

Units of some group algebras of groups of order 12 over any finite field of characteristic 3

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ABSTRACT. The structure of the unit groups of the group algebra of the groups D_{12} and $C_3 \rtimes C_4$ over any finite field of characteristic 3 is established.

1. Introduction

Let $\mathcal{U}(RG)$ be the group of units of the group ring RG of the group G over the commutative ring R . It is well known that

$$\mathcal{U}(RG) = V(RG) \times U(R),$$

where $V(RG) = \{\sum_{g \in G} \alpha_g g \in \mathcal{U}(RG) \mid \sum_{g \in G} \alpha_g = 1\}$ is the group of normalized units of RG and $U(R)$ is the group of units of R . For further details and background see Polcino Milies and Sehgal [9].

We are interested in the structure of $\mathcal{U}(KG)$ when of order ap^k where p is a prime, $a, k \in \mathbb{N}_0$ and $(a, p) = 1$. It is well known that $|V(KG)| = |K|^{|G|-1}$ if G is a finite p -group and K is a finite field of characteristic p . A basis for $V(\mathbb{F}_p G)$ is determined in [10], where \mathbb{F}_p is the Galois field of p elements and G is an abelian p -group. Note that several results were obtained in the case where K is a field of characteristic p and G is a non-abelian p -group, see [1, 2, 3] for further details.

In [6], the order of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})$ is determined where \mathbb{F}_{p^k} is the Galois field of p^k -elements, $D_{2p^m} = \langle a, b \mid a^{p^n} = 1, b^2 = 1, a^b = a^{-1} \rangle$ is the

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dihedral group of order $2p^m$ and p is an odd prime. In [4], the structure of $U(\mathbb{F}_{3^k}D_6)$ was determined in terms of split extensions of elementary abelian groups. Additionally in [8], it is shown that $Z(V_1)$ and $V_1/Z(V_1)$ are elementary abelian 3-groups where $V_1 = 1 + J(\mathbb{F}_{3^k}D_6)$, $J(\mathbb{F}_{3^k}D_6)$ is the Jacobson Radical of $\mathbb{F}_{3^k}D_6$ and $Z(V_1)$ is the center of V_1 . The structure of FA_4 is established in [11] where F is any finite field and A_4 is the alternating group of degree 4. Our main result is the following:

Theorem 1. *Let $V(FG)$ be the group of normalized units of the group algebra FG of a group G over a finite field F of 3^k elements. The following conditions hold:*

(i) *If $G = \langle x, y \mid x^6 = y^2 = 1, \quad xy = x^{-1} \rangle \cong D_{12}$, then*

$$V(FG) \cong (C_3^{6k} \rtimes C_3^{2k}) \rtimes C_{3^{k-1}}^3.$$

(ii) *If $G = \langle x, y \mid x^4 = y^3 = 1, \quad yxy = x \rangle \cong C_3 \rtimes C_4$, then*

$$V(FG) \cong \begin{cases} (C_3^{6k} \rtimes C_3^{2k}) \rtimes C_{3^{k-1}}^3 & \text{if } 4 \mid (3^k - 1) \\ (C_3^{6k} \rtimes C_3^{2k}) \rtimes (C_{3^{k-1}} \times C_{3^{2k-1}}) & \text{if } 4 \nmid (3^k - 1). \end{cases}$$

The next two results can be found in [7].

Theorem 2. *Let F be an arbitrary field of characteristic $p > 0$, let G be a p -solvable group and c be the sum of all p -elements of G including 1, then*

$$J(FG) = \text{lann}(c)$$

where $J(FG)$ is the Jacobson Radical of FG and $\text{lann}(c)$ is the left annihilator of c .

Theorem 3. *Let N be a normal subgroup of G such that G/N is p -solvable. If $|G/N| = np^a$ where $(n, p) = 1$, then*

$$J(FG)^{p^a} \subseteq FG \cdot J(FN) \subseteq J(FG)$$

where F is a field of characteristic $p > 0$. In particular, if G is p -solvable of order np^a where $(n, p) = 1$, then

$$J(FG)^{p^a} = 0.$$

Denote by $\hat{g} = \sum_{h \in \langle g \rangle} h \in RG$.

