subgroup that is contranormal in all its intermediate subgroups [21]. The condition "every contranormal subgroup is abnormal" (the CA-property) is an amplification of the transitivity of abnormality (the TA-property). Some simple examples show that the class of TA-groups is wider than the class of CA-groups and does not coincide with the class of #**U**-groups. The description of soluble CA-groups having #**U**-property was obtained in [47].

2. Pronormal subgroups, their interplays and generalizations

The pronormality is an indirect union of abnormality and normality. Besides of normal and abnormal subgroups, Sylow p-subgroups and Hall π-subgroups of normal soluble subgroups of a finite group are other examples of pronormal subgroups.

As for abnormality, we shall consider a weak variant of pronormality

Let G be a group. A subgroup H is called *weakly pronormal* in G if the subgroups H and H^x conjugate in $H^{\langle x\rangle}$ for each element $x \in G$ [2].

The inclusion $\langle H, H^x\rangle \leq H^{\langle x\rangle}$ shows that *every pronormal subgroup is weakly pronormal*. The converse statement is not true – the correspondent example can be found in [2].

Let G be a group and H be a subgroup of G. We say that H has *the Frattini property* if for every subgroups K, L such that $H \leq K$ and K is normal in L we have $L = N_L(H)K$.

Theorem 2.1 [2]. *Let G be a group and H be a subgroup of G. Then H is weakly pronormal in G if and only if H has the Frattini property.*

Proof. Suppose first that H is a weakly pronormal subgroup of G. Let K, L be subgroups of G with the properties $K \geq H$ and $K \trianglelefteq L$. For each element $x \in L$, $H^x = H^u$ for some element $u \in H^{\langle x\rangle}$. It follows that $H = ux^{-1}Hxu^{-1}$. By the choice of the subgroups K and L we have $H^{\langle x\rangle} \leq K^L = K$. Thus $v = xu^{-1} \in N_L(H)$, or $x = vu \in N_L(H)K$, which implies that $L = N_L(H)K$.

Conversely, assume that H has the Frattini property. If x is an arbitrary element of G, then put $K = H^{\langle x\rangle}$ and $L = \langle H, x\rangle$. By such choice, K is normal in L, so that $L = N_L(H)K$. It follows that $x = yz$ where $y \in N_L(H)$ and $z \in K$. Then $H^x = H^{yz} = H^z$ what proves that H is weakly pronormal. □

Corollary 2.2. *Let G be a group and H be a pronormal subgroup of G. Then H has the Frattini property.*

Corollary 2.3. *Let G be a group and H be a weakly pronormal subgroup of G. Then H is weakly abnormal in G if and only if $H = N_G(H)$.*

Proof. If H is weakly abnormal subgroup of G, then $H = N_G(H)$ by Theorem 1.6. Suppose that H is weakly pronormal and self-normalizing. For each element $x \in G$ we have $H^x = H^u$ for some element $u \in H^{\langle x\rangle}$. It follows that $H = ux^{-1}Hxu^{-1}$, and therefore, $v = xu^{-1} \in N_G(H) = H$. Hence

$$x = vu \in \left\langle H, H^{\langle x \rangle} \right\rangle = H^{\langle x \rangle}$$

which shows that H is weakly abnormal in G. □

The following result clarifies relationships between weakly pronormal and pronormal subgroups.

Theorem 2.4. [2]. Let G be a group and H be a subgroup of G. Then H is pronormal in G if and only if the following conditions hold:

(i) H is weakly pronormal;

(ii) if L is a intermediate subgroup for H and g is an element of G such that $H \leq L^g$, then there exists an element $x \in N_G(H)$ with the property $\mathrm{L}^x = \mathrm{L}^g$

*Proof***.** Suppose first that H is pronormal subgroup of G. Corollary 2.2 shows that H satisfies (i). Let g be an element of G such that $L^g \geq H$. It follows that $gHg^{-1} \leq L$, so that and $\left\langle H, gHg^{-1} \right\rangle \leq L$. Since H is pronormal, there exists an element $y \in \left\langle H, gHg^{-1} \right\rangle$ such that $y^{-1}Hy = gHg^{-1}$. It follows that $x = y^g \in N_G(H)$. Observe that $y \in L$, and therefore $L^x = L^{yg} = L^g$.

Conversely, suppose that H satisfies the conditions (i) and (ii). Let g be an arbitrary element of G and put $K = \langle H, H^g \rangle$. Then $gKg^{-1} = \left\langle gHg^{-1}, H \right\rangle \geq H$. By (ii), there exists an element $x \in N_G(H)$ such that $x^{-1}Kx = gKg^{-1}$. It follows that $xg \in N_G(K)$, thus K is normal in the subgroup $\langle K, xg \rangle = L$. By (i), $L \leq N_L(H)K$, in particular, $xg = vy$ where $v \in N_L(H), y \in K$. Hence

$$H^g = H^{xg} = H^{vy} = H^y.$$

And finally just recall that $y \in K = \langle H, H^g \rangle$. □

Corollary 2.5. *Let G be a group and H be a pronormal subgroup of G. Then H is abnormal in G if and only if $H = N_G(H)$.*

Proposition 2.6. *Let G be a group and H be a subgroup of G. If H is pronormal in G, then $N_G(H)$ is abnormal in G.*

Proof. Let g be an arbitrary element of G and $L = N_G(H)$. Since H is pronormal, then there is an element $x \in \langle H, H^g \rangle$ such that $H^g = H^x$. It follows that $gx^{-1} = u \in N_G(H) = L$. Thus $g = ux \in \langle L, \langle H, H^g \rangle \rangle \leq \langle L, L^g \rangle$. □

For finite soluble groups, T.A. Peng obtained the following characterization of pronormal subgroups.

Theorem 2.7 (T. A. Peng [61]). *Let G be a finite soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

This characterization of pronormal subgroups could be extended on infinite groups in the following way.

Theorem 2.8 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [35]). *Let G be a hyper-N-group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

The following results are direct amplification of this theorem.

Corollary 2.9 (F. de Giovanni, G. Vincenzi [26]). *Let G be a hyperabelian group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

Corollary 2.10 *Let G be a soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

If H is a pronormal subgroup of a group G and L is an intermediate subgroup for H, then H is pronormal in L. So Proposition 2.6 shows that $N_L(H)$ is abnormal in L. We observe that every abnormal subgroup is contranormal.

A subgroup H of a group G is called *nearly pronormal* if $N_L(H)$ is contranormal in L for every intermediate for H subgroup L.

As we can see, every pronormal subgroup is nearly pronormal but the converse statement is not true.

