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On fibers and accessibility of groups acting on trees with inversions

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Abstract. Throughout this paper the actions of groups on graphs with inversions are allowed. An element g of a group G is called inverter if there exists a tree X where G acts such that g transfers an edge of X into its inverse. A group G is called accessible if G is finitely generated and there exists a tree on which G acts such that each edge group is finite, no vertex is stabilized by G, and each vertex group has at most one end.

In this paper we show that if G is a group acting on a tree X such that if for each vertex v of X, the vertex group G_v of v acts on a tree X_v, the edge group G_e of each edge e of X is finite and contains no inverter elements of the vertex group $G_{t(e)}$ of the terminal $t(e)$ of e, then we obtain a new tree denoted $\widetilde{X}$ and is called a fiber tree such that G acts on $\widetilde{X}$. As an application, we show that if G is a group acting on a tree X such that the edge group G_e for each edge e of X is finite and contains no inverter elements of $G_{t(e)}$, the vertex G_v group of each vertex v of X is accessible, and the quotient graph G/X for the action of G on X is finite, then G is an accessible group.

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Introduction

The theory of groups acting on trees without inversions known Bass-Serre theory is introduced in [2] and [10], and with inversions is introduced in [9]. The concepts of the fibers of groups acting on trees without inversions were introduced in ([2], p. 78). In this paper we generalize such concepts to the case where the actions of groups on trees with inversions are allowed, and have applications. This paper is divided into 3 sections. In section 1, we introduce the concept of groups acting on trees with inversions. In section 2, we use the results of section 1 to obtain new trees called the fibers of groups acting on trees with inversions. In section 3, we use the results of section 2 to have applications.

1. Groups acting on trees

We begin with general background. A graph X consists of two disjoint sets $V(X)$, (the set of vertices of X) and $E(X)$, (the set of edges of X), with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions that $\partial_0\eta = \partial_1$ and $\partial_1\eta = \partial_0$. For simplicity, if $e \in E(X)$, we write $\partial_0(e) = o(e)$, $\partial_1(e) = t(e)$, and $\eta(e) = \overline{e}$. This implies that $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$, and $\overline{\overline{e}} = e$. The case $\overline{e} = e$ is allowed. For the edge e, $o(e)$ and $t(e)$ are called the ends of e, and $\bar{e}$ is called the inverse of e. By a path P of X we mean a sequence $y_1, ..., y_n$ of edges of X such that $t(y_j) = o(y_{j+1})$ for $j = 1, \ldots, n-1$. P is reduced if $y_{i+1} \neq \overline{y}_i$, $i = 1, \ldots, n-1$.

The origin $o(P)$ and the terminal $t(P)$ of P are defined as $o(P) = o(y_1)$, and $t(P) = t(y_n)$. There are obvious definitions of subgraphs, circuits, morphisms of graphs and $Aut(X)$, the set of all automorphisms of the graph X which is a group under the composition of morphisms of graphs. For more details, the interested readers are referred to [2], [9], and [10]. We say that a group G acts on a graph X, (or X is a G-graph) if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. Thus, if $g \in G$, and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\overline{y}) = \overline{g(y)}$. The case the actions with inversions are allowed. That is; $g(y) = \overline{y}$ is allowed for some $g \in G$, and $y \in E(X)$. In this case we say that g is an inverter element of G and y is called an inverted edge of X.

If X and Y are G-graphs, and $\mu : V(X) \to V(Y)$ is a map, then μ is called G-map if $\mu(g(x)) = g(\mu(x))$ for all $x \in V(X)$.

Convention. If the group G acts on the graph X and $x \in X$, (x is a vertex or edge), then

1. The stabilizer of x, (or the x group) denoted G_x is defined to be the set $G_x = \{g \in G\colon g(x) = x\}$. It is clear that $G_x \leqslant G$, and if $x \in E(X)$, and $u \in \{o(x),\ t(x)\}$, then $G_{\overline{x}} = G_x$ and $G_x \leqslant G_u$.

2. The orbit of x denoted $G(x)$ and is defined to be the set $G(x) = \{g(x) : g \in G\}$. It is clear that G acts on the graph X without inversions if and only if $G(\overline{e}) \neq G(e)$ for any $e \in E(X)$.

3. The set of the orbits G/X of the action of G on X is defined as $G/X = \{G(x) : x \in X\}$. G/X forms a graph called the quotient graph of the action of G on X, where $V(G/X) = \{G(v) : v \in V(X)\}$, $E(G/X) = \{G(e) : e \in E(X)\}$, and if $e \in E(X)$, then $o(G(e)) = G(o(e))$, $t(G(e)) = G(t(e))$, and $\overline{G(e)} = G(\overline{e})$. The map $p : X \to G/X$ given by $p(x) = G(x)$ is an onto morphism of graphs. If X is connected, then G/X is connected.

4. The set of elements of X fixed by G is the set $X^G = \{x \in X : G_x = G\}$.

Definition 1. Let G be a group acting on a tree X with inversions and let T and Y be two subtrees of X such that $T \subseteq Y$, and each edge of Y has at least one end in T. Assume that T and Y are satisfying the following.

(i) T contains exactly one vertex from each vertex orbit.

(ii) Y contains exactly one edge y(say) from edge orbit if $G(y) \neq G(\overline{y})$ and exactly one pair x, $\overline{x}$ from each edge orbit if $G(x) = G(\overline{x})$. Then

(1) T is called a tree of representatives for the action of G on X,

(2) Y is called a transversal for the action of G on X.