2. Proof of Main Theorem

Case (i). Let $G = D_{12}$. Consider the ring homomorphism $\theta : \mathbb{F}_{3^k} D_{12} \rightarrow \mathbb{F}_{3^k}(C_2 \times C_2)$ given by

$$\sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3 + a_{i+6}y + a_{i+9}x^3y) \mapsto \sum_{i=0}^2 (a_i + a_{i+3}\bar{x} + a_{i+6}\bar{y} + a_{i+9}\bar{x}\bar{y}).$$

where $C_2 \times C_2 = \langle \bar{x}, \bar{y} \mid \bar{x}^2 = \bar{y}^2 = 1, \bar{x}\bar{y} = \bar{y}\bar{x} \rangle$ and $\alpha_i \in \mathbb{F}_{3^k}$.

We can now construct the group homomorphism $\theta' : \mathcal{U}(\mathbb{F}_{3^k} D_{12}) \rightarrow \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$, since units get mapped to units. We define the group epimorphism $\phi : \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2)) \rightarrow \mathcal{U}(\mathbb{F}_{3^k} D_{12})$ given by $a + b\bar{x} + c\bar{y} + d\bar{x}\bar{y} \mapsto a + bx^3 + cy + dx^3y$ where $a, b, c, d \in \mathbb{F}_{3^k}$. Clearly $\theta' \circ \phi = 1$ and therefore $\mathcal{U}(\mathbb{F}_{3^k} D_{12}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$ where $H = \ker(\theta')$. Let

$$\alpha = \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3 + a_{i+6}y + a_{i+9}x^3y) \in \mathbb{F}_{3^k} D_{12},$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Then $\alpha \in H$ iff

$$\sum_{i=0}^2 a_i = 1, \sum_{j=0}^2 a_{j+3} = \sum_{k=0}^2 a_{k+6} = \sum_{l=0}^2 a_{l+9} = 0.$$

Lemma 1. H has exponent 3.

Proof. D_{12} is solvable and hence p -solvable. Clearly $|D_{12}| = 4 \cdot 3$ and by Theorem 3, $J(\mathbb{F}_{3^k} D_{12})^3 = 0$ and $1 + J(\mathbb{F}_{3^k} D_{12})$ has exponent 3. Now by Theorem 2, $J(\mathbb{F}_{3^k} D_{12}) = \{\alpha \in \mathbb{F}_{3^k} D_{12} \mid \alpha x^2 = 0\}$. Let

$$\alpha = \sum_{i=0}^2 x^{2i} [\alpha_{i+1} + \alpha_{i+4}x^3 + \alpha_{i+7}y + \alpha_{i+10}x^3y] \in \mathcal{U}(\mathbb{F}_{3^k} D_{12}),$$

then

$$\alpha \in J(\mathbb{F}_{3^k} D_{12}) \iff \sum_{i=1}^3 \alpha_i = \sum_{j=1}^3 \alpha_{j+3} = \sum_{l=1}^3 \alpha_{l+6} = \sum_{m=1}^3 \alpha_{m+9} = 0$$

where $\alpha_i \in \mathbb{F}_{3^k}$. Therefore $H \cong 1 + J(\mathbb{F}_{3^k} D_{12})$. \square

Lemma 2. *Every element of the centralizer of x^2 in H ($C_H(x^2)$) has the form $\sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Also $C_H(x^2) \cong C_3^{6k}$.*

Proof. $C_H(x^2) = \{h \in H \mid hx^2 = x^2h\}$. Let $h = \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3 + \alpha_{i+6}y + \alpha_{i+9}x^3y) \in H$ where $\alpha_i \in \mathbb{F}_{3^k}$, $\sum_{i=0}^2 \alpha_i = 1$ and $\sum_{i=0}^2 \alpha_{i+3} = \sum_{i=0}^2 \alpha_{i+6} = \sum_{i=0}^2 \alpha_{i+9} = 0$.