Let G be a special unitary group of 3×3 matrices over the field $\mathbb{F}_9$ of order 9. This group is simple and its order is 6048. The multiplicative group $\mathbf{U}(\mathbb{F}_9)$ is cyclic. Let g be an element such that $\langle g \rangle = \mathbf{U}(\mathbb{F}_9)$. Let K be a subgroup generated by the following matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} g & g^2 & g^5 \\ g^5 & 0 & g^5 \\ g & g^6 & g^5 \end{pmatrix}, \begin{pmatrix} 1 & g^2 & 1 \\ g^2 & 0 & g^6 \\ 1 & g^6 & 1 \end{pmatrix}.$$

This subgroup is nearly pronormal, but not pronormal. The order of K is 24. This group is soluble, but not nilpotent. We observe that K is isomorphic to $\mathbf{Sym}(4)$.

Nevertheless, for some classes of generalized soluble groups the nearly pronormallity coincides with pronormality.

Proposition 2.11 (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [48]). *Let G be a group having an ascending series whose factors are abelian. Then every nearly pronormal subgroup of G is weakly pronormal in G.*

Theorem 2.12 (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin[48]). *Let G be a hyper-N-group. Then every nearly pronormal subgroup of G is pronormal in G.*

Corollary 2.13 [48]. *Let G be a soluble group. Then every nearly pronormal subgroup of G is pronormal in G.*

Corollary 2.14. *Let G be a soluble group. Suppose that a subgroup H satisfies the following condition:*

If K is a subgroup containing H, then $N_K(H)$ is abnormal in K.

Then K is pronormal in G.

Corollary 2.15. (Wood G.J. [77]). *Let G be a finite soluble group. Suppose that a subgroup H satisfies the following condition:*

if K is a subgroup containing H, then $N_K(H)$ is abnormal in K.

Then K is pronormal in G.

Remark also that for generalized pronormal subgroups the class of an $\tilde{N}$-group plays a very special role. The following Proposition justifies this.

Proposition 2.16. *Let G be an $\tilde{N}$-group and H be a nearly pronormal subgroup of G. Then H is normal in G.*

Proof. Suppose the contrary. Then $N_G(H) \neq G$. In this case, there exists an element $x \notin N_G(H)$. Put $L = \langle x, N_G(H)\rangle$, and pick in L a subgroup M which is maximal relative to the properties $N_G(H) \leq M$ and $x \notin M$. By such choice, M is a maximal subgroup of L. Since G is an $\tilde{N}$-group, M is normal in L. Since H is nearly pronormal in G, $N_L(H)$ is contranormal in L, that is $(N_L(H))^L = L$. On the other hand, the inclusion $N_G(H) \leq M$ and the fact that M is normal in L imply that $(N_L(H))^L \leq M \neq L$. This contradiction shows that $N_G(H) = G$, i.e. H is normal in G. □

Corollary 2.17. *Let G be a locally nilpotent group and H be a nearly pronormal subgroup of G. Then H is normal in G.*

Proposition 2.18. *Let G be an $\tilde{N}$-group and H be a weakly pronormal subgroup of G. Then H is normal in G.*

Proof. Suppose the contrary. Then $N_G(H) \neq G$. In this case, there exists an element $x \notin N_G(H)$. Put $L = \langle x, N_G(H)\rangle$, and choose in L a subgroup M which is maximal relative to the properties $N_G(H) \geq M$ and $x \notin M$. By such choice, M is a maximal subgroup of L. Since G is an $\tilde{N}$-group,

M is normal in L. The inclusion $H \leq M$ and the fact that H is weakly pronormal imply the equality $L = MN_L(H) = MN_G(H)$. On the other hand, $N_G(H) \leq M$, so that $L = M$, and we obtain a contradiction. This contradiction shows that $N_G(H) = G$, i.e. H is normal in G. $\square$

Corollary 2.19. *Let G be a locally nilpotent group and H be a weakly pronormal subgroup of G. Then H is normal in G.*

Corollary 2.20. Let G be an $\tilde{N}$-group and H be a pronormal subgroup of G. Then H is normal in G.

Corollary 2.21 (N.F. Kuzennyi, I.Ya Subbotin [52]). *Let G be a locally nilpotent group and H be a pronormal subgroup of G. Then H is normal in G.*

T.A. Peng has considered finite groups whose all subgroups are pronormal. He proved it to be the groups in which the relation"to be normal subgroup" is transitive.

A group G is said to be *a T-group* if every subnormal subgroup of G is normal. A group G is said to be *a $\bar{T}$-group*, if every subgroup of G is a T-group.

It should be noted that T-groups have been investigating for a long period of time (see e.g. [1, 8, 24, 31, 30, 63, SK1988]). The structure of finite soluble T-groups has been described by W. Gaschütz [24]. In particular, he proved that every finite soluble T-group is a $\bar{T}$-group. Observe that a finite T-group is metabelian. The infinite soluble T-groups have been studied by D.J.S. Robinson [63]. A locally soluble $\bar{T}$-group has the following structure.

Theorem 2.22 (D.J.S. Robinson [63]). *Let G be a locally soluble $\bar{T}$-group.*

(i) If G is not periodic, then G is abelian.

(ii) If G is periodic and L is the locally nilpotent residual of G, then G satisfies the following conditions:

(a) G/L is a Dedekind group;

(b) $\Pi(L) \cap \Pi(G/L) = \varnothing$;

(c) $2 \notin \Pi(L)$;

(d) every subgroup of L is G-invariant.

In particular, if L is non-identity, then $L = [L, G]$.

Note that in general case, the locally nilpotent residual has no complement. In the paper [28], an related well-known sophisticated construction has been developed. This construction, in particular, allows us to develop some examples of periodic groups that are non-splitting extensions of its abelian Hall derived subgroup by an uncountable elementary abelian 2-group.

T.A. Peng proved the following result for finite soluble groups.

Theorem 2.23 (T.A. Peng [60]). *Let G be a finite soluble group. Then every subgroup of G is pronormal is and only if G is a T-group.*

As the following theorem shows, the infinite case is much more complicated.

Recall that a group is called *locally graded*, if every its non-identity finitely generated subgroup has a proper subgroup of finite index.

Theorem 2.24 (N.F. Kuzennyi and I.Ya. Subbotin [51]). *Let G be a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent:*

(i) every cyclic subgroup of G is pronormal in G;

(ii) G is a soluble $\bar{T}$-group.

The infinite groups whose subgroups are pronormal firstly have been considered in [50]. The authors completely described such infinite locally soluble non-periodic and infinite locally graded periodic groups. The main result of that paper is the following appealing theorem.

Theorem 2.25 (N.F. Kuzennyi and I.Ya. Subbotin [50]). *Let G be a group whose subgroups are pronormal, and L be a locally nilpotent residual of G.*

(i) If G is periodic and locally graded, then G is a soluble $\bar{T}$-group in which L is a complement to every Sylow $\Pi(G/L)$-subgroup.

(ii) If G is not periodic and locally soluble, then G is abelian.

Conversely, if G has this structure (i) – (ii), then every subgroup of G is pronormal in G.

