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A generalization of supplemented modules

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Abstract. Let R be an arbitrary ring with identity and M a right R-module. In this paper, we introduce a class of modules which is an analogous of δ-supplemented modules defined by Kosan. The module M is called *principally δ-supplemented*, for all $m \in M$ there exists a submodule A of M with $M = mR + A$ and $(mR) \cap A$ δ-small in A. We prove that some results of δ-supplemented modules can be extended to principally δ-supplemented modules for this general settings. We supply some examples showing that there are principally δ-supplemented modules but not δ-supplemented. We also introduce principally δ-semiperfect modules as a generalization of δ-semiperfect modules and investigate their properties.

1. Introduction

Throughout this paper all rings have an identity, all modules considered are unital right modules. Let M be a module, N and P be submodules of M. We call P *a supplement* of N in M if $M = P + N$ and $P \cap N$ is small in P. A module M is called *supplemented* if every submodule of M has a supplement in M. A module M is called *lifting* if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M. Supplemented and lifting modules have been discussed by several authors (see [4, 8]) and these modules are useful in characterizing semiperfect and right perfect rings (see [8, 14]). A submodule L is called a *δ-supplement* of N in M if $M = N + L$ and $N \cap L$ is δ-small in L(therefore

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in M), and M is called *δ-supplemented* in case every submodule of M has a δ-supplement in M. Principally supplemented modules are introduced and studied in [3]. A module M is said to be *principally supplemented* if for any cyclic submodule has a supplement in M. Principally supplemented modules generalizes principally lifting modules([9]), supplemented modules and weakly supplemented modules(see [1], [8], [14]).

In this paper, we introduce principally δ-supplemented modules and investigate their properties. A module M is called *principally δ-supplemented* if for each cyclic submodule has the *principally δ-supplement property*, i.e., for each $m \in M$, there exists a submodule N such that $M = mR + N$ with $(mR) \cap N$ is δ-small submodule in N. A module M is called *principally δ-semiperfect* if, for each $m \in M$, M/mR has a projective δ-cover[12]. New characterizations of principally δ-semiperfect rings are obtained using principally δ-supplemented modules.

In what follows, by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, natural numbers, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n. For unexplained concepts and notations, we refer the reader to [2] and [8].

2. Preliminaries

In this section we establish the notation and state some results on δ-small submodules which are required later. Following Zhou [16], a submodule N of a module M is called a *δ-small* if, whenever $M = N + X$ with M/X singular, we have $M = X$.

We state the next lemma which is contained in [16, Lemma 1.2 and 1.3].

Lemma 2.1. *Let M be a module. Then we have the following.*

1. *If N is δ-small in M and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \leq N$.*

2. *If K is δ-small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is δ-small in N. In particular, if K is δ-small in $M \leq N$, then K is δ-small in N.*

3. *Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is δ-small in $M_1 \oplus M_2$ if and only if K_1 is δ-small in M_1 and K_2 is δ-small in M_2.*

4. *Let N, K be submodules of M with K is δ-small in M and $N \leq K$. Then N is also δ-small in M.*

Lemma 2.2. *Let M be a module and $m \in M$. Then the following are equivalent.*

1. *mR is not δ-small in M.*
2. *There is a maximal submodule N of M such that $m \notin N$ and M/N singular.*

Lemma 2.3. *Let M be a module and K, L, H be submodules of M. If L is δ-small in K, then L is δ-small in $K + H$.*

Proof. Assume that L is δ-small in K. Let U be a submodule of M with $K+H = L+U$ and $(K+H)/U$ singular. Then $K/(U\cap K) \cong (K+U)/U = (K + H)/U$ is singular. On the other hand we have $K = L + (K \cap U)$. Since L is δ-small in K, $K = K \cap U \leq U$. Hence $K + H = U$. □

Lemma 2.4. *Let L be a δ-supplement submodule of a module M. If U is a δ-small submodule of M with $U \leq L$, then U is δ-small in L.*

Proof. Let $M = K + L$ with $K \cap L$ δ-small in L and $L = U + V$ and L/V singular. We prove that $L = V$. Then $M = K + U + V$ and $M/(K+V) = (K+L)/(K+V) = ((K+V)+L)/(K+V) \cong L/(L\cap(K+V))$ which is a homomorphic image of singular module L/V. By hypothesis $M = K + V$. Then $L = (L \cap K) + V$ and so $L = V$. □

Lemma 2.5. *Let $A \leq B$ and K be submodules of M and $M = A + K$. If $B \cap K$ is δ-small in M, then B/A is δ-small submodule of M/A.*

Proof. Let $M/A = B/A + L/A$ with M/L singular. We have $M = B + L$ and $B = A + B \cap K$. Then $M = A + B \cap K + L = B \cap K + L$. Hence $M = L$ since $B \cap K$ is δ-small in M and M/L is singular. □

Lemma 2.6. *Let M be an R-module and K, L, N be submodules of M. Then we have the followings.*
(1) *If K is a δ-supplement of N in M and T is δ-small in M, then K is a δ-supplement of $N + T$ in M.*
(2) *Let $M \xrightarrow{f} N$ be an epimorphism with $Kerf$ δ-small in M. If the submodule L of M is a δ-supplement in M, then $f(L)$ is a δ-supplement in N. The converse holds if $Kerf$ is a δ-small submodule of L.*