$$\begin{aligned} (hx^2 - x^2h) &= \left(\sum_{i=0}^2 \alpha_{i+6}x^{2i}y + \sum_{i=0}^2 \alpha_{i+9}x^{2i+3}y \right) x^2 - \\ &\quad - x^2 \left(\sum_{i=0}^2 \alpha_{i+6}x^{2i}y + \sum_{i=0}^2 \alpha_{i+9}x^{2i+3}y \right) \\ &= \sum_{i=0}^2 \alpha_{i+6}x^{2i+4}y + \sum_{i=0}^2 \alpha_{i+9}x^{2i+1}y - \sum_{i=0}^2 \alpha_{i+6}x^{2i+2}y - \sum_{i=0}^2 \alpha_{i+9}x^{2i+5}y \\ &= \sum_{i=0}^2 [\alpha_{i+6}x^{2i+4}y - \alpha_{i+6}x^{2i+2}y] + \sum_{i=0}^2 [\alpha_{i+9}x^{2i+1}y - \alpha_{i+9}x^{2i+5}y] \\ &= (\alpha_6 - \alpha_7)\widehat{x^2}y + (\alpha_9 - \alpha_{10})x^3\widehat{x^2}y. \end{aligned}$$

Therefore $h \in C_H(x^2)$ if and only if $\alpha_6 = \alpha_7 = \alpha_8$ and $\alpha_9 = \alpha_{10} = \alpha_{11}$. Hence every element of $C_H(x^2)$ has the form $\sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. It remains to show that $C_H(x^2)$ is abelian. Let $\alpha = \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y \in C_H(x^2)$ and $\beta = \sum_{i=0}^2 x^{2i}(\beta_i + \beta_{i+3}x^3) + (\beta_6 + \beta_7x^3)\widehat{x^2}y \in C_H(x^2)$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$, $\sum_{i=0}^2 \beta_i = 1$, $\sum_{i=0}^2 \beta_{i+3} = 0$ and $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$. Note that $(\widehat{x^2})^2 = 0$. Then

$$\begin{aligned} \alpha\beta &= \sum_{i=0}^2 x^{2i}(a_i + a_{i+3}x^3) \sum_{i=0}^2 x^{2i}(b_i + b_{i+3}x^3) \\ &\quad + \sum_{i=0}^2 x^{2i}(a_i + a_{i+3}x^3)(b_6 + b_7x^3)\widehat{x^2}y \\ &\quad + (a_6 + a_7x^3)\widehat{x^2}y \sum_{i=0}^2 (b_i x^{2i} + b_{i+3}x^{2i+3}) \end{aligned}$$

$$\begin{aligned}
& + (a_6 + a_7x^3)\widehat{x^2y}(b_6 + b_7x^3)\widehat{x^2y} \\
& = \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \\
& + \sum_{i=0}^2 (a_i + a_{i+3}x^3) \widehat{x^2}(b_6 + b_7x^3)y \\
& + \sum_{i=0}^2 (b_i + b_{i+3}x^3) \widehat{x^2}(a_6 + a_7x^3)y \\
& = \sum_{i=0}^2 x^{2i} (a_i + a_3x^{2i+3}) \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \\
& + \widehat{x^2}(b_6 + b_7x^3)y + \widehat{x^2}(a_6 + a_7x^3)y \\
& = \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \sum_{i=0}^2 x^{2i} (b_i x^{2i} + b_{i+3}x^{2i+3}) \\
& + ((a_6 + b_6) + (a_7 + b_7)x^3)\widehat{x^2y}
\end{aligned}$$

and

$$\begin{aligned}
\beta\alpha & = \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \\
& + \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) (a_6 + a_7x^3)\widehat{x^2y} \\
& + (b_6 + b_7x^3)\widehat{x^2y} \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \\
& + (b_6 + b_7x^3)\widehat{x^2y}(a_6 + a_7x^3)\widehat{x^2y} \\
& = \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \\
& + \sum_{i=0}^2 (b_i + b_{i+3}x^3) \widehat{x^2}(a_6 + a_7x^3)y \\
& + \sum_{i=0}^2 (a_i + a_{i+3}x^3) \widehat{x^2}(b_6 + b_7x^3)y \\
& = \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3)
\end{aligned}$$