In the paper [65], the assertion (ii) has been extended to non-periodic locally graded groups proving that in this case such groups still to be abelian.

N.F. Kuzennyi and I. Ya. Subbotin have also completely described the locally graded periodic groups in which all primary subgroups are pronormal [53] and the infinite locally soluble groups in which all infinite subgroups are pronormal [51]. They proved that in the infinite case, the class of groups whose all subgroups are pronormal is a proper subclass of the class of groups with the transitivity of normality. Moreover, it is also a proper subclass of the class of groups whose primary subgroups are pronormal. However, the pronormality for all subgroups can be weakened to the pronormality for only abelian subgroups [54].

In the paper [39], the groups whose subgroups are nearly pronormal have been considered.

Theorem 2.26 (L.A. Kurdachenko, A. Russo, G. Vincenzi [39]). Let G be a locally radical group.

(i) If every cyclic subgroup of G is nearly pronormal, then G is a $\bar{T}$-group.

(ii) If every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.

If G is a finite group, then for each subgroup H there is a chain of subgroups

$$H = H_0 \leq H_1 \leq ... \leq H_{n-1} \leq H_n = G$$

such that H_j is maximal in H_{j+1}, $0 \leq j \leq n-1$. Generalizing this, J. Rose has introduced *the balanced chain connecting a subgroup H to a group G*, that is, the chain of subgroups

$$H = H_0 \leq H_1 \leq ... \leq H_{n-1} \leq H_n = G$$

such that for each $j, 0 \leq j \leq n-1$, either H_j is normal in H_{j+1}, or H_j is abnormal in H_{j+1}. The number n is *the length* of this chain. He refers appropriately to two consecutive subgroups $H_j \leq H_{j+1}$ as forming *a normal link* or respectively *an abnormal link* of this chain [66]. In finite groups, every subgroup can be connected to a group by some balanced chain.

It is natural to consider the case when all of these balanced chains are quite short, i.e. their lengths are bounded by small numbers. If these lengths are 1, then every subgroup is either normal or abnormal in a group. Such finite groups were studied in [22]. Infinite groups of this kind and some their generalizations were described in [75] and [21]. Moreover, in the last paper, the groups whose subgroups are either abnormal or subnormal have been considered. More general situation was considered in the paper of L.A. Kurdachenko and H. Smith [41]. They investigated the groups, whose subgroups are either self-normalizing or subnormal.

Observe that in the groups in which a normalizer of any subgroup is abnormal, and in the groups in which every subgroup is abnormal in its normal closure. the mentioned lengths are at most 2. It is logical to choose these groups as the subjects of investigation.

It is interesting to observe that if G is a soluble $\bar{T}$-group, then every subgroup of G is abnormal in its normal closure. As we mentioned above, for any pronormal subgroup H of a group G, the normalizer $N_G(H)$ is an abnormal subgroup of G. So a subgroup having abnormal normalizers is a generalization of a pronormal subgroup. There are examples showing that this generalization is non-trivial.

The article [40] initiated the study of groups whose subgroups are connected to a group by balanced chains of length at most 2. As we recently mentioned, such groups are naturally related to the T-groups.

Theorem 2.27 (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [40]).

(i) Let G be a radical group. Then G is a $\bar{T}$-group if and only if every cyclic subgroup of G is abnormal in its normal closure.

(ii) Let G be a periodic soluble group. Then G is a $\bar{T}$-group if and only if its locally nilpotent residual L is abelian and the normalizer of each cyclic subgroup of G is abnormal in G.

The following result from [40] provides us with the following new interesting and useful characterization of groups with all pronormal subgroups.

Theorem 2.28 (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [40]). *Let G be a periodic soluble group. Then every subgroup is pronormal in G if and only if its locally nilpotent residual L is abelian and a normalizer of every subgroup of G is abnormal in G.*

For the non-periodic case, there exist non-periodic non-abelian groups in which normalizers of all subgroups are abnormal [40]. On the other hand, the non-periodic locally soluble groups in which all subgroups are pronormal are abelian [50]. So, in the non-periodic case, we cannot count on a similar to above characterization. However, we have the following result.

Theorem 2.29 (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [40]). *Let G be a non-periodic group with the abelian locally nilpotent residual L. If a normalizer of every cyclic subgroup is abnormal and for each prime $p \in \Pi(L)$ the Sylow p-subgroup of L is bounded, then G is abelian.*

3. Generalized normality and criteria of generalized nilpotency

The following well-known characterizations of finite nilpotent groups are tightly connected to abnormality and pronormality.

A finite group G is nilpotent if and only if G has no proper abnormal subgroups.

A finite group G is nilpotent if and only if its every pronormal subgroup is normal.

Note that since a normalizer of a pronormal subgroup is abnormal, *the absence of abnormal subgroups is equivalent to the normality of all pronormal subgroups.*

The mentioned above results 2.18-2.21 can be reformulated in the following way.

Let G be an $\tilde{N}$-group, Then G has no proper abnormal subgroups.

Let G be a locally nilpotent group, then G has no proper abnormal subgroups.

There exists an example of an $\tilde{N}$-group which is not locally nilpotent [78]. It follows that the absence of abnormal subgroups does not need necessary imply the locally nilpotency of a group. Therefore the following question is natural:

I*n what groups the absences of abnormal subgroups is equivalent to locally nilpotency?*

In other words, it would be interesting to obtain some criteria of nilpotency in terms of abnormality and pronormality. In the paper [34], the first such criterion was obtained.

Theorem 3.1 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [34]). *Let G be a soluble generalized minimax group. If every pronormal subgroup of G is normal (or, what is equivalent, G has no proper abnormal subgroups), then G is hypercentral.*

Let G be a group. Then the set

$$\mathbf{FC}(G) = \{x \in G \mid x^G \text{ is finite}\}$$

is a characteristic subgroup of G which is called the *FC-center* of G. A group G is an FC-group if and only if $G = \mathbf{FC}(G)$. Starting from the FC-center, we construct the upper FC-central series of a group G

$$\langle 1 \rangle = C_0 \leq C_1 \leq ...C_\alpha \leq C_{\alpha+1} \leq ...C_\gamma$$

where $C_1 = \mathbf{FC}(G), C_{\alpha+1}/C_\alpha = \mathbf{FC}(G/C_\alpha)$ for all $\alpha < \gamma$, and $\mathbf{FC}(G/C_\alpha) = \langle 1 \rangle$.

The term C_α is called the *$\alpha - FC$-hypercenter* of G, while the last term C_γ of this series is called *the upper FC-hypercenter* of G. If $C_\gamma = G$, then the group G is called *FC-hypercentral*, and, if γ is finite, then G is called *FC-nilpotent.*

The following criteria of hypercentrality have been obtained in [38].