For simplicity we say that $(T; Y)$ is a fundamental domain for the action of G on X.

For the existence of fundamental domains we refer the readers to [5]. For the rest of this section, G is a group acting on a tree X with inversions, and $(T; Y)$ is a fundamental domain for the action of G on X.

The properties of T and Y imply the following that for any $v \in V(X)$ there exists a unique vertex denoted v^* of T and an element g (not unique) of G such that $g(v^*) = v$; that is, $G(v^*) = G(v)$. Moreover, if $v \in V(T)$, then $v^* = v$.

Definition 2. For each $y \in E(Y)$, let $[y]$ be an element of G chosen as follows.

(a) if $o(y) \in V(T)$, then $[y]((t(y))^*) = t(y)$, $[y] = 1$ in case $y \in E(T)$, and $[y](y) = \overline{y}$ if $G(y) = G(\overline{y})$,

(b) if $t(y) \in V(T)$, then $[y](o(y)) = (o(y))^*$, $[y] = [\overline{y}]^{-1}$ if $G(y) \neq G(\overline{y})$, and $[y] = [\overline{y}]$ if $G(y) = G(\overline{y})$.

Proposition 1. *G is generated by G_v and $[e]$, where v runs over $V(T)$ and e runs over $E(Y)$.*

Proof. See Lemma 4.4 of [9]. □

The proof of the following proposition is clear.

Proposition 2. *For each edge $y \in E(Y)$,let $[y][\overline{y}] = \delta_y$. Then $\delta_y = 1$ if $G(y) \neq G(\overline{y})$, and $\delta_y = [y]^2 \in G_y$ if $G(y) = G(\overline{y})$. Moreover $[y] \notin G_{(t(y))^*}$, if $y \notin E(T)$.*

Definition 3. For each $y \in E(Y)$, let $+y$ be the edge $+y = y$ if $o(y) \in V(T)$, and $+y = [y](y)$ if $t(y) \in V(T)$.

It is clear that if $G(y) = G(\overline{y})$ or $y \in E(T)$, then $G_{+y} = G_y$. Furthermore, if x and y are two edges of Y such that $+x = +y$, then $x = y$ or $x = \overline{y}$.

Definition 4. By a word w of G we mean an expression of the form $w = g_0$, $g_0 \in G_v$, $v \in V(T)$, or, $w = g_0.y_1.g_1...y_n.g_n$, $n > 0$, $y_i \in E(Y)$ for $i = 1, \ldots, n$ such that the following hold.

(1) $g_0 \in G_{\ (o(y_1))^*}$,

(2) $(t(y_i))^* = (o(y_{i+1}))^*$, for $i = 1, 2, \ldots, n-1$,

(3) $g_i \in G_{\ (t(y_i))^*}$, for $i = 1, 2, \ldots, n$.

We define $o(w) = (o(y_1))^*$ and $t(w) = (t(y_n))^*$. If $o(w) = t(w) = v$, then w is called a closed word of G.

We have the following concepts related to the word w defined above.

(i) The value of w is denoted by $[w]$ and defined to be the element of

$$[w] = g_0[y_1]g_1...[y_n]g_n \text{ of } G.$$

(ii) w reduced if either $n = 0$ and $g_0 \neq 1$, or else $n > 0$ and w contains no subword of the following forms:

$$y_i.g_i.\overline{y}_i \text{ if } g_i \in G_{+(y_i)}, \text{and } +y_{i+1} = +(\overline{y}_i),\ i = 1, \ldots n.$$

(iii) For each $i, i = 1, \ldots, n$, let $w_i = g_0.y_1.g_1...y_{i-1}.g_{i-1}$with convention $w_1 = g_0$.

Definition 5. For $g \in G$ and $e \in E(Y)$ let $[g; e]$ be the ordered pair $[g; e] = (gG_{+e}; +e)$.

Remark 1. If w is a reduced word of G and $y \in E(Y)$, no confusion will be confused by $[w]$, the value of w, and the ordered pair $[[w];y]$.

Proposition 3. *Let $w = g_0.y_1.g_1...y_n.g_n$ and $w' = h_0.x_1.h_1...x_m.h_m$ be two reduced words of G such that $o(w) = o(w')$, $t(w) = t(w')$, and $[w] = [w']$. Then $m = n$ and, $[[w_i];y_i] = [[w_i'];x_i]$ for $i = 1,\dots,n$.*

Proof. We have $[w'][w]^{-1} = 1$. Let $\widetilde{w} = g_n^{-1}\delta_{y_n}^{-1}.\overline{y}_n...g_1^{-1}\delta_{y_1}^{-1}.\overline{y}_1.g_0^{-1}$.

It is clear that $\widetilde{w}$ is a reduced word of G and $[\widetilde{w}] = [w]^{-1}$. Then $w_0 = \widetilde{w}w' = g_n^{-1}\delta_{y_n}^{-1}.\overline{y}_n.....g_1^{-1}\delta_{y_1}^{-1}.\overline{y}_1.g_0^{-1}h_0.x_1.h_1.....x_m.h_m$ is a word of G.