$$\begin{aligned}
 & + \widehat{x^2}(a_6 + a_7x^3)y + \widehat{x^2}(b_6 + b_7x^3)y \\
 & = \sum_{i=0}^2 x^{2i} (b_i + b_{i+3}x^3) \sum_{i=0}^2 x^{2i} (a_i + a_{i+3}x^3) \\
 & + ((a_6 + b_6) + (a_7 + b_7)x^3)\widehat{x^2}y.
 \end{aligned}$$

Let $\gamma = \sum_{i=0}^2 x^{2i} (\beta_i + \beta_{i+3}x^3) \in H$ and $\delta = \sum_{i=0}^2 x^{2i} (\alpha_i + \alpha_{i+3}x^3) \in H$. Clearly $\gamma, \delta \in V(\mathbb{F}_{3^k}C_6)$ and $\gamma\delta = \delta\gamma$ since $V(\mathbb{F}_{3^k}C_6)$ is abelian. Therefore $C_H(x^2)$ is abelian. \square

Lemma 3. *Let S be the subgroup of H consisting of elements of the form $1 + \check{x}(r + r_1x^3)(1 + y)$ where $\check{x} = \sum_{i=0}^2 ix^{2i}$ and $r, r_1 \in \mathbb{F}_{3^k}$. Then $S \cong C_3^{2k}$.*

Proof. Let $s_1 = 1 + \check{x}(r + r_1x^3)(1 + y) \in S$ and $s_2 = 1 + \check{x}(s + s_1x^3)(1 + y) \in S$ where $r, s, r_1, s_1 \in \mathbb{F}_{3^k}$, then

$$\begin{aligned}
 s_1s_2 & = 1 + \check{x}(s + s_1x^3)(1 + y) + \check{x}(r + r_1x^3)(1 + y) \\
 & + \check{x}(r + r_1x^3)(1 + y)\check{x}(s + s_1x^3)(1 + y) \\
 & = 1 + \check{x}((s + r) + (s_1 + r_1)x^3)(1 + y) \\
 & + \check{x}(r + r_1x^3)(1 + y)(1 - y)\check{x}(s + s_1x^3) \\
 & = 1 + \check{x}((s + r) + (s_1 + r_1)x^3)(1 + y) \\
 & + \check{x}(r + r_1x^3)(1 - 1)\check{x}(s + s_1x^3) \\
 & = 1 + \check{x}((s + r) + (s_1 + r_1)x^3)(1 + y).
 \end{aligned}$$

Therefore S is closed. \square

Lemma 4. $H \cong C_H(x^2) \rtimes S$.

Proof. Clearly $H = C_H(x^2).S$ since $C_H(x^2) \cap S = \{1\}$. It remains to show that S normalizes $C_H(x^2)$. Let $s = 1 + \check{x}(r + r_1x^3)(1 + y)$, then $s^{-1} = 1 - \check{x}(r + r_1x^3)(1 + y)$ where $r, r_1 \in \mathbb{F}_{3^k}$. Also let $c = \sum_{i=0}^2 x^{2i} (\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y \in C_H(x^2)$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Note that $\check{x}^2 = \widehat{x^2}\check{x} = 0$. Then

$$\begin{aligned}
 c^s & = \left(1 - \check{x}(r + r_1x^3)(1 + y)\right) \left(\sum_{i=0}^2 x^{2i} (\alpha_i + \alpha_{i+3}x^3) + (\alpha_6 + \alpha_7x^3)\widehat{x^2}y\right) \\
 & \times \left(1 + \check{x}(r + r_1x^3)(1 + y)\right)
 \end{aligned}$$