Proof. (1) Let K be a δ-supplement of N in M. Then $M = N + K$ and $N \cap K$ is δ-small in K. We prove $(N + T) \cap K$ is δ-small in K. For if, let $L \leq K$ with $K = L+(N+T)\cap K$ and K/L singular, then $M = L+N+T$ and $M/(L + N) = (K + N)/(L + N) \cong K/(K + (L \cap N))$ is singular as

an homomorphic image of the singular module K/L. Since T is δ-small in M, $M = L + N$. Hence $K = L + K \cap N$. Since $K \cap N$ is δ-small in K and K/L is singular we have $K = L$.
(2) Let L be a δ-supplement of K in M. Then L is a δ-supplement of $K + Kerf$ by (1). By Lemma 3.4, $f(L) = f(L + Kerf)$ is also a δ-supplement of $f(K) = f(K + Kerf)$ in N. Conversely, let $N = f(L) + U$ with $f(L) \cap U$ is δ-small in $f(L)$ and $K = f^{-1}(U)$. Then $M = L + K$. To complete the proof we prove that $L \cap K$ is δ-small in L. For if $L = V + L \cap K$ with L/V singular, then $f(L) = f(V) + f(L) \cap f(K) = f(V) + f(L) \cap U$ since $Kerf \leq K$, $f(L \cap K) = f(L) \cap f(K)$. $f(L)/f(V)$ is singular as an homomorphic image of singular module L/V. Hence $f(L) = f(V)$. So $L = V + Kerf$. Thus $L = V$. □

3. Principally δ-supplemented modules

In this section we introduce principally δ-supplemented modules and investigate some properties of these modules. We prove that some results of supplemented and δ-supplemented modules can be extended to principally δ-supplemented modules.

Lemma 3.1. *Let $m \in M$ and L a submodule of M. Then the following are equivalent.*

1. *$M = mR + L$ and $mR \cap L$ is δ-small in L.*
2. *$M = mR + L$ and for any proper submodule K of L with L/K singular, $M \neq mR + K$.*

Proof. (1) $\Rightarrow$ (2) Let $K \leq L$ and $M = mR + K$ where L/K singular. Then $L = (L \cap mR) + K$. Since $L \cap mR$ is δ-small in L, $L = K$.
(2) $\Rightarrow$ (1) If $L = (mR \cap L) + K$ where $K \leq L$ and L/K singular, then $M = mR + L = mR + K$. By (2), $K = L$. So $mR \cap L$ is δ-small in L. □

Lemma 3.2. *Let M be a module and K, L, H be submodules of M. If L is a δ-supplement of K in M and K is a δ-supplement of H in M, then K is a δ-supplement of L in M.*

Proof. Let $M = K + L = K + H$, $K \cap L$ and $K \cap H$ are δ-small in L and K respectively. We prove $K \cap L$ is δ-small in K. Let $X \leq M$ be such that $K \cap L + X = K$ and K/X is singular. Then $M = (K \cap L) + X + H$. Since $K \cap L$ is δ-small in M, by Lemma 2.1 there exists a projective semisimple submodule Y in $K \cap L$ such that $M = Y \oplus (X + H)$. Hence $K = (Y \oplus X) + (K \cap H)$. Since $K/(X + Y)$ is singular and $K \cap H$ is

δ-small in K, again by Lemma 2.1, $K = X \oplus Y$. Thus $Y = 0$ as K/X is singular and Y is projective semisimple. □

Let M be a module and $m \in M$. A submodule L is called a *principally δ-supplement* of mR in M, if mR and L satisfy Lemma 3.1 and the module M is called *principally δ-supplemented* if every cyclic submodule of M has a principally δ-supplement in M, equivalently, for all $m \in M$ there exists a submodule A of M with $M = mR + A$ and $mR \cap A$ δ-small in A. In [12], a module M is defined to be *principally δ-lifting* if, for all $m \in M$, there exists a decomposition $M = A \oplus B$ such that $A \leq mR$ and $mR \cap B$ is δ-small in B (equivalently, in M).

Clearly, every supplemented module and every principally δ-lifting module is principally δ-supplemented. Since every factor module of a singular module is singular, every singular δ-supplemented module is supplemented. There are principally δ-supplemented modules but not supplemented and so not δ-supplemented.

Example 3.3. (1) The $\mathbb{Z}$-module $\mathbb{Q}$ has no maximal submodules. Every cyclic submodule of $\mathbb{Q}$ is small, therefore $\mathbb{Q}$ is principally δ-supplemented. But $\mathbb{Q}$ is not supplemented, and so not δ-supplemented since it is singular $\mathbb{Z}$-module.
(2) Let $R = \mathbb{Z}$ and $M = \bigoplus_{i=1}^{\infty} M_i$ with each $M_i = \mathbb{Z}_{p^\infty}$, where p is prime number. Then $\delta(M) = \bigoplus_{i=1}^{\infty} \delta(M_i) = M$ is essential in M. In [10], it is proved that M is neither supplemented nor δ-supplemented. We prove M is principally δ-supplemented. For if $m = (m_i) \in M$ then m is contained in a finite direct sum of copies of $\mathbb{Z}_{p^\infty}$. Since any submodule of a small submodule is small and finite sum of small submodules is small, $m\mathbb{Z}$ is small in M. Hence M is principally δ-supplemented.