For each $i = 0,1,\dots,n$, let

$$L_i = g_i^{-1}\delta_{y_i}^{-1}\left[\overline{y}_i\right].....g_1^{-1}\delta_{y_1}^{-1}\left[\overline{y}_1\right]g_0^{-1}h_0\left[x_1\right]h_1...\left[x_i\right]h_i$$

with convention that $L_0 = g_0^{-1}h_0$. Since $[y]\left[\overline{y}\right] = \delta_y$ for every $y \in E(Y)$, therefore $L_i = g_i^{-1}\left[y_i\right]^{-1}.....g_1^{-1}\left[y_1\right]^{-1}g_0^{-1}h_0\left[x_1\right]h_1...\left[x_i\right]h_i$. Moreover, $L_i = g_i^{-1}\left[y_i\right]^{-1}L_{i-1}\left[x_i\right]h_i$. Since $[w_0] = 1$, the identity element of G, therefore by Corollary 1 of [8], w_0 is not reduced. Since $\widetilde{w}$ and w' are reduced, the only way that the indicated word w_0 can fail to be reduced is that $m = n$, and for $i = 1,\dots,n$, $+x_i = +\overline{\overline{y_i}} = +y_i$ and $L_{i-1} \in G_{+(x_i)} = G_{+(y_i)}$.

The case $L_{i-1} \in G_{+(x_i)} = G_{+(y_i)}$ implies that $[w_i]^{-1}[w_i'] \in G_{+(\overline{x_i})} = G_{+(\overline{y_i})}$. Then $[w_i]G_{+(y_i)} = [w_i']G_{+(x_i)}$. Consequently $[[w_i];y_i] = [[w_i'];x_i]$, $i = 1,\dots,n-1$. This completes the proof. □

2. Fibers of groups acting on trees

We begin some general background taken from ([2], p. 78).

Definition 6. Let H be a subgroup of the group G and H acts on the set X. Define $\equiv$ to be the relation on $G \times X$ defined as $(f,u) \equiv (g,v)$, if there exists $h \in H$ such that $f = gh$ and $u = h^{-1}(v)$. It is easy to show that $\equiv$ is an equivalence relation on $G \times X$. The equivalence class containing (f,u) is denoted by $f \otimes_H u$. Thus, $f \otimes_H u = \left\{(fh, h^{-1}(u)) : h \in H\right\}$.

Consequently, if $f \otimes_H u = g \otimes_H v$, then $f = gh$ and $u = h^{-1}(v)$, $h \in H$. So $f \otimes_H u = fh \otimes_H h^{-1}(u)$ for all $h \in H$.

Let $g \in G$ and $A \subseteq H$. Define $g \otimes_H A = \{g \otimes_H a : a \in A\}$, and

$$G \otimes_H X = \{g \otimes_H x : g \in G, x \in X\}.$$

It is clear that $1 \otimes_H x = h \otimes_H x$ for all $h \in H_x$, the stabilizer of x under the action of H on X. It is easy to show that the rule $f(g \otimes_H x) = fg \otimes_H x$ for all $f, g \in G$, and all $x \in X$ defines an action of G on $G \otimes_H X$. The stabilizer $G_{g \otimes_H} x$ of $g \otimes_H x$ under the action of G on $G \otimes_H X$ is $G_{g \otimes_H} x = gH_x g^{-1}$ and the orbit $G(g \otimes_H x)$ of $g \otimes_H x$ under the action of G on $G \otimes_H X$ is $G \otimes_H H(x)$ where $H(x)$ is the orbit of x under the action of H on X.

Remark 2. $x \in X$ means x is a vertex or an edge of X.

Definition 7. Let G be a group acting on a tree X and $(T; Y)$ be a fundamental domain for the action of G on X. For each $v \in V(T)$, let X_v be a tree on which G_v acts; (X_v could consist of the single vertex $\{v\}$) and let $\widehat{X}$ be the set $\widehat{X} = \{[g; e] : g \in G, e \in E(Y)\}$, and $\widetilde{X}$ be the set $\widetilde{X} = \widehat{X} \cup (\underset{v \in V(T)}{\cup} (G \otimes_{G_v} X_v))$.

The following lemma is a generalization of Corollary 4.9 of ([2], p. 18) and is essential for the proof of the main result of this section.

Lemma 1. *Let G be a group acting on a tree X and H be a finite subgroup of G such that H contains no inverter elements of G. Then H is in G_v for some $v \in V(X)$.*

Proof. If G acts on X without inversions, then G contains no inverter elements and by ([2], p. 18) H is in G_v for some v∈V(X). Let G act on X with inversions and $g \in H$ be an inverter element. Then $g(e) = \overline{e}$ for some $e \in E(X)$. This implies that $g(o(e)) = t(e)$. Now we show that $g \notin G_v$ for any $v \in V(X)$. If $g \in G_v$, then there is a unique reduced path $e_1, e_2, ..., e_n$ in X joining $o(e)$ and v. Then $g(e_1), g(e_2), ..., g(e_n)$ is a unique reduced path in X joining $g(o(e)) = t(e)$ and $g(v) = v$. Then $\overline{e}, g(e_1), g(e_2), ..., g(e_n)$ is a path in X joining $g(o(e)) = t(e)$ and $g(v) = v$ but not reduced because X is a tree. Therefore $e = g(e_1)$ and $g(e_2), ..., g(e_n)$ is a reduced path in X joining $t(e)$ and v. Thus, the vertices $t(e)$ and v are joined in X by two distinct reduced paths. This contradicts the assumption that X is tree. This completes the proof. □

Remark 3. In Lemma 1 if $g \in G$ and $e \in E(X)$ such that $g(e) = \overline{e}$, then $g^2(e) = g(\overline{e}) = \overline{g(e)} = \overline{\overline{e}} = e$. This implies that $g \notin G_e$ and $g^2 \in G_e$.