Theorem 3.2 (L.A. Kurdachenko, A. Russo, G. Vincenzi [38]). *Let G be a group whose pronormal subgroups are normal. Then every FC-hypercenter of G having finite number is hypercentral.*

Let G be an FC-nilpotent group. If all pronormal subgroups of G are normal, then G is hypercentral.

Let G be a group whose pronormal subgroups are normal. Suppose that H is an FC-hypercenter of G having finite number. If C is a normal subgroup of G such that $C \geq H$ and C/H is hypercentral, then C is a hypercentral.

For periodic groups, the above results were obtained in [44].

Observe that the abnormal subgroups are an important particular case of the contranormal subgroups: *in the soluble groups, the abnormal subgroups are exactly the subgroups that are contranormal in each subgroup containing them.* On the other hand, the abnormal subgroups are a particular case of pronormal subgroups.

Pronormal subgroups are connected to contranormal subgroups in the following way. If H is a pronormal subgroup of a group G and $H \leq K$, then $N_K(H)$ is an abnormal and hence contranormal subgroup of K.

Starting from the normal closure of H we can construct *the normal closure series of H in G*

$$H^G = H_0 \leq H_1 \leq ... \, H_\alpha \leq H_{\alpha+1} \leq ... H_\gamma$$

by the following rule: $H_{\alpha+1} = H^{H_\alpha}$ for every $\alpha < \gamma$, and $H_\lambda = \bigcap\limits_{\mu<\lambda} H_\mu$ for a limit ordinal λ. The term H_α of this series is called *the α-th normal closure of H in G* and will be further denoted by $H^{G,\alpha}$. The last term H_γ of this series is called *the lower normal closure of H in G* and will be denoted by $H^{G,\infty}$. Observe that every subgroup H is contranormal in its lower normal closure.

The subgroup H of a group G is called descendant (in G), if H coincides with its lower normal closure $H^{G,\infty}$. An important particular case of descendant subgroups are subnormal subgroups. *A subnormal subgroup is exactly a descending subgroup having finite normal closure series.* These subgroups strongly affect structure of a group. For example, it is not hard to prove that if every subgroup of a locally (soluble-by-finite) group is descendant, then this group is locally nilpotent. If every subgroup of a group G is subnormal, then, by a remarkable result due to W. Möhres [56], G is soluble. Subnormal subgroups have been studied very thoroughly for quite a long period of time. We are not going to consider this topic here since it has been excellently presented in the survey of C. Casolo [15]. However, we need to admit that, with the exception of subnormal subgroups, we have no significant information regarding descendant subgroups. The next results connect the conditions of generalized nilpotency to descendant subgroups.

Theorem 3.3 (L.A. Kurdachenko, I.Ya. Subbotin [44]). *Let G be a group every subgroup of which is descendant. If G is FC-hypercentral, then G is hypercentral.*

Theorem 3.4 (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [35]). *Let G be a soluble generalized minimax group. Then every subgroup of G is descendant if and only if G is nilpotent.*

If every subgroup of a group G is descendant, then G does not include proper contranormal subgroups. The study of groups without contranormal subgroups is a next logical step. We observe that *every non-normal maximal subgroup of an arbitrary group is contranormal*. Since a finite group whose maximal subgroups are normal is nilpotent, we come to the following criterion of nilpotency of finite groups in terms of contranormal subgroups:

A finite group G is nilpotent if and only if G does not contain proper contranormal subgroups.

The question on existence of similar criterion for infinite groups is very natural. However, in general, the absence of contranormal subgroups does not imply nilpotency. In fact, there exist non-nilpotent groups all subgroups of which are subnormal. The first such example has been constructed by H. Heineken and I.J. Mohamed [32]. Nevertheless, for some classes of infinite groups the absence of contranormal subgroups does imply nilpotency of a group. The groups without proper contranormal subgroups have been considered in papers [36, 37]. We present the main results of these articles here.

Theorem 3.5. *Let G be a group and H be a normal soluble-by-finite subgroup such that G/H is nilpotent. Suppose that H satisfies $Min-G$. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if a soluble-by-finite group G without proper contranormal subgroups satisfies minimal condition for normal subgroups, then G is nilpotent.*

Theorem 3.6. *Let G be a group and H be a normal Chernikov subgroup such that G/H is nilpotent. If G has no proper contranormal subgroups, then G is nilpotent. In particular, a Chernikov group without proper contranormal subgroups is nilpotent.*

Theorem 3.7. *Let G be a group and C be a normal subgroup of G such that G/C is nilpotent. Suppose that C has a finite series of G-invariant subgroups*

$$\langle 1\rangle = C_0 \le C_1 \le ... \le C_n = C$$

whose factors $C_j/C_{j-1}, 1 \le j \le n$, satisfy one of the following conditions:

(i) C_j/C_{j-1} is finite;

(ii) C_j/C_{j-1} is hyperabelian and minimax;

(iii) C_j/C_{j-1} is hyperabelian and finitely generated;

(iv) C_j/C_{j-1} is abelian and satisfies $Min - G$.

If G has no proper contranormal subgroups, then G is nilpotent.

Let G be a group and let A be an infinite normal abelian subgroup of G. We say that A is a *G-quasifinite subgroup*, if every proper G-invariant subgroup of A is finite. This means that either A contains a proper finite G-invariant subgroup B such that A/B is G-simple, or A is a union of all finite proper G-invariant subgroups.

Theorem 3.8. *Suppose that a group G contains a normal subgroup C such that G/C is nilpotent. Suppose that C has a finite series of G-invariant subgroups*

$$\langle 1\rangle = C_0 \le C_1 \le ... \le C_n = C$$

every factor $C_j/C_{j-1}, 1 \le j \le n$, of which satisfies one of the following conditions:

(i) C_j/C_{j-1} is finite;

(ii) C_j/C_{j-1} is hyperabelian and minimax;

(iii) C_j/C_{j-1} is hyperabelian and finitely generated;

(iv) C_j/C_{j-1} is abelian and G-quasifinite.

If G has no proper contranormal subgroups, then G is nilpotent.

The following useful assertions are almost direct corollaries of this theorem.

Theorem 3.9. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C has a finite G-chief series. If G has no proper contranormal subgroups, then G is nilpotent.*

Theorem 3.10. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian* *minimax subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is hyperabelian minimax group without proper contranormal subgroups, then G is nilpotent.*

Theorem 3.11. Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a Chernikov subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is a Chernikov group without proper contranormal subgroups, then G is nilpotent.

Theorem 3.12. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian finitely generated subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is a hyperabelian finitely generated group without proper contranormal subgroups, then G is nilpotent.*

Theorem 3.13. *Suppose that the group G contains a normal G-minimax subgroup C such that G/C is a nilpotent group of finite section rank. If G has no proper contranormal subgroups, then G is nilpotent.*

4. Groups with transitivity of some generalized normal properties

We mentioned already some important results on transitivity of normality. Transitivity of such important subgroup properties as pronormality, abnormality and other related to them properties have been studied by L.A. Kurdachenko, I.Ya. Subbotin, and J.Otal (see, [43, 35]).