$$\begin{aligned}
&= (1 - \check{x}(r + r_1x^3)(1 + y)) \left(\sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) \right. \\
&\quad \left. + \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\check{x}(r + r_1x^3)(1 + y)) + (\alpha_6 + \alpha_7x^3)\widehat{x^2y} \right) \\
&= \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\check{x}(r + r_1x^3)(1 + y)) \\
&\quad + (\alpha_6 + \alpha_7x^3)\widehat{x^2y} - \check{x}(r + r_1x^3)(1 + y) \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) \\
&= \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3)(\check{x}(r + r_1x^3)(y)) \\
&\quad + (\alpha_6 + \alpha_7x^3)\widehat{x^2y} - \check{x}(r + r_1x^3)(y) \sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) \\
&= \left(\sum_{i=0}^2 x^{2i}(\alpha_i + \alpha_{i+3}x^3) + ((\alpha_6 + \delta) + (\alpha_7 + \delta)x^3)\widehat{x^2y} \right) \in C_H(x^2)
\end{aligned}$$

where $\delta = r(\alpha_1 - \alpha_2) + r_1(\alpha_4 - \alpha_5)$ and $\alpha_i, r, r_1 \in \mathbb{F}_{3^k}$. \square

Recall that $\mathcal{U}(\mathbb{F}_{3^k}D_{12}) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2))$. Consider $\mathbb{F}_{3^k}(C_2 \times C_2)$.

$$\begin{aligned}
\mathbb{F}_{3^k}(C_2 \times C_2) &\cong (\mathbb{F}_{3^k}C_2)C_2 \\
&\cong (\mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k})C_2 \\
&\cong \mathbb{F}_{3^k}C_2 \oplus \mathbb{F}_{3^k}C_2 \\
&\cong \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k} \oplus \mathbb{F}_{3^k}.
\end{aligned}$$

Thus $\mathcal{U}(\mathbb{F}_{3^k}(C_2 \times C_2)) \cong C_{3^{k-1}}^4$. Therefore

$$\begin{aligned}
\mathcal{U}(\mathbb{F}_{3^k}D_{12}) &\cong [C_3^{6k} \times C_3^{2k}] \rtimes C_{3^{k-1}}^4 \\
&\cong [(C_3^{6k} \times C_3^{2k}) \rtimes C_{3^{k-1}}^3] \times \mathcal{U}(\mathbb{F}_{3^k})
\end{aligned}$$

Case (ii). Let $G \cong C_3 \rtimes C_4$. Define the group epimorphism

$$\tau : \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \longrightarrow \mathcal{U}(\mathbb{F}_{3^k}C_4)$$

given by

$$\sum_{i=0}^2 (\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \mapsto \sum_{i=0}^2 (\alpha_i + \alpha_{i+3}\bar{x} + \alpha_{i+6}\bar{x}^2 + \alpha_{i+9}\bar{x}^3)$$

where \bar{x} generates C_4 .

Define the group homomorphism $\phi : \mathcal{U}(\mathbb{F}_{3^k}C_4) \longrightarrow \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$ by $a + b\bar{x} + c\bar{x}^2 + d\bar{x}^3 \mapsto a + bx + cx^2 + dx^3$ where $a, b, c, d \in \mathbb{F}_{3^k}$. Then $\tau \circ \phi = 1$, therefore $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}C_4)$ where $L \cong \ker(\tau)$. Let

$$l = \sum_{i=0}^2 (\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \in \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)),$$

then $l \in L$ if and only if $\sum_{i=0}^2 \alpha_i = 1$ and

$$\sum_{j=0}^2 \alpha_{j+3} = \sum_{l=0}^2 \alpha_{l+6} = \sum_{m=0}^2 \alpha_{m+9} = 0 \quad (\alpha_i \in \mathbb{F}_{3^k})$$

Therefore $|L| = 3^{8k}$.