If $G_e = \{1\}$, then the subgroup $H = \{1, g\}$ is finite, but H is not contained in G_e for any $v \in V(X)$.

Theorem 1. *Let G be a group acting on a tree X and $(T; Y)$ be a fundamental domain for the action of G on X. For each $v \in V(T)$, let X_v be a tree on which G_v acts such that for each $e \in E(X)$, $o(e) \in V(T)$, the stabilizer G_e is in a vertex stabilizer $(G_{o(e)})_w$, $w \in V(X_{o(e)})$.*

Then $\widetilde{X}$ forms a tree and G acts on $\widetilde{X}$. Furthermore, if G acts on X with inversions, or for some $v \in V(T), G_v$ acts on X_v with inversions, then G acts on $\widetilde{X}$ with inversions.

Proof. For each edge $e \in E(Y)$ it is clear that $o(+e) = (o(e))^* \in V(T)$ and $G_{+y} \leqslant G_{(o(y))^*}$. By assumption there exists a vertex denoted v_e such that $v_e \in V(X_{o(e)})$ and $G_e \leqslant (G_{o(e)})_{v_e}$, where $(G_{o(e)})_{v_e}$ is the vertex stabilizer of the vertex v_e under the action of $G_{o(e)}$ on $X_{o(e)}$. Now we show that $\widetilde{X}$ forms a graph. The set of vertices $V(\widetilde{X})$ of $\widetilde{X}$ is defined to be the set $V(\widetilde{X}) = \underset{v\in V(T)}{\cup}(G \otimes_{G_v} V(X_v))$ and the set of edges $E(\widetilde{X})$ of $\widetilde{X}$ is defined to be the set $E(\widetilde{X}) = \widehat{X} \cup (\underset{v\in V(T)}{\cup}(G \otimes_{G_v} E(X_v))$. It is clear that $V(\widetilde{X}) \neq \phi$ and $V(\widetilde{X}) \cap E(\widetilde{X}) = \phi$. The ends and the inverses of the edges of $\widetilde{X}$ are defined as follows. Let $g \in G$, $v \in V(T)$, and $e \in E(X_v)$.

Define the ends and the inverse of the edge $g \otimes_{G_v} e$ as follows.

$$t(g \otimes_{G_v} e) = g \otimes_{G_v} t(e), o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e) \text{ and } \overline{g \otimes_{G_v} e} = g \otimes_{G_v} \overline{e},$$

where $t(e), o(e)$, and $\overline{e}$ are the ends and the inverse of the edge e in X_v.

If $e \in E(Y)$, we define the ends and the inverse of the edge $[g; e]$ as follows. $o[g; e] = g \otimes_{G_{(o(e))^*}} v_e, t[g; e] = g[e] \otimes_{G_{(t(e))^*}} v_{\overline{e}}$ and $\overline{[g; e]} = [g[e]; \overline{e}]$. Then $\overline{\overline{[g; e]}} = [g[e][\overline{e}]; \overline{\overline{e}}] = [g; e]$ because $[e][\overline{e}] \in G_{+e}$. These definitions show that $\widetilde{X}$ forms a graph. For $g \in G$ and $v \in V(T)$, let $g \otimes_{G_v} X_v = \{g \otimes_v u : u \in X_v\}$. It is clear that the elements of $g \otimes_{G_v} X_v$ are distinct and $g \otimes_{G_v} X_v$ forms a subtree of $\widetilde{X}$, where $V(g \otimes_{G_v} X_v) = g \otimes_{G_v} V(X_v)$ and $E(g \otimes_{G_v} X_v\ ) = g \otimes_{G_v} E(X_v)$. Then $g \otimes_{G_v} X_v = 1 \otimes_{G_v} X_v$, $g \in G_v$. We observe that if $g \in G$, $v \in V(T)$, v_1 and v_2 are two vertices of $V(X_v)$, and $P : e_1, e_2, ..., e_n$ is a reduced path in X_v joining v_1 and v_2 then it is clear that $g \otimes_{G_v} P : g \otimes_{G_v} e_1, g \otimes_{G_v} e_2, ..., g \otimes_{G_v} e_n$ is a reduced path in $g \otimes_{G_v} X_v$ joining the vertices $g \otimes_{G_v} v_1$ and $g \otimes_{G_v} v_2$ of $g \otimes_{G_v} X_v$. We call $g \otimes_{G_v} P$ the reduced path in $g \otimes_{G_v} X_v$ joining the vertices $g \otimes_{G_v} v_1$ and $g \otimes_{G_v} v_2$ in $g \otimes_{G_v} X_v$ induced by the reduced path in X_v joining v_1 and v_2. We note that P could consist of a single vertex. Now we show that $\widetilde{X}$ forms a tree. First we show that $\widetilde{X}$ contains no loops.

For, if $g \in G$ and $e \in E(Y)$ such that $o[g; e] = t[g; e]$, then $g \otimes_{G_{(o(e))^*}} v_e = g[e] \otimes_{G_{(t(e))^*}} v_{\overline{e}}$. This implies that $(o(e))^* = (t(e))^*$ and $[e] \in G_{(o(e))^*}$. If $e \in E(T)$ then $[e] = 1$ and the case $(o(e))^* = (t(e))^*$ implies that $o(e) = t(e)$. So e is a loop. This is impossible because X is a tree. So $e \notin E(T)$ and $[e] \in G_{(o(e))^*}$. This contradicts Proposition 2. If $g \in G$ and $e \in E(X_v)$ such that $t(g \otimes_{G_{(t(e))^*}} e) = o(g \otimes_{G_{(o(e))^*}} e)$, then $g \otimes_{G_{(t(e))^*}} t(e) = g \otimes_{G_{(o(e))^*}} o(e)$.