The groups in *which pronormality is transitive, are called TP-groups,* and the groups in which *all subgroups are TP-groups, are called $\bar{T}P$-groups.* For the $\bar{T}P$-groups the following description has been obtained.

Theorem 4.1 (L.A. Kurdachenko, I.Ya. Subbotin [43]). *Let G be a locally soluble group. Then G is a $\bar{T}P$-group if and only if G is a $\bar{T}$-group.*

Theorem 4.2 (L.A. Kurdachenko, I.Ya. Subbotin [43]). *Let G be a periodic soluble group. Then G is a TP-group if and only if $G = A \leftthreetimes (B \times P)$ where*

(i) A, B are abelian $2'$-subgroup, and P is a 2-subgroup (if P is non-identity);

(ii) $\Pi(A) \cap \Pi(B) = \varnothing$;

(iii) P is a T-group;

(iv) $[G, G] = A \times [P, P]$;

(v) every subgroup of $[G, G]$ is G-invariant;

(vi) A is a complement to every Sylow $\Pi(B \times P)$-subgroup of G.

In [42], the authors were able to list all types of periodic soluble TP-groups.

The following theorem completes the description of soluble TP-groups.

Theorem 4.3 (L.A. Kurdachenko, I.Ya. Subbotin [43]). *Let G be a non-periodic soluble group.*

(i) If $C_G([G,G])$ is non-periodic, then G is a TP-group if and only if G is a T-group.

(ii) If $C_G([G,G])$ is periodic, then G is a TP-group if and only if G is a hypercentral T-group.

Recall the following interesting property of pronormal subgroups:

Let G be a group, H, K be the subgroups of G and $H \le K$. If H is a subnormal and pronormal subgroup of K, then H is normal in K.

We say that a subgroup H of a group G is *transitively normal* if H is normal in every subgroup $K \ge H$ in which H is subnormal [46]. In [59], these subgroups have been introduced under a different name. Namely, a subgroup H of a group G is said to *satisfy the subnormalizer condition* in G if for every subgroup K such that H is normal in K we have $N_G(K) \le N_G(H)$.

We say that a subgroup H of a group G is *strong transitively normal*, if HA/A is transitively normal for every normal subgroup A of the group G [46]. Since a homomorphic image of a pronormal subgroup is pronormal, we can conclude that every pronormal subgroup is a strong transitively normal subgroup.

Theorem 4.4 (L.A. Kurdachenko, I.Ya. Subbotin [46]). *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal soluble subgroup R such that G/R is hypercentral. If H is strong transitively normal in G and R satisfies $Min-H$, then H is a pronormal subgroup of G.*

As direct corollaries we can mention the following results from [46].

Corollary 4.5 [46]. *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal soluble Chernikov subgroup R such that G/R is hypercentral. If H is strong transitively normal in G, then H is a pronormal subgroup of G. In particular, if G is a soluble Chernikov group and H is a hypercentral strong transitively normal subgroup of G, then H is pronormal in G.*

A subgroup H is said to be *polynormal* in a group G, if for each intermediate for H subgroup S the subgroup H is contranormal in H^S ([2]).

Corollary 4.6 [46]. *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal soluble subgroup R such that G/R is hypercentral. If H is a polynormal in G and R satisfies $Min - H$ (in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

Corollary 4.7 [59]. *Let G be a soluble finite group and H be a nilpotent subgroup of G. If H is a polynormal in G, then H is a pronormal subgroup of G.*

A subgroup H is said to be *paranormal* in a group G if H is contranormal in $\langle H, H^g \rangle$ for all elements $g \in G$ [2]). *Every pronormal subgroup is paranormal, and every paranormal subgroup is polynormal* [2].

Corollary 4.8 [46]. *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal soluble subgroup R such that G/R is hypercentral. If H is a paranormal in G and R satisfies $Min - H$ (in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

As a corollary we obtain

Corollary 4.9. *Let G be a soluble finite group and H be a nilpotent subgroup of G. If H is a paranormal in G, then H is a pronormal subgroup of G.*

In [61] the following criterion of pronormality of a nilpotent subgroup in a finite group has been established.

Theorem 4.10 (T.A. Peng [61]). *Let G be a finite nilpotent-by-abelian group and H be a nilpotent subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

The article [46] contains the following useful strong generalization of this criterion to some infinite cases.

Theorem 4.11 (L.A. Kurdachenko, I.Ya. Subbotin [46]). *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal nilpotent subgroup R such that G/R is hypercentral. If H is transitively normal in G and R satisfies $Min - H$ (in particular, if R is a Chernikov subgroup), then H is a pronormal subgroup of G.*

As a corollary we obtain

Corollary 4.12 [46]. *Let G be a nilpotent-by-hypercentral Chernikov group and H be a hypercentral subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

A subgroup H of a group G is called *weakly normal* if $H^g \leq N_G(H)$ implies $g \in N_G(H)$ (K.H. Müller [58]). We note that every pronormal subgroup is weakly normal [4], every weakly normal subgroup satisfies the subnormalizer condition [4], and hence it is transitively normal in a group. Thus from above result we obtain

Corollary 4.13 [46]. *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal nilpotent subgroup R such that G/R is hypercentral. If H is weakly normal in G and R satisfies $Min-H$ (in particular, if R is a Chernikov subgroup), then H is a pronormal subgroup of G.*

A subgroup H of a group G is called an $\mathfrak{H}$*-subgroup* if $N_G(H) \cap H^g \leq H$ for all elements $g \in G$ [9]. Note that every $\mathfrak{H}$-subgroup is transitively normal [9]. Therefore, from the above result we obtain

Corollary 4.14 [46]. *Let G be a group and H be a hypercentral subgroup of G. Suppose that G contains a normal nilpotent subgroup R such that G/R is hypercentral. If H is an $\mathfrak{H}$-subgroup of G and R satisfies $Min-H$ (in particular, if R is a Chernikov subgroup), then H is a pronormal subgroup of G.*

Some properties of transitively normal subgroups (under another name) have been considered in the paper [27], which, in particular, contains the following result.

Theorem 4.15 (F. de Giovanni, G. Vincenzi [27]). *Let G be an FC-group and H be a transitively normal subgroup of G. If H is a p-subgroup for some prime p, then H is a pronormal subgroup of G.*

The pronormal subgroups play very important role in a following interesting class of groups connected to the following essential generalization of normal subgroups. A subgroup H of a group G is called *permutable* in G, if $HK = KH$ for every subgroup K of G.