Lemma 3.4. *If $M \xrightarrow{f} M'$ is a homomorphism and N is a δ-supplement in M with $Ker(f) \leq N$, then $f(N)$ is a δ-supplement in $f(M)$.*

Proof. Let $M = N + K$ with $N \cap K$ δ-small in N. Then $f(M) = f(N + K) = f(N) + f(K)$. Since $Kerf \leq N$, we have $f(N) \cap f(K) = f(N \cap K)$. By Lemma 2.1 (2), $f(N \cap K) = f(N) \cap f(K)$ is δ-small in $f(N)$. Hence $f(N)$ is a δ-supplement of $f(K)$ in $f(M)$. □

Lemma 3.5. *Let M be a principally δ-supplemented module and $N \leq M$. If every cyclic submodule mR has a δ-supplement A with $N \leq A$, then M/N is principally δ-supplemented.*

Proof. Let K/N be a cyclic submodule of M/N. Then $K = mR + N$ for some $m \in M$. There exists $L \leq M$ such that $N \leq L$, $M = mR + L$ with $mR \cap L$ δ-small in L. Let $M \xrightarrow{\pi} M/N$ natural epimorphism. By Lemma 3.4, $\pi(L)$ is δ-supplement of $\pi(mR) = K/N$, indeed $M/N = L/N + (mR+N)/N = L/N + K/N$ and $(N + (L \cap mR))/N$ is δ-small in L/N as it is a homomorphic image of $L \cap mR$ which is δ-small in L. $\square$

Lemma 3.6. *Let M be a module, N a δ-supplemented submodule of M and K a cyclic submodule of M. If $N + K$ has a δ-supplement T in M, then $N \cap (T + K)$ has a δ-supplement U in N. In particular, $T + U$ is a δ-supplement of K in M.*

Proof. We have $M = (N + K) + T$ and $(N + K) \cap T$ is δ-small in T, $N \cap (K + T) + U = N$ and $(K + T) \cap U$ is δ-small in U. Then $M = N + K + T = K + N \cap (K + T) + U = K + T + U$. Since finite sum of δ-small submodules is δ-small by Lemma 2.1 (3), $K \cap (T + U) \leq T \cap (K + U) + U \cap (K + T) \leq T \cap (K + N) + U \cap (K + T)$ is δ-small in $T + U$. $\square$

Recall that a module M is called *distributive*, if for all submodules K, L and N, $N \cap (K+L) = N \cap K + N \cap L$ or $N + (K \cap L) = (N+K) \cap (N+L)$. Lemma 3.7 is well known and obvious but we prove it for the sake of easy reference.

Lemma 3.7. *Let $M = M_1 \oplus M_2 = K + N$ and $K \leq M_1$. If M is distributive and $K \cap N$ is δ-small in N, then $K \cap N$ is δ-small in $M_1 \cap N$.*

Proof. Let $M_1 \cap N = (K \cap N) + L$ with $(M_1 \cap N)/L$ singular. Since M is distributive, $N = M_1 \cap N \oplus M_2 \cap N$. We have $M = K + N = K + M_1 \cap N + M_2 \cap N = K + L + (M_2 \cap N)$ and $N = K \cap N + L + (M_2 \cap N)$. Now $N/(L \oplus (M_2 \cap N)) = ((N \cap M_1) \oplus (N \cap M_2))/(L \oplus (M_2 \cap N)) \cong (N \cap M_1)/L$ is singular. Hence $N = L \oplus (M_2 \cap N)$. This and $N = (N \cap M_1) \oplus (N \cap M_2)$ and $L \leq M_1 \cap N$ imply $L = M_1 \cap N$. Hence $K \cap N$ is δ-small in $M_1 \cap N$. $\square$

Theorem 3.8. *Every direct summand of a distributive principally δ-supplemented module is principally δ-supplemented.*

Proof. Let $M = M_1 \oplus M_2$ and $m \in M_1$. There exists $N \leq M$ such that $M = mR + N$ and $mR \cap N$ is δ-small in N. Then $M_1 = mR + (M_1 \cap N)$ and by Lemma 3.7, $mR \cap (M_1 \cap N)$ is δ-small in $(M_1 \cap N)$. $\square$

Proposition 3.9. *Let M_1 and M_2 be principally δ-supplemented modules and $M = M_1 \oplus M_2$. If M is a distributive module, then M is principally δ-supplemented.*