This implies that $t(e) = o(e)$. So e is a loop in X_v. This contradicts the fact that X_v is a tree. Let $g \in G$ and, u and v be two vertices of T. We need to show that the subtrees $1 \otimes_{G_u} X_u$ and $g \otimes_{G_v} X_v$ of $\widetilde{X}$ are joined by exactly one reduced path in $\widetilde{X}$. By Lemma 2.7 of [7], there exists a reduced word $w = g_0.y_1.g_1.....y_n.g_n$ of G such that $o(w) = u$, $t(w) = v$, and $[w] = g = g_0[y_1]g_1.....[y_n]g_n$. Then $(o(y_1))^* = u$, $(t(y_n))^* = v$, $g_0 \in G_u$, $g_i \in G_{(t(y_i))^*}$, $i = 1, \ldots, n$.

Furthermore, $(t(y_i))^* = (o(y_{i+1}))^*$, and, v_{y_i} and $v_{\overline{y}_{i+1}}$ are in $X_{(o(y_{i+1}))^*}$ for $i = 1, \ldots, n-1$. For $i = 1, \ldots, n$, let $[w_i] = g_0[y_1]g_1.....[y_{i-1}]g_{i-1}$ with convention that $[w_1] = g_0$, and let p_i be the edge $p_i = [[w_i]; y_i]$. Let P_i be the unique reduced path in $[w_{i+1}] \otimes_{G_{(o(y_{i+1}))^*}} X_{(o(y_{i+1}))^*}$ joining the vertices and $[w_{i+1}] \otimes_{G_{(o(y_{i+1}))^*}} v_{\overline{y}_i}$ and $[w_{i+1}] \otimes_{G_{(o(y_{i+1}))^*}} v_{y_{i+1}}$ induced by the unique reduced path in $X_{(o(y_{i+1}))^*}$ joining the vertices $v_{\overline{y}_i}$ and $v_{y_{i+1}}$ for $i = 1, \ldots, n-1$. Let P be the sequence of edges $P : p_1, P_1, p_2, P_2, ..., p_{n-1}, P_{n-1}, p_n$. We need to show that P is a unique reduced path in $\widetilde{X}$ joining the subtrees $1 \otimes_{G_u} X_u$ and $g \otimes_{G_v} X_v$.

$$
\begin{aligned}
o(p_1) &= o[[w_1]; y_1] = o[g_0; y_1] = g_0 \otimes_{G_{(o(y_1))^*}} v_{y_1} \in 1 \otimes_{G_u} X_u, \\
t(p_n) &= t[[w_n]; y_n] = [w_n][y_n] \otimes_{G_{(t(y_n))^*}} v_{\overline{y}_n} = [w_n][y_n]g_n \otimes_{G_{(t(y_n))^*}} v_{\overline{y}_n} \\
&= g \otimes_{G_v} v_{\overline{y}_n} \in g \otimes_{G_v} X_v. \\
t(p_i) &= t[[w_i]; y_i] = [w_i][y_i] \otimes_{G_{(t(y_i))^*}} v_{\overline{y}_i} = [w_i][y_i]g_i \otimes_{G_{(t(y_i))^*}} v_{\overline{y}_i} \\
&= [w_{i+1}] \otimes_{G_{(o(y_{i+1}))^*}} v_{\overline{y}_i} = o(p_i).t(p_i) = [w_{i+1}] \otimes_{G_{(o(y_{i+1}))^*}} v_{y_{i+1}} \\
&= o(p_{i+1}).
\end{aligned}
$$

Thus, P is a path in $\widetilde{X}$ joining the subtrees $1 \otimes_{G_u} X_u$ and $g \otimes_{G_v} X_v$. Now we show that P is reduced. Since the paths $p_1, p_2, ..., p_{n-1}$ are reduced and $Y \cap X_z = \phi$ for all $z \in V(T)$, we need to show that $p_{i+1} \neq \overline{p}_i$ for $i = 1, \ldots, n-1$. For if $p_{i+1} = \overline{p}_i$, then $[g_0[y_1]g_1.....[y_i]g_i; y_{i+1}] = [g_0[y_1]g_1.....[y_{i-1}]g_{i-1}; \overline{y}_i]$.

This implies that $g_i G_{+y_{i+1}} = G_{+(y_i)}$ and $+y_{i+1} = +(\overline{y}_i)$. So $g_i \in G_{+y_{i+1}}$.

This contradicts above that w is a reduced word of G. Hence P is a reduced path in $\widetilde{X}$ joining the vertices $1 \otimes_{G_{(o(y_1))^*}} v_{y_1}$ and $g \otimes_{G_v} v_{\overline{y}_i}$.

Now we show that P is unique.