Investigation of permutable subgroups begun rather long time ago (see, for example the book [71]). In particular, the groups (finite and infinite) whose every subgroup is permutable have been described (see, for example, [71, 2.4]). According to a well-known theorem of Stonehewer ([74, Theorem A]) permutable subgroups are always ascendant. In this connection, it is natural to consider the opposite situation, namely, the groups whose ascendant subgroups are permutable. A group G is said to be an *AP-group* if every ascendant subgroup of G is permutable in G. These groups are quite close to the groups in which the relation "to be a permutable subgroup" is transitive. A group G is said to be a *PT-group* if *permutability is a transitive relation in* G, that is, if K is a permutable subgroup of H and H is a permutable subgroup of G, then K is a permutable subgroup of G. *If G is a finite group, then G is a PT-group if and only if every subnormal subgroup is permutable.* The study of finite PT-groups has been initiated in the paper of G. Zacher [79]. He determined the structure of finite soluble PT-groups in a manner corresponding to W. Gaschütz's [24] characterization of finite soluble groups, in which normality is a transitive relation.

Theorem 4.15 (G. Zacher [79]). *Let G be a finite soluble group and L be a nilpotent residual of G. Then G is a AP-group (and hence a PT-group) if and only if it satisfies the following conditions:*
(a) every subgroup of G/L is permutable;
(b) L is abelian Hall subgroup of G;
(c) $2 \notin \Pi(L)$;
(d) every subgroup of L is G-invariant.

Consider now infinite AP-groups. A paper [64, Lemma 4] contains the following result: *every ascendant subgroup of an arbitrary PT-group is permutable.* But this result is incorrect. The following example shows this. Let $G = A \leftthreetimes \langle b \rangle$ where A is a Prüfer 2-group (that is $A = \langle a_n \mid a_1^2 = 1, a_{n+1}^2 = a_n, n \in \mathbb{N} \rangle$), $|b| = 2$ and $ab = a^{-1}$ for each $a \in A$. If $x = ab$ for some $a \in A$, then $x^2 = abab = aa^{-1} = 1$. Let H be a proper subgroup of G. If A contains H, then H is G-invariant. Suppose that A does not include H. Then H is finite and $H = K \langle x \rangle$ where K is a proper subgroup of A, $x \in A$ and hence $|x| = 2$. Every subgroup of A is cyclic, so $K = \langle c \rangle$ where $c \in A$. Let $|c| = 2^m$ for some $m \in \mathbb{N}$. Then $|H| = 2^m \cdot 2 = 2^{m+1}$. Since the subgroup A is divisible, we may choose an element $d \in A$ such that $d^t = c$ where $t = 2^5$. Put $y = dx$. Clearly $y \notin A$, thus $|y| = 2$. Suppose that $H \langle y \rangle = \langle y \rangle H$. Then $|H \langle y \rangle| = 2^{m+1} \cdot 2 = 2^{m+2}$. On the other hand, $(dx)x \in \langle H, y \rangle$ and $(dx)x = d$, but $|d| = 2^{m+5}$. This shows that H can not be permutable in G. In other words, if H is a permutable subgroup of G, then $H \leq L$. But in this case, every permutable subgroup of G is G-invariant. It follows that G is a PT-group. A subgroup $\langle b \rangle$ is ascendant in G, but it is not permutable in G.

This example shows that for infinite groups the classes of AP-groups and PT-groups do not coincide. Infinite AP-group have been studied in the paper [5]. Obviously, groups with no permutable subgroups are AP-groups, but no sense in study of properties of permutable subgroups in groups with no permutable subgroups. This justifies the necessity of imposing some restrictions on the group in this study. The natural framework for considering AP-groups are the classes of groups that have many ascendant subgroups. As the first step we consider radical hyperfinite groups.

Theorem 4.16 [5]. *Let G be a radical hyperfinite group and L be a locally nilpotent residual of G. Then the following conditions hold:*

(i) L is abelian;

(ii) if R is the locally nilpotent radical of G, then $R = L \times Z$ where Z is the upper hypercenter of G;

(iii) $\Pi(L) \cap \Pi(G/L) = \varnothing$;

(iv) every subgroup of L is G-invariant;

(v) G/L is hypercentral and every subgroup of G/L is permutable.

Conversely, let G be a periodic group satisfying the conditions (i)-(v). Then G is a soluble AP-group.

The following corollary shows the illustrates the connections between AP-groups and $\bar{T}$-groups.

Corollary 4.17 [5]. *Let G be a locally soluble hyperfinite AP-group. If the Sylow 2-subgroups of G are Dedekind and the Sylow p-subgroups of G are abelian for $p \neq 2$, then G is a metabelian $\bar{T}$-group.*

The description of AP-groups from Theorem 4.16 can be extend to ather classes of groups.

Theorem 4.18 [5]. *Let G be a periodic AP-group. If G is a hyper-N-group, then G is hyperfinite.*

Corollary 4.19 [5]. *Let G be a periodic AP-group. If G is a hyper-Gruenberg group, then G is a hypercyclic metabelian AP-group.*

Corollary 4.20 [5]. *Let G be a periodic AP-group. If G is hyperabelian, then G is a hypercyclic metabelian AP-group.*

Corollary 4.21 [5]. *Let G be a periodic AP-group. If G is residually soluble, then G is a hypercyclic metabelian AP-group.*

In the paper [7]. the role of pronormal subgroups in AP-groups has been studied.

Let p be a prime, G be a group and P be an arbitrary Sylow p-subgroup of G. We say that a group G belongs to the class $\mathfrak{P}_p$ if G satisfies the following two conditions:

(i) every subgroup of P is permutable;

(ii) each normal subgroup of P is pronormal in G.

Theorem 4.22 [7]. *Let G be a finite soluble group. Then G is an AP-group if and only if G belong to the class $\mathfrak{P}_p$ for all primes p.*

In the paper [6] this result was extend to infinite groups in the following way.

Theorem 4.23 [6]. *Let G be a periodic locally soluble group. If $G \in \mathfrak{P}_p$ for all primes p, then G is a AP-group. Moreover, if L is the locally nilpotent residual, then there exists a hypercentral subgroup T such that $G = L \leftthreetimes T$.*

As we can see, the properties which define the class $\mathfrak{P}_p$ for all primes p, are stronger than the property AP, because in an arbitrary soluble AP-group the locally nilpotent residual is not always complemented. So, unlike in the case of finite soluble groups, the class of infinite soluble AP-groups does not coincide with the class of infinite soluble groups that belong to the class $\mathfrak{P}_p$ for all primes p.

References

[1] Abramovskii, N. Locally generalized Hamiltonian groups, *Sibir. Math. J.* **7** (1966), 481-485.

[2] Ba M.S., Borevich Z. I. On arrangement of intermediate subgroups. *Rings and Linear Groups*, Kubanskij Univ., Krasnodar, 1988, 14-41.

[3] Baer R. Situation der Untergruppen und Struktur der Gruppe. *S.-B. Heidelberg Akad.* **2** (1933), 12-17.