Proof. Let $M = M_1 \oplus M_2$ be a distributive module and mR be a submodule of M. Then $mR = (mR \cap M_1) \oplus (mR \cap M_2)$. Since $mR \cap M_1$ and $mR \cap M_2$ are cyclic submodules of M_1 and M_2 respectively, there exist A a submodule of M_1 such that $M_1 = (mR \cap M_1) + A$ and $A \cap (mR \cap M_1) = A \cap mR$ is δ-small in A, and $B \leq M_2$ such that $M_2 = (mR \cap M_2) + B$, $B \cap (mR \cap M_2) = B \cap mR$ is δ-small in B. Then $M = mR + A + B$. Now we claim $mR \cap (A + B) = (mR \cap A) + (mR \cap B)$. The inclusion $(mR \cap A) + (mR \cap B) \leq mR \cap (A + B)$ always holds. For the inverse inclusion, $mR \cap (A+B) \leq A \cap (mR+B) + B \cap (mR+A) = A \cap ((mR \cap M_1) + M_2) + B \cap (M_1 + (mR \cap M_2))$. On the other hand $A \cap ((mR \cap M_1) + M_2) \leq (mR \cap M_1) \cap (A + M_2) + M_2 \cap ((mR \cap M_1) + A) = mR \cap A$. Similarly $B \cap (M_1 + (mR \cap M_2)) \leq mR \cap B$. Hence $(mR \cap (A+B) \leq mR \cap A + mR \cap B$. So the claim $(mR \cap (A + B) = mR \cap A + mR \cap B$ is justified. Since $mR \cap A$ is δ-small in A and $mR \cap B$ is δ-small in B, by Lemma 2.1 (3), we have $mR \cap (A + B)$ is δ-small in $A + B$. Hence M is principally δ-supplemented. □

Let M be a module with $S = \mathrm{End}(M_R)$. A submodule N is called *fully invariant* if for each $f \in S$, $f(N) \leq N$. Then M is an (S, R)-module and a principal submodule N of the right R-module M is fully invariant if and only if N is an (S, R)-submodule of M. Clearly 0 and M are fully invariant submodules of M. The right R-module M is called *duo* provided every submodule of M is fully invariant. For the readers' convenience we state and prove Lemma 3.10 which is proved in [11].

Lemma 3.10. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ and N a fully invariant submodule of M. Then $N = \bigoplus_{i \in I}(N \cap M_i)$.*

Proof. For each $j \in I$, let $p_j : M \to M_j$ denote the canonical projection and let $i_j : M_j \to M$ denote inclusion. Then $i_j p_j$ is an endomorphism of M and hence $i_j p_j(N) \subseteq N$ for each $j \in I$. It follows that $N \subseteq \bigoplus_{j \in I} i_j p_j(N) \subseteq \bigoplus_{j \in I}(N \cap M_j) \subseteq N$, so that $N = \bigoplus_{j \in I}(N \cap M_j)$. □

We can not prove that any direct sum of principally δ-supplemented modules need not be principally δ-supplemented. Note the following fact.

Proposition 3.11. *Let M_1 and M_2 be principally δ-supplemented modules and $M = M_1 \oplus M_2$. If M is a duo module, then M is principally δ-supplemented.*

Proof. Same as the proof of Proposition 3.9. □

A module M is said to be *principally semisimple* if every cyclic submodule is a direct summand of M. Tuganbayev calls a principally semisimple module as a regular module in [7]. Every semisimple module is principally semisimple. Every principally semisimple module is principally δ-lifting, and so principally δ-supplemented. For a module M, we write $\operatorname{Rad}_\delta(M) = \sum\{L \mid L \text{ is a } \delta\text{-small submodule of } M\}$.

Lemma 3.12. *Let M be a distributive principally δ-supplemented module. Then $M/\operatorname{Rad}_\delta(M)$ is a principally semisimple module.*

Proof. Let $\overline{m} \in M/\operatorname{Rad}_\delta(M)$. There exists a submodule A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A, so is δ-small in M. By the distributivity of M we have $mR \cap (A + \operatorname{Rad}_\delta(M)) = (mR \cap A) + mR \cap \operatorname{Rad}_\delta(M) = \operatorname{Rad}_\delta(M)$.

$$\begin{aligned} M/\operatorname{Rad}_\delta(M) &= ((mR + \operatorname{Rad}_\delta(M))/\operatorname{Rad}_\delta(M)) + \\ &\quad + ((A + \operatorname{Rad}_\delta(M))/\operatorname{Rad}_\delta(M) = \\ &= ((\overline{m}R)/\operatorname{Rad}_\delta(M)) \oplus ((A + \operatorname{Rad}_\delta(M))/\operatorname{Rad}_\delta(M). \end{aligned}$$ □

Theorem 3.13 may be proved easily by making use of Lemma 3.12 for distributive modules. But we prove it in another way in general.

Theorem 3.13. *Let M be a principally δ-supplemented module. Then M has a submodule M_1 such that M_1 has an essential socle and $\operatorname{Rad}_\delta(M) \oplus M_1$ is essential in M.*