Let $Q : q_1, Q_1, q_2, Q_2, ..., q_{m-1}, Q_{m-1}, q_m$ be a reduced path in $\widetilde{X}$ joining the vertices $1 \otimes_{G_{(o(y_1))^*}} v_{y_1}$ and $g \otimes_{G_v} v_{\overline{y}_i}$, where $q_j = [a_j; x_j]$, $a_j \in G$, $x_j \in E(Y)$, $j = 1, \ldots, m$, and Q_i is defined similarly as P_i above. We need to show that $Q = P$. We have $o[a_1; x_1] = 1 \otimes_{G_u} v_{y_1}$, $t[a_i; x_i] =$

$o[a_{i+1};x_{i+1}]$, $[a_{i+1};x_{i+1}] \neq \overline{[a_i;x_i]}$ for $i = 1,\ldots,n-1$, and $t[a_m;x_m] = g \otimes_{G_v} v_{\overline{y}_n}$. This implies that $a_1 \otimes_{G_{(o(x_1))^*}} v_{x_1} = 1 \otimes_{G_u} v_{y_1}$, $a_i[x_i] \otimes_{G_{(t(x_i))^*}} v_{\overline{x}_i} = a_{i+1} \otimes_{G_{(o(x_{i+1}))^*}} v_{x_{i+1}}$, $a_{i+1}G_{+x_{i+1}} \neq a_i[x_i]G_{+x_i}$ or $x_{i+1} \neq +\overline{x}_i$, and $a_m[x_m] \otimes_{G_{(t(x_m))^*}} v_{\overline{x}_m} = g \otimes_{G_v} v_{\overline{y}_n}$. Consequently $(o(x_1))^* = u$, $(t(x_i))^* = (o(x_{i+1}))^*$, $(t(x_m))^* = v$, $a_1 = h_0 \in G_u$, $a_{i+1} = a_i[x_i]h_i, h_i \in G_{(t(x_i))^*}$ and $g = a_m[x_m]h_m, h_m \in G_v$. We get the word $w\prime = h_0.x_1.h_1.....x_m.h_m$ such that $o(w\prime) = u, t(w\prime) = v$, and $[w\prime] = g$. $w\prime$ is reduced because $x_{i+1} \neq +\overline{x}_i$ or $h_i \notin G_{+x_i}$. By Proposition 3 we have $m = n$ and $[[w_i];y_i] = [[w\prime_i];x_i]$, $i = 1,\ldots,n-1$. So $Q = P$. Consequently $\widetilde{X}$ forms a tree. If G acts on X with inversions, then there exists $y \in E(Y)$ such that $G(y) = G(\overline{y})$ and $[y](y) = \overline{y}$. Then $+y = +\overline{y}$ and $\overline{[1;y]} = [[y];\overline{y}] = [y][1;y]$. So the element $[y]$ transfers the edge $[1;y]$ into its inverse $[[y];y]$. If $v \in V(T)$ and G_v acts on X_v with inversions, there exist $g \in G_v$ and $e \in E(X_v)$ such that $g(e) = \overline{e}$. The definition of $\otimes$ implies that $g \otimes_{G_v} e = 1 \otimes_{G_v} \overline{e}$. Then $g \otimes_{G_v} e = g(1 \otimes_{G_v} e) = 1 \otimes_{G_v} \overline{e} = \overline{1 \otimes_{G_v} e}$. Consequently, G acts on $\widetilde{X}$ with inversions. This completes the proof. □

Corollary 1. *Let G, X, and X_v, $v \in V(T)$ be as in Theorem 1. For each $e \in E(X)$, let G_e be finite and contains no inverter elements of $G_{t(e)}$. Then the conclusions of Theorem 1 hold. Moreover, the mapping $\mu : V(\widetilde{X}) \to V(X)$ given by $\mu(g \otimes_{G_v} w) = g(v)$, for all $w \in X_v$ is surjective, and is a G-map.*

Proof. Since G_e is finite and contains no inverter elements of $G_{t(e)}$, therefore by Lemma 1, there exists a vertex $w \in V(X_{t(e)})$ such that $G_e \leqslant (G_{t(e)})_w$. Then by Theorem 1, G acts on $\widetilde{X}$, and if G acts on X with inversions, or for some $v \in V(T)$, G_v acts on X_v with inversions, then G acts on $\widetilde{X}$ with inversions. Now if $f, g \in G$, and $u, w \in V(X_v)$ such that $f \otimes_{G_v} u = g \otimes_{G_v} w$, then $g^{-1}f \in G_v$. This implies that $g^{-1}f(v) = v$, or equivalently, $f(v) = g(v)$. Then $\mu(f \otimes_{G_v} u) = \mu(g \otimes_{G_v} w)$, and μ is well-defined. If $v \in V(X)$, and $u \in V(X_v)$, then it is clear that $\mu(1 \otimes_{G_v} u) = v$. So μ is surjective. If $f, g \in G$, $v \in V(X)$ and $u \in V(X_v)$, then $\mu(f(g \otimes_{G_v} u)) = \mu(fg \otimes_{G_v} u) = fg(v) = f(\mu(g \otimes_{G_v} u))$. This implies that μ is surjective, and is a G-map. This completes the proof. □

Corollary 2. *Let G, X, and X_v, $v \in V(T)$ be as in Corollary 1. If the stabilizer of each edge of X_v is finite, then the stabilizer of each edge of $\widetilde{X}$ is finite.*

Proof. $E(\widetilde{X}) = \widehat{X} \cup (\bigcup_{v \in V(T)} (G \otimes_{G_v} E(X_v)))$. Let $g \in G$, $v \in V(T), p \in E(X_v)$, and $e \in E(Y)$. It is clear that the stabilizer $G_{g \otimes_{G_v} p}$ of the edge

$g \otimes_{G_v} p$ under the action of G on $\widetilde{X}$ is $G_{g\otimes_{G_v}p} = g(G_v)_p g^{-1}$, where $(G_v)_p$ is the stabilizer of the edge p under the action of G_v on X_v. Since $(G_v)_p$ is finite, therefore $G_{g\otimes_{G_v}p}$ is finite. Similarly, that the stabilizer $G_{[g;e]}$ of the edge $[g;e]$ under the action of G on $\widetilde{X}$ is $G_{[g;e]} = gG_{+e}g^{-1}$. This completes the proof. □

Now we end this section the following definition.