[4] Balester-Bollinches, A., Esteban-Romero, R. On finite $\mathfrak{F}$-groups. *Journal Austral. Math. Soc.* **75** (2003), 1-11.

[5] Ballester-Bolinches A., Kurdachenko L.A., Otal J., Pedraza T. Infinite groups with many permutable subgroups. *Rev. Mat. Iberoamericana*, **24** (2008), no. 3, 745-764

[6] Ballester-Bolinches A., Kurdachenko L.A., Otal J., Pedraza T. Local characterizations of infinite groups whose ascendant subgroups are permutable. *Forum Mathematicum*, **22** (2010), no. 1, 187-201

[7] Beidleman J.C., Brewster B. and Robinson D.J.S. Criteria for permutability to be transitive infinite groups. *Journal Algebra.* **222** (1999), 400-412.

[8] Best, E., Taussky, O. A class of groups, *Proc. Roy. Irish. Acad. Sect. A.* **47** (1942), 55-62.

[9] Bianchi M., Mauri A.G.B., Herzog M., Verardi L. On finite solvable groups in which normality is a transitive relation. *Journal Group Theory,* **3** (2000),147-156.

[10] Borevich Z.I. On arrangement of subgroups. *Zap. Nauchn. Sem. LOMI,* **94** (1979), 5-12.

[11] Borevich Z.I., Macedoncska O.N. On the lattice of subgroups. *Zap. Nauchn. Sern. LOMI,* **103** (1980), 13-19.

[12] Borevich Z.I., Vavilov N.A. Arrangement of subgroups in the general linear group over a commutative ring. *Trudy Mat. Inst. Steklov.(LOMI),* **165** (1984), 24-42.

[13] Brandl, R. Groups with few non-normal subgroups. *Communications in Algebra*, **23**, (1995), Issue 6, 2091-2098.

[14] Carter, R.W. Nilpotent self-normalizing subgroups of soluble groups. *Math. Z.* **75** (1961), 136-139.

[15] Casolo C. Groups with all subgroups subnormal. *Conference "Advances in Group Theory and Applications 2007"*, Otranto, 4-8 June 2007, Italy, 148 pp.

[16] Chernikov, S.N. Groups with prescribed properties of system of infinite subgroups. *Ukrain Math. J.* **19** (1967) v. 6, 111-131

[17] Chernikov, S.N. Groups with normal infinite abelian subgroups. *Groups with Restrictions on Subgroups.* Naukova Dumka. Kiev. 1971, 47-64.

[18] Chernikov, S.N. Infinite groups, in which infinite non-abelian suvgroups are normal. *Ukrain Math. J.* **23** (1971), 604-628.

[19] Dedekind, R. Über Gruppen,deren sammtliche Teiler Normalteiler sind, *Math. Annalen,* **48** (1897), 548-561.

[20] Dixon M.R. A conjugacy theorem for Carter subgroups in groups with $min - p$ for all p. *Lecture Notes in Math.* **848** (1981), 161-168.

[21] De Falco M, Kurdachenko L.A., Subbotin I.Ya. Groups with only abnormal and subnormal subgroups. *Atti del Seminario Matematico e Fisico dell'Universita di Modena,* **2** (1998), 46, 435-442.

[22] Fatahi, A. Groups with only normal and abnormal subgroups. *Journal Algebra,* **28** (1974), 15-19.

[23] Gardiner A.D., Hartley B. and Tomkinson M.J. Saturated formations and Sylow structure in locally finite groups. *Journal Algebra,* **17** (1971),177-211.

[24] Gaschütz W. Gruppen in denen das Normalreilersein transitiv ist. *J. Reine.Angew. Math.* **198** (1957), 87-92.

[25] Gavron P.V., Kolotilina N. Yu. On subgroups of T-groups. *Zap. Nauchn. Sem. LOMI,* **103** (1980), 62-65.

[26] de Giovanni F., Vincenzi G. Some topics in the theory of pronormal subgroups of groups. *Topics in Infinite Groups, Quaderni di Matematica,* **8** (2001), 175-202.

[27] de Giovanni F., Vincenzi G. Pseudonormal subgroups of groups. *Ricerche Mat.* **52** (2003), no. 1, 91-101.

[28] Gorchakov, Yu.M. On primitively factorized groups. *Ukrain. Math. J.* **14** (1962), 3-9.

[29] Hall P. On the system normalizers of a soluble groups. *Proc. London Math.* Soc. **43** (1937), 507-528.

[30] Heineken, H. Groups with restriction on their infinite subnormal subgroups, *Proc. Edinburgh. Math. Soc.* **31** (1988), 231-241.

[31] Heineken, H. C., Lennox, J. Subgroups of finite index in T-groups. *Boll. Unione Mat. Ital.* **134** (1985), 829-841.

[32] Heineken H., Mohamed I.J. Groups with normalizer condition. *Math. Annalen,* **198** (1972), no 3, 178-187.

[33] Huppert B. *Endliche Gruppen I.* SPINGER: Berlin-1967.

[34] Kurdachenko L.A., Otal J., Subbotin I.Ya. Of some criteria of nilpotency. *Comm. Algebra,* **30** (2002), no 8, 3755-3776.

[35] Kurdachenko L.A., Otal J., Subbotin I.Ya. Abnormal, pronormal, contranormal and Carter subgroups in some generalized minimax groups. *Comm. Algebra*, **33** (2005), no 12, 4595-4616.

[36] Kurdachenko L.A., Otal J., Subbotin I.Ya. Criteria of nilpotency and influence of contranormal subgroups on the structure of infinite groups. *Turkish J. Math.* **33** (2009), 227-237.

[37] Kurdachenko L.A., Otal J., Subbotin I.Ya. Influence of contranormal subgroups on the structure of infinite groups. Comm. Algebra, **37** (2009), no 12, 4542-4557.

[38] Kurdachenko L.A., Russo A., Vincenzi G. Groups without proper abnormal subgroups. *Journal Group Theory* **9** (2006), no 4, 507-518.

[39] Kurdachenko L.A., Russo A., Vincenzi G. On some groups all subgroups of which are near to pronormal. *Ukrain. Math. J.* **59** (2007), 1332-1339.

[40] Kurdachenko L.A., Russo A., Subbotin I.Ya., Vincenzi G. Infinite groups with short balanced chains of subgroups. J*ournal Algebra,* **319** (2008), issue 9, pp. 3901-3917.

[41] Kurdachenko L.A., Smith H. Groups with all subgroups either subnormal or self-normalizing. *Journal Pure and Applied Algebra*, **196** (2005), 271-278.

[42] Kurdachenko L.A., Subbotin I.Ya. On U-normal subgroups, *Southeast Asian Bulletin of Mathematics,* **26** (2002), 789-801.