Proof. By Zorn's Lemma we may find a submodule M_1 of M such that $\operatorname{Rad}_\delta(M) \oplus M_1$ is essential in M. To prove $\operatorname{Soc}(M_1)$ is essential in M_1, we show that every cyclic submodule of M_1 has a simple submodule. Let $m \in M_1$. Since M is principally δ-supplemented, there exists a submodule A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in A. Then $mR \cap A = 0$. Let K be a maximal submodule of mR. If K is unique maximal submodule in mR, then it is small, therefore δ-small in mR and so in M. This is not possible since $mR \cap \operatorname{Rad}_\delta(M) = 0$. Hence there exists $x \in mR$ such that $mR = K + xR$. We claim that $K \cap xR = 0$. Otherwise let $0 \neq x_1 \in K \cap xR$. By hypothesis there exists C_1 such that $M = x_1R + C_1$ with $(x_1R) \cap C_1$ is δ-small in M. So $M = x_1R \oplus C_1$ since $(x_1R) \cap C_1 \leq K \cap \operatorname{Rad}_\delta(M) = 0$. Hence $mR = x_1R \oplus (mR \cap C_1)$ and $K = x_1R \oplus (K \cap C_1)$. If $K \cap C_1$ is nonzero, let $0 \neq x_2 \in K \cap C_1$. By hypothesis there exists C_2 such that $M = x_2R + C_2$ with $(x_2R) \cap C_2$ is δ-small in M. So $M = x_2R \oplus C_2$ since $(x_2R) \cap C_2 \leq K \cap \operatorname{Rad}_\delta(M) = 0$. Then $K \cap C_1 = (x_2R) \oplus (K \cap C_1 \cap C_2)$. Hence $mR = x_1R \oplus x_2R \oplus (mR \cap C_1 \cap C_2)$

and $K = x_1R \oplus x_2R \oplus (K \cap C_1 \cap C_2)$. If $K \cap C_1 \cap C_2$ is nonzero, similarly there exists $0 \neq x_3 \in K \cap C_1 \cap C_2$ and $C_3 \leq M$ such that $M = x_3R \oplus C_3$. Then $mR = x_1R \oplus x_2R \oplus x_3R \oplus (mR \cap C_1 \cap C_2 \cap C_3)$ and $K = x_1R \oplus x_2R \oplus x_3R \oplus (K \cap C_1 \cap C_2 \cap C_3)$. This process must terminate at a finite step, say t. At this step $mR = x_1R \oplus x_2R \oplus x_3R \oplus ... \oplus x_tR$ and so $mR = K$ since at t^{th} step we must have $K \cap C_1 \cap C_2 \cap ... \cap C_t \leq mR \cap C_1 \cap C_2 \cap ... \cap C_t = 0$. This is a contradiction. There exists $x \in mR$ such that $mR = K \oplus xR$. Then xR is a simple module. □

In the following we investigate under what conditions direct summands of principally δ-supplemented modules are principally δ-supplemented.

Lemma 3.14. *Let $M = M_1 \oplus M_2$ be a decomposition of M. Then M_2 is principally δ-supplemented if and only if for every cyclic submodule N/M_1 of M/M_1, there exists a submodule K of M_2 such that $M = K + N$ and $N \cap K$ is δ-small in K.*

Proof. Suppose that M_2 is principally-supplemented. Let N/M_1 be a cyclic submodule of M/M_1. Let $N/M_1 = (xR+M_1)/M_1$ and $x = m_1+m_2$ where $m_1 \in M_1$, $m_2 \in M_2$. Then $N/M_1 = (m_2R + M_1)/M_1$. By supposition there exists a submodule $K \leq M_2$ such that $M_2 = (m_2R) + K$ with $(m_2R) \cap K$ is δ-small in K. Then $N = m_2R + M_1$ and $M = N + K$. Now $N \cap K = ((m_2R) + M_1) \cap K \leq (m_2R) \cap (M_1 + K) + M_1 \cap (K + (m_2R)) \leq K \cap (M_1 + (m_2R)) + M_1 \cap (m_2R + K)$. $M_1 \cap (m_2R + K) = 0$ implies $(M_1 + m_2R) \cap K = (m_2R) \cap ((m_1R) + K)$. Hence $N \cap K \leq m_2R$. Since $(m_2R) \cap K$ is δ-small in K, $N \cap K$ is δ-small in K.

Conversely, let N be a cyclic submodule of M_2. Consider the cyclic submodule $(N + M_1)/M_1$ of M/M_1. By hypothesis, there exists a submodule K of M_2 such that $M = (N + M_1) + K$ and $K \cap (N + M_1)$ is δ-small submodule of K. Then $M_2 = N + K$. To complete the proof it is enough to show $K \cap (M_1 + N) = N \cap (M_1 + K) = N \cap K$. Now $N \cap (M_1 + K) \leq M_1 \cap (K + N) + K \cap (N + M_1) = K \cap (N + M_1) \leq N \cap (M_1 + K) + M_1 \cap (K + N) = N \cap (M_1 + K)$ since $M_1 \cap (K + N) = 0$. Then $N \cap (M_1 + K) = K \cap (N + M_1)$. But $(M_1 + K) \cap N = K \cap (N + M_1) = N \cap K$ is obvious now. Hence $N \cap K$ is δ-small submodule of K. □

Proposition 3.15. *Let M_1 and M_2 be principally δ-supplemented modules with $M = M_1 \oplus M_2$. Then M is principally δ-supplemented if and only if every cyclic submodule N of M with $M = N + K$ for any proper submodule K of M has a supplement in M.*