Definition 8. Let G be a group acting on a tree X and $(T;Y)$ be a fundamental domain for the action of G on X. For each $v \in V(T)$, let X_v be a tree on which G_v acts, and for each $e \in E(Y)$, let G_e be finite and contains no inverter elements of $G_{t(e)}$. Then $\widetilde{X}$ is called a fibered G-tree of base X and fibers X_v, $v \in V(T)$.

3. Accessibility of groups acting on trees

For the study of the concepts of the ends of groups we refer the readers to ([1], p. 17), or ([2], p. 124, 126), or ([11], p. 171).

The number of the ends of a group G is denoted by $e(G)$.

A finitely generated group G is called accessible on the tree X if G acts on X and satisfies the following.

1. $X^G = \phi$,
2. G_e is finite for any $e \in E(X)$,
3. $e(G_v) \leqslant 1$ for all $v \in V(X)$.

A group is G called accessible if there exists a tree X on which G is accessible on X.

If G is an accessible group on the tree X, then by Proposition 7.4 ([2], p. 132), there exists a tree X' such that G acts on X' and G is not accessible on X'. In this case we say that G is inaccessible.

The main result of this section is the following theorem.

Theorem 2. *Let G be a group acting on the tree X such that for each edge e of X, G_e is finite and contains no elements of $G_{t(e)}$, and for each vertex v of X, G_v is an accessible, and the quotient graph G/X is finite. Then G is an accessible group, and G is inaccessible on X.*

Proof. The accessibility of G_v, $v \in V(X)$ implies that G_v is finitely generated. Since the quotient graph G/X is finite, therefore similar to the proof of Theorem 4.1 of [2, p. 15], we can show that G is finitely generated. Let $(T;Y)$ be a fundamental domain for the action of G on X. Then there exists a tree X_v on which G_v acts such that $X_v^{G_v} = \phi$,

$(G_v)_y$ is finite for every $y \in E(X_v)$, and $e(G_v) \leqslant 1$. The condition G_e is finite and contains no inverter elements of $G_{t(e)}$, $e \in E(Y)$ implies that G acts on the fiber tree $\widetilde{X}$. If $g \in G$ and $u \in V(X_v)$ such that $G_{g \otimes_{G_v} u} = g(G_v)_u g^{-1} = G$, then $(G_v)_u = G_v$. This contradicts the condition that $X_v^{G_v} = \phi$. So $\widetilde{X}^G = \phi$. If $e \in E(Y)$ and $p \in E(X_v), v \in V(X)$, then G_e and $(G_v)_p$ are finite. Then for every $g \in G$, $G_{[g;e]} = gG_{+e}g^{-1}$ and $G_{g \otimes_{G_v} p} = g(G_v)_p g^{-1}$ are finite. For $g \in G$, $v \in V(T)$ and $u \in V(X_v)$, $e(G_{g \otimes u}) = e(g(G_v)_u g^{-1}) = e((G_v)_u) \leqslant 1$. This implies that G is accessible on $\widetilde{X}$. Consequently G is accessible. If G is accessible on X, then for every $v \in V(T)$, $e(G_v) \leqslant 1$. Since G_v is accessible, then by Theorem 6.10 of ([2], p. 128), $e(G_v) \geqslant 2$. Contradiction. So G is inaccessible on X. This completes the proof. $\square$

Now we apply Theorem 2 to tree product of groups $A = \prod_{i \in I}^{*}(A_i; U_{ij} = U_{ji})$ of the groups A_i, $i \in I$, with amalgamation subgroups U_{ij}, $i, j \in I$ introduced in [3], and to a new class of groups called quasi-HNN groups introduced in [4], and defined as follows.

Let G be a group, I and J be two indexed sets such that $I \cap J = \phi$ and $I \cup J \neq \phi$. Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$, and $\{C_j : j \in J\}$ be families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism and for each $j \in J$, let $\alpha_j : C_j \to C_j$ be an automorphism such that α_j^2 is an inner automorphism determined by $c_j \in C$ and c_j is fixed by α_j; that is, $\alpha_j(c_j) = c_j$ and $\alpha_j^2(c) = c_j c c_j^{-1}$ for all $c \in C_j$.

The group G^* of the presentation

$$\Big\langle gen(G), t_i, t_j \mid rel(G), t_i a t_i^{-1} = \phi_i(a),\, t_j c t_j^{-1} = \alpha_j(c),\\ t_j^2 = c_j,\, a \in A_i,\, c \in C_j \Big\rangle,$$

where $i \in I, j \in J$, or simply,

$$G^* = \Big\langle gen(G), t_i, t_j \mid rel(G), t_i A_i t_i^{-1} = B_i,\, t_j C_j t_j^{-1} = C_j,\\ t_j^2 = c_i,\ i \in I, j \in J \Big\rangle$$

is called a quasi HNN group of base H and associated pairs (A_i, B_i), and (C_j, C_j) of subgroups of G.

The tree product $A = \prod_{i \in I}^{*}(A_i; U_{ij} = U_{ji})$ of the groups $A_i, i \in I$, acts on the tree X without inversions defined as follow.

$$V(X) = \{(gA_i, i) : g \in A, i \in I\},\\ \text{and } E(X) = \{(gU_{ij}, ij) : g \in A, i, j \in I\}.$$

If y is the edge $y = (gU_{ij}, ij)$, then $o(y) = (gA_i, i), t(y) = (gA_j, j)$, and $\overline{y} = (gU_{ji}, ji)$. A acts on X as follows.