[43] Kurdachenko L.A., Subbotin I.Ya. On transitivity of pronormality. *Comment. Matemat. Univ. Caroline,* **43** (2002), no. 4, 583-594.

[44] Kurdachenko L.A., Subbotin I.Ya. Pronormality, contranormality and generalized nilpotency in infinite groups. *Publicacions Matemàtiques*, **47** (2003), no 2, 389-414

[45] Kurdachenko L.A., Subbotin I.Ya. Abnormal subgroups and Carter subgroups in some infinite groups. *Algebra and Discrete Mathematics,* 2005, no 1, 63-77.

[46] Kurdachenko L.A. and Subbotin I.Ya. Transitivity of normality and pronormal subgroups. *Combinatorial group Theory, discrete groups, and number theory. Contemporary Mathematics.* **421** (2006), 201-212.

[47] Kurdachenko L.A., Subbotin I.Ya. On some soluble groups in which U-subgroups form a lattice. Comment. Math. Univ. Carolinae, **48** (2007), no. 4, 585-593.

[48] Kurdachenko L.A., Pypka A.A., Subbotin I.Ya. On some properties of pronormal subgroups. *Cent. Eur. J. Math.* **8** (5) 2010, 840-845.

[49] Kurosh A.G., Chernikov S.N. Soluble and nilpotent groups. Uspehi Mat. Nauk. **2** (1947) no. 3, 18-59. Amer. Math. Soc. Transl. No. 80 (1953).

[50] Kyzennyi N.F., Subbotin, I.Ya. Groups in which all subgroups are pronormal. Ukrain Math. J. 39 (1987), no. 3, 325-329.

[51] Kuzennyi N.F., Subbotin I.Ya. Locally soluble groups in which all infinite subgroups are pronormal. *Izv. Vyssh. Ucheb. Zaved., Mat.* **11** (1988), 77-79.

[52] Kuzennyi, N.F., Subbotin I.Ya. New characterization of locally nilpotent $\bar{I}H$-groups. *Ukranian. Mat. Journal,* **40** (1988), 274-277.

[53] Kuzennyi N.F., Subbotin I.Ya. Groups with pronormal primary subgroups. *Ukrain. Mat. J.* **41** (1989), 323-327.

[54] Kuzennyi N.F., Subbotin I.Ya. On groups with fan subgroups. *Contemporary Mathematics,* **131** (1992), part 1, 383-388

[55] Miller G.A, Moreno H. Non-abelian groups in which every subgroup is abelian. *Trans. Amer. Math. Soc.* **4** (1903), 389-404.

[56] Möhres W. Auflösbarkeit von Gruppen deren Untergruppen alle subnormal sind. *Archiv Math.* **54** (1990), 232-235

[57] Mousavi H. On finite groups with few non-normal subgroups. *Comm. Algebra* **27** (1999), no. 7, 3143-3151.

[58] Müller K.H. Schwachnormale Untergruppen: Eine gemeinsame Verallgemeinerung der normalen und normalisatorgleichen Untergruppen. *Rend. Semin. Mat. Univ. Padova.* **36** (1966), 129-157.

[59] Mysovskikh V.I. Subnormalizers and properties of embedding of subgroups in finite groups. *Zapiski Nauchnyh Semin. POMI,* St. Peterburgh, **265** (1999), 258-280.

[60] Peng T.A. Finite groups with pronormal subgroups. *Proc. Amer. Math, Soc.* **20** (1969), 232-234.

[61] Peng, T.A. Pronormality in finite groups. *J. London Math. Soc.* **3** (1971), no. 2, 301-306.

[62] Poland, J., Rhemtulla, A. The number of conjugacy classes of non-normal subgroups in nilpotent groups. *Commu. Algebra*, **24**, (1996), Issue 10, 3237-3245

[63] Robinson, D.J.S. Groups in which normality is a transitive relation. *Proc. Cambridge Philos. Soc.* **60** (1964), 21-38.

[64] Robinson D.J.S. Minimality and Sylow-permutability in locally finite groups. *Ukrain. Math. J.* **54** (2002), no 6, 1038-1049.

[65] Robinson D. J. S., Russo A., Vincenzi G. On groups which contain no HNN-extensions. *International J. Algebra Computation,* **17** (2007), no. 7, 1377-1387.

[66] Rose J.S. Abnormal depth and hypereccentric length in finite soluble groups. *Math. Z.* **90** (1965), 29-49.

[67] Rose J.S. Nilpotent subgroups of finite soluble groups. *Math. Z.***106** (1968), 97-112.

[68] Schmidt O.Yu. Groups whose all subgroups are special. *Mat. Sbornik,* **31** (1924), 366-372.

[69] Schmidt, O. Yu. Groups having only one class on non normal subgroups. *Mat. Sbornik,* **33** (1926), 161-172

[70] Schmidt, O. Yu. Groups with two classes on non normal subgroups. *Proceeding of the Group Theory Seminar*, 1938, 7-26

[71] Schmidt R. *Subgroups Lattices of Groups.* Walter de Gruyter, Berlin, 1994.

[72] Stonehewer S.E. Abnormal subgroup of a class of periodic locally soluble groups. *Proc. London Math. Soc.* **14** (1964), 520-536.

[73] Stonehewer S.E. Locally soluble FC-groups. *Archiv Math.* **16** (1965), 158-177.

[74] Stonehewer S.E. Permutable subgroups of infinite groups. *Math. Z.* 1972, 125, 1-16

[75] Subbotin I.Ya. Groups with alternatively normal subgroups. *Izv. Vyssh. Ucheb. Zaved., Mat.* **3** (1992), 86-88.

[76] Sytnikov, V.M., Ustyuzaninov, A.D. Finite groups with three classes of non-normal subgroups. *Matem. Zapiski Ural. Universit.* **1** (1957), 94-102.

[77] Wood G.J. On pronormal subgroups of finite soluble groups. *Archiv Math.* **25** (1974), 578-585.

[78] Wilson J.S. On periodic generalized nilpotent groups. *Bulletin London Math. Soc.* **9** (1977), 81-85.

[79] Zacher G. I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **37** (1964), no. 8, 150-154.

Contact information

V. V. Kirichenko Department of Geometry, School of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, Volodymyrska, 64, Kyiv, 01033, Ukraine
E-Mail: `vkir@univ.kiev.ua`

L. A. Kurdachenko Department of Algebra, School of Mathematics and Mechanics, National University of Dnepropetrovsk, Gagarin Prospect 72, Dnepropetrovsk 10, 49010, Ukraine
E-Mail: `lkurdachenko@i.ua`

I. Ya. Subbotin Department of Mathematics and Natural Sciences, National University, 5245 Pacific Concourse Drive, Los Angeles, CA 90045-6904, USA
E-Mail: `isubboti@nu.edu`

Received by the editors: 14.03.2011
and in final form 18.03.2011.