Proof. Necessity is clear. Conversely, suppose that for every cyclic submodule N of M with $M = N + K$ for any proper direct summand K of M has a supplement in M. Let $N = mR$ be a cyclic submodule. If $M = N + M_i$ or $N \leq M_i$ we have done. Otherwise we may assume $m = m_1 + m_2$ and m_1 and m_2 are nonzero. By supposition there are $K_1 \leq M_1$ and $K_2 \leq M_2$ such that $M_1 = (m_1R) + K_1$, $M_2 = (m_2R) + K_2$ and $(m_1R) \cap K_1$ is δ-small in K_1 and $(m_2R) \cap K_2$ is δ-small in K_2. $m_1R + m_2R = N + m_2R = N + m_1R$ and $M = N + m_1R + K_1 + K_2 = N + M_1 + K_2$. Similarly $M = N + M_2 + K_1$. Assume $M = M_1 + K_2$. Then $M_2 = K_2$ and so $m_2 = 0$ and $N \leq M_1$. It leads us to a contradiction. Hence $M_1 + K_2$ is a proper submodule of M. Similarly $M_2 + K_1$ is proper. Thus N has a supplement in M. □

Principally δ-hollow modules and principally δ-lifting modules are defined in [12] and properties of these modules are investigated. A nonzero module M is called *δ-hollow* if every proper submodule is δ-small in M, and M is called *principally δ-hollow* if every proper cyclic submodule is δ-small in M, and M is said to be *finitely δ-hollow* if every proper finitely generated submodule is δ-small in M. Since finite direct sum of δ-small submodules is δ-small, M is principally δ-hollow if and only if it is finitely δ-hollow. There are principally δ-hollow modules but not δ-hollow. Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and rational numbers respectively. Then the $\mathbb{Z}$-module $\mathbb{Q}$ is principally δ-hollow since each finitely generated submodule of $\mathbb{Q}$ is small, therefore δ-small in $\mathbb{Q}$. Let $\mathbb{Q}_1 = \{a/b \in \mathbb{Q} \mid 2$ does not divide $b\}$ and $\mathbb{Q}_2 = \{a/b \in \mathbb{Q} \mid 2 \text{ divides } b\}$. Then $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$. Since $\mathbb{Q}/\mathbb{Q}_1$ and $\mathbb{Q}/\mathbb{Q}_2$ are singular $\mathbb{Z}$-modules, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are not δ-small submodules in $\mathbb{Q}$.

Recall that a nonzero module M is called *principally δ-lifting* if for each cyclic submodule has the *δ-lifting property*, i.e., for each $m \in M$, M has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ is δ-small in B (see [12] for detail). It is obvious that every principally δ-lifting module is principally δ-supplemented. There are principally δ-supplemented modules but not principally δ-lifting. As an illustration we record here Example 3.16.

Example 3.16. Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}/2\mathbb{Z}$ and $M_2 = \mathbb{Z}/8\mathbb{Z}$. As $\mathbb{Z}$-modules M_1 and M_2 are principally δ-hollow, therefore principally δ-supplemented modules. Let $M = M_1 \oplus M_2$. It is mentioned in [12] that M is not a principally δ-lifting $\mathbb{Z}$-module. The submodules $N_1 = (\overline{1}, \overline{2})\mathbb{Z}$ and $N_2 = (\overline{1}, \overline{1})\mathbb{Z}$, $N_3 = (\overline{0}, \overline{4})\mathbb{Z}$ and $N_4 = (\overline{0}, \overline{2})\mathbb{Z}$ are the only proper submodules of M and all of them are cyclic. N_3 and N_4 are δ-small in M and $M = N_1 + N_2$. Now $N_1 \cap N_2 = N_3$ is δ-small in both N_1 and N_2. Hence M is principally δ-supplemented. By the same reasoning, for any prime integer p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ is principally

δ-supplemented but not principally δ-lifting.

Lemma 3.17. *Let M be an indecomposable module. Consider the following conditions.*

1. *M is a principally δ-lifting module.*

2. *M is a principally δ-hollow module.*

3. *M is a principally δ-supplemented module.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Leftrightarrow$ (2) is proved in [12]. (2) $\Rightarrow$ (3) Let $m \in M$. By (2) each cyclic submodule is δ-hollow. Then $M = mR + M$ and $mR \cap M$ is δ-small in M. So M is principally δ-supplemented. □

Note that Lemma 3.17 (3) $\Rightarrow$ (2) does not hold in general.

In a subsequent paper the authors continue studying some generalizations of supplemented modules. In [8], the module M is called $\oplus$-supplemented if for every submodule N of M there is a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K, and M is called *$\oplus$-δ-supplemented module* if for each submodule N of M there exists a direct summand A such that $M = N + A$ and $N \cap A$ is δ-small in A. In the same way *δ-$\oplus$-supplemented module* means for each submodule N of M there exists a direct summand A such that $M = N + A$ and $N \cap A$ is δ-small in A. It is the same as $\oplus$-δ-supplemented module. Hence we introduce M is called *principally $\oplus$-δ-supplemented module* if for each $m \in M$ there exists a direct summand A such that $M = mR + A$ and $mR \cap A$ is δ-small in A.