Let $f \in A$. Then $f((gA_i, i)) = (fgA_i, i)$ and $f((gU_{ij}, ij)) = (fgU_{ij}, ij)$.

If $v = (gA_i, i) \in V(X)$ and $y = (gU_{ji}, ij) \in E(X)$, then the stabilizer of v is $A_v = gA_ig^{-1} \cong A_i$, a conjugate of A_i, and then the stabilizer of y is $A_y = gU_{ij}g^{-1} \cong U_{ij}$, a conjugate of U_{ij}. The orbit of v is $A(v) = \{(agA_i, i) : a \in A, i \in I\}$, and the orbit of y is $A(y) = \{(agU_{ij}, ij) : a \in A, i, j \in I\}$.

So the quotient graph A/X is finite if I is finite. This leads the following proposition as an application to Theorem 2.

Proposition 4. *Let $A = \prod^*_{i \in I}(A_i; U_{ij} = U_{ji})$ be a tree product of the groups A_i, $i \in I$, such that A_i is accessible, and U_{ij} is finite and contains no inverter element of A_i for all i, $j \in I$. If I is finite, then A is accessible.*

A free product of groups with amalgamated subgroup is a special case of tree product of the groups, we state the following corollary of Proposition 4.

Corollary 3. *Let $A = *_c A_i$, $i \in I$, be the free product of the groups A_i, $i \in I$ with amalgamation subgroup C such that A_i is accessible, and C is finite and contains no inverter element of A_i for all $i, j \in I$. If I is finite, then A is accessible.*

It is shown in [6] that the quasi-HNN group

$$G^* = \Big\langle gen(G), t_i, t_j \mid rel(G), t_iA_it_i^{-1} = B_i,\ t_jC_jt_j^{-1} = C_j,$$
$$t_j^2 = c_i,\ i \in I, j \in J \Big\rangle$$

acts on the tree X with inversions defined as follow.

$$V(X) = \{gG : g \in G^*\},\ \text{and}\ E(X) = \{(gB_i, t_i), (gA_i, t_i), (gC_j, t_j)\},$$

where $g \in G^*, i \in I$,and $j \in J$. For the edges $(gB_i,\ t_i), (gA_i, t_i)$, and (gC_j, t_j), $i \in I, j \in J$, define $o(gB_i,\ t_i) = o(gA_i, t_i) = o(gC_j, t_j) = gG$, $t(gB_i, t_i) = gt_iG, t(gA_i, t_i) = gt_i^{-1}G$, and $t(gC_j, t_j) = gt_jG$, and $\overline{(gB_i, t_i)} = (gt_iA_i, t_i^{-1})$, $\overline{(gA_i, t_i^{-1})} = (gt_i^{-1}B_i, t_i)$, and $\overline{(gC_j, t_j)} = (gt_jC_j, t_j)$.

G^* acts on X as follows. Let $f \in G^*$. Then for the vertex gG and the edges $(gB_i,\ t_i), (gA_i, t_i^{-1})$, and (gC_j, t_j) of X, define $f(gG) = fgG$, $f(gB_i, t_i) = (fgB_i, t_i)$, $f(gA_i, t_i^{-1}) = (fgA_i, t_i^{-1})$, and $f(gC_j, t_j) = (fgC_j, t_j)$.

The action of G^* on X is with inversions because the element $t_j \in G^*$ maps the edge (C_j, t_j) to its inverse $\overline{(C_j, t_j)}$; that is, $t_j(C_j, t_j) = (t_jC_j, t_j) = \overline{(C_j, t_j)}$

The stabilizer of the vertex $v = gG$ is, $G^*_v = gGg^{-1}$, a conjugate of G, the stabilizers of the edges (gB_i, t_i), $f(gA_i, t_i^{-1})$, and (gC_j, t_j) are gB_ig^{-1}, conjugates of B_i, gA_ig^{-1}, a conjugate of A_i, and gC_jg^{-1}, a conjugate of C_j respectively, for all $i \in I$, and all $j \in J$.

The orbits of gG, (gB_i, t_i), $f(gA_i, t_i^{-1})$, and (gC_j, t_j) are $\{fG : f \in G^*\}$, $\{(fB_i, t_i) : f \in G^*\}$, and $\{(fC_j, t_j) : f \in G^*\}$. Then the quotient graph $G^*\diagup X$ is finite if $I \cup J$ is finite. This leads the following proposition as an application to Theorem 2.

Proposition 5. *Let G^* be the quasi-HNN group*

$$G^* = \Big\langle gen(G), t_i, t_j \mid rel(G),\ t_iA_it_i^{-1} = B_i,\ t_jC_jt_j^{-1} = C_j,\ t_j^2 = c_i,\ i \in I, j \in J \Big\rangle$$

such that G is accessible, A_i, B_i, and C_j are finite and contain no inverter elements of G. If $I \cup J$ is finite, then G^* is accessible.

By taking $J = \phi$ in the group G^* defined above, yields the the following corollary of Proposition 5.

Corollary 4. *Let G^* be the HNN group*

$$G^* = \Big\langle gen(G), t_i \mid rel(G), t_iA_it_i^{-1} = B_i,\ i \in I \Big\rangle$$

such that G is accessible, A_i, and B_i are finite and contain no inverter elements of G. If I is finite, then G^* is accessible.

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