The module M is called *a weak principally δ-supplemented* if for each $m \in M$ there exists a submodule A such that $M = mR + A$ and $mR \cap A$ is δ-small in M. Every weakly supplemented module is weak principally δ-supplemented. The module M is called *principally $\oplus$-supplemented* if for each $m \in M$ there exists a direct summand A of M such that $M = mR+A$ and $mR \cap A$ is small in A. $\oplus$-supplemented modules are studied in [6]. Every $\oplus$-supplemented module is principally $\oplus$-δ-supplemented and it is evident that every principally $\oplus$-supplemented is weak principally δ-supplemented. In a subsequent paper the authors investigates the interconnections between principally δ-supplemented modules, weakly principally δ-supplemented modules and principally $\oplus$-δ-supplemented modules in detail.

Recall that a module M is said to have the *summand intersection property* if the intersection of any two direct summands of M is again a direct summand of M. The summand intersection property was studied by J. L. Garcia [5], who characterized modules with the summand intersection property. A module M is called *refinable* if for any submodule U, V of M with $M = U + V$ there is a direct summand U' of M such that $U' \subseteq U$ and $M = U' + V$ (see namely [15]).

Theorem 3.18. *Let M be a refinable module. Consider the following conditions.*

(1) *M is principally δ-lifting.*

(2) *M is principally $\oplus$-δ-supplemented.*

(3) *M is principally δ-supplemented.*

(4) *M is weak principally δ-supplemented.*
Then (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). If M has the summand intersection property then (4) $\Rightarrow$ (1).

Proof. By definitions (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) always hold.

(4) $\Rightarrow$ (2) Let M be a weakly principally δ-supplemented module and $m \in M$. By (4) there exists a submodule A of M such that $M = mR + A$ and $mR \cap A$ is δ-small in M. By hypothesis, there exists a direct summand U of M with $U \leq A$ and $M = mR + U = U' \oplus U$ for some submodule U' of M. We claim that $mR \cap U$ is δ-small in U. Assume that $mR \cap U + L = U$ for some submodule L of U with U/L singular. Since $M/(U' + L)$ is singular as it is isomorphic to the singular U/L. Then $M = U' + (mR \cap U) + L$ implies $M = U' \oplus L$ as $mR \cap U$ is δ-small in M. Hence $L = U$. So M is a principally $\oplus$-δ-supplemented module.

(4) $\Rightarrow$ (1) Assume that M has the summand intersection property and let $m \in M$. By (4) there exists a submodule A such that $M = mR + A$ and $mR \cap A$ is δ-small in M. By hypothesis, there exists a direct summand U_1 of M such that U_1 is contained in A and $M = mR + U_1 = U_1' \oplus U_1$. Since U_1 is direct summand and $mR \cap A$ is δ-small in M, $mR \cap U_1$ is δ-small in U_1 by Lemma 2.1 (3). Again by hypothesis, there exists a direct summand U_2 of M such that U_2 is contained in mR and $M = U_2 + U_1 = U_2 \oplus U_2'$. By the summand intersection property $U_2 \cap U_1$ is a direct summand of M, $M = (U_2 \cap U_1) \oplus K$ for some submodule K of M. Then $U_1 = (U_2 \cap U_1) \oplus (K \cap U_1)$ and $M = U_2 \oplus (K \cap U_1)$. By Lemma 2.1 (4), $mR \cap (K \cap U_1)$ is δ-small in U_1 since $mR \cap (K \cap U_1) \leq mR \cap U_1 \leq U_1$ and $mR \cap U_1$ is δ-small in U_1. By Lemma 2.1 (3), $mR \cap (K \cap U_1)$ is δ-small in $K \cap U_1$ as $K \cap U_1$ is direct summand of U_1. $\square$

Theorem 3.19 is proved in [12]. We state without proof for the convenience of the reader.

Theorem 3.19. *Let M be a principally δ-semiperfect module. Then*

1. *M is principally δ-supplemented.*
2. *Each factor module of M is principally δ-semiperfect, hence any homomorphic image and any direct summand of M is principally δ-semiperfect.*

Theorem 3.20. *Let M be a projective module. The following conditions are equivalent.*

1. *M is principally δ-semiperfect.*
2. *M is principally δ-lifting.*
3. *M is principally δ-supplemented.*

Proof. (1) $\Leftrightarrow$ (2) is proved in [12].

(1) $\Rightarrow$ (3) By Theorem 3.19.

(3) $\Rightarrow$ (1) Let $m \in M$. By (3) there exists a submodule A such that $M = mR + A$ such that $mR \cap A$ is δ-small in A. Let $M \xrightarrow{f} M/mR$ defined by $f(y) = a + mR$, where $y = mr + a \in M$ with $mr \in mR$, $a \in A$, and $M \xrightarrow{\pi} M/mR$ the natural epimorphism. There exists $M \xrightarrow{g} M$ such that $fg = \pi$. Then $M = g(M) + mR \cap A$. Since $mR \cap A$ is δ-small in A, it is δ-small in M. By Lemma 2.1 (1), there exists a projective semisimple submodule Y of $mR \cap A$ such that $M = g(M) \oplus Y$ and so that $g(M)$ is projective. Hence $g(M) \cong M/\mathrm{Ker}(g)$ implies $M =\mathrm{Ker}(g) \oplus B$ for some submodule B of M and B is projective. Let $(fg)_{|B}$ denote the restriction of fg on B. Then $\mathrm{Ker}(fg)_{|B} \leq mR \cap A$. Hence $\mathrm{Ker}(fg)_{|B}$ is δ-small in B and so $B \xrightarrow{(fg)_{|B}} M/mR$ is a projective δ-cover of M. □

4. Applications

Recall that *projective δ-cover* of a module M is a projective R-module P with an epimorphism f from P to M such that $Kerf$ is δ-small in P. The next result is a well known fact about the relation between projective δ-cover and a δ-supplement and we prove for completeness.

Lemma 4.1. *Let M be a module and $m \in M$. If M/mR has a projective δ-cover, then N contains a δ-supplement of mR.*

Proof. Let $f : P \to M/mR$ be a projective δ-cover of M/mR and $\pi : M \to M/mR$ natural epimorphism. There exists an $g : P \to M$ such that $f = \pi g$. Then $M = mR + g(P)$ and $mR \cap g(P) = g(\mathrm{Ker}(f))$. It is δ-small in $g(P)$ as an homomorphic image of δ-small submodule $Kerf$ in P by Lemma 2.1 (2). $\square$

In [12] principally δ-semiperfect modules are introduced and some properties are studied. By [16], a ring is called *δ-perfect* (or *δ-semiperfect*) if every R-module (or every simple R-module) has a projective δ-cover. For more detailed discussion on δ-small submodules, δ-perfect and δ-semiperfect rings, we refer to [16]. A module M is called *principally δ-semiperfect* if every factor module of M by a cyclic submodule has a projective δ-cover. A ring R is called *principally δ-semiperfect* in case the right R-module R is principally δ-semiperfect. Every δ-semiperfect module is principally δ-semiperfect. In Example 4.2, we see that there is a principally δ-semiperfect module but not semiperfect. In [16], a ring R is called *δ-semiregular* if every cyclically presented R-module has a projective δ-cover.

We recall some well known examples for motivation.

Example 4.2. Let $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x, y, z \in \mathbb{Z}_4 \right\}$ denote the ring of upper triangular matrices over the ring of integers modulo 4. It is easy to check that principal right ideals of R are either small in R or direct summands of R. Hence R is principally δ-supplemented right R-module. By Theorem 4.3, R is principally δ-semiperfect. Let e_{12} denote the matrix unit having 1 at $(1, 2)$ entry and zero elsewhere. Let $I = e_{12}R$. Then I is small, therefore δ-small right ideal and Jacobson radical $J(R)$ of R is equal to I. Hence $R/J(R)$ is not semisimple. Therefore R is not a semiperfect ring.

Theorem 4.3. *Let R be a ring. The following conditions are equivalent.*

1. *R is principally δ-semiperfect.*
2. *R is principally δ-lifting.*
3. *R is δ-semiregular.*
4. *R is principally δ-supplemented.*

Proof. (1) $\Rightarrow$ (2) Clear from Theorem 3.20.

(2) $\Rightarrow$ (3) Assume that R is principally δ-lifting and $x \in R$. Then there exists a direct summand right ideal A of R such that $R = A \oplus B$,

$A \leq xR$ and $xR \cap B$ is δ-small in B. Then $xR = A \oplus xR \cap B$ and $xR \cap B \leq \mathrm{Rad}_\delta(M)$. By [16, Theorem 3.5], R is δ-semiregular.

(3) $\Rightarrow$ (4) Assume that R is δ-semiregular. Let $x \in R$ and $\pi : R \to R/xR$ natural epimorphism. By hypothesis, R/xR has a projective δ-cover $f : P \to R/xR$ since R/xR is cyclically presented. There exists $g : P \to R$ such that $f = \pi g$. Then $R = g(P) + xR$ and $g(P) \cap xR$ is δ-small in $g(P)$ since $g(P) \cap xR = g(\mathrm{Ker} f)$ and $Ker f$ is δ-small in P. Hence R is principally δ-supplemented.

(4) $\Rightarrow$ (1) Clear from Theorem 3.20. $\square$

Theorem 4.4. *Let M be a refinable projective module with $\mathrm{Rad}_\delta(M)$ is δ-small in M. If $M/\mathrm{Rad}_\delta(M)$ is principally semisimple, then M is principally δ-supplemented.*

Proof. Let xR be any cyclic submodule of M. Then we have $M/\mathrm{Rad}_\delta(M) = [(xR + \mathrm{Rad}_\delta(M))/\mathrm{Rad}_\delta(M)] \oplus [U/\mathrm{Rad}_\delta(M)]$ for some $U \leq M$. Then $M = xR + U$ and $\mathrm{Rad}_\delta(M) = xR \cap U + \mathrm{Rad}_\delta(M)$. Hence $xR \cap U$ is δ-small in M and $xR \cap U \leq \mathrm{Rad}_\delta(M)$. Since $M = xR + U$ there exists a direct summand A of M such that $A \leq U$ and $M = xR + U = xR + A = B \oplus A$. Since $xR \cap A$ is δ-small in M, so it is δ-small in A since A is direct summand. This completes the proof. $\square$

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