Lemma 5. *L has exponent 3.*

Proof. By Theorem 3, $J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))^3 = 0$ and $1 + J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$ has exponent 3. By Theorem 2, $J(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) = \{\alpha \in \mathbb{F}_{3^k}(C_3 \rtimes C_4) \mid \alpha\hat{y} = 0\}$. Therefore $L \cong 1 + J(\mathbb{F}_{3^k}(C_3 \rtimes C_4))$. \square

Lemma 6. *Every element of the centralizer of y in L ($C_L(y)$) has the form $\sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\hat{y}$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Also $C_L(y) \cong C_3^{6k}$.*

Proof. $C_L(y) = \{l \in L \mid ly = yl\}$. Let

$$l = \sum_{i=0}^2 (\alpha_i + \alpha_{i+3}x + \alpha_{i+6}x^2 + \alpha_{i+9}x^3)y^i \in L.$$

where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$, $\sum_{i=0}^2 \alpha_{i+6} = 0$, $\sum_{i=0}^2 \alpha_{i+9} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$.

Then

$$\begin{aligned}
 ly - yl &= \left(\sum_{i=0}^2 (\alpha_{i+3}x + \alpha_{i+9}x^3)y^i \right) y - y \left(\sum_{i=0}^2 (\alpha_{i+3}x + \alpha_{i+9}x^3)y^i \right) \\
 &= \sum_{i=0}^2 (\alpha_{i+3}x + \alpha_{i+9}x^3)y^{i+1} - \sum_{i=0}^2 (\alpha_{i+3}yx + \alpha_{i+9}yx^3)y^i \\
 &= (\alpha_5 - \alpha_4)x\widehat{y} + (\alpha_{11} - \alpha_{10})x^3\widehat{y}.
 \end{aligned}$$

Therefore $l \in C_L(y)$ if and only if $\alpha_3 = \alpha_4 = \alpha_5$ and $\alpha_9 = \alpha_{10} = \alpha_{11}$. Thus every element of $C_L(y)$ is of the form

$$\sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y}$$

where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and $\alpha_i \in \mathbb{F}_{3^k}$. Let $\alpha = \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y} \in C_L(y)$ and $\beta = \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i + (\beta_3x + \beta_7x^3)\widehat{y} \in C_L(y)$ $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$, $\sum_{i=0}^2 \beta_i = 1$, $\sum_{i=0}^2 \beta_{i+3} = 0$ and $\alpha_i, \beta_j \in \mathbb{F}_{3^k}$.

Then

$$\begin{aligned}
 \alpha\beta &= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \\
 &+ \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i (\beta_3x + \beta_7x^3)\widehat{y} \\
 &+ (\alpha_3x + \alpha_7x^3)\widehat{y} \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y} (\beta_3x + \beta_7x^3)\widehat{y} \\
 &= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \\
 &+ \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)\widehat{y} (\beta_3x + \beta_7x^3) + (\alpha_3x + \alpha_7x^3) \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)\widehat{y} \\
 &= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \\
 &+ \widehat{y} (\beta_3x + \beta_7x^3) + (\alpha_3x + \alpha_7x^3)\widehat{y}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \\
&\quad + ((\alpha_3 + \beta_3)x + (\alpha_7 + \beta_7)x^3)\widehat{y} \\
\beta\alpha &= \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \\
&\quad + \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i (\alpha_3x + \alpha_7x^3)\widehat{y} \\
&\quad + (\beta_3x + \beta_7x^3)\widehat{y} \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (\beta_3x + \beta_7x^3)\widehat{y}(\alpha_3x + \alpha_7x^3)\widehat{y} \\
&= \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \\
&\quad + \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)\widehat{y}(\alpha_3x + \alpha_7x^3) \\
&\quad + (\beta_3x + \beta_7x^3) \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)\widehat{y} \\
&= \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \\
&\quad + \widehat{y}(\alpha_3x + \alpha_7x^3) + (\beta_3x + \beta_7x^3)\widehat{y} \\
&= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \\
&\quad + ((\alpha_3 + \beta_3)x + (\alpha_7 + \beta_7)x^3)\widehat{y}
\end{aligned}$$

Let $\gamma = \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i \in L$ and $\delta = \sum_{i=0}^2 (\beta_i + \beta_{i+4}x^2)y^i \in L$. Clearly $\gamma, \delta \in V(\mathbb{F}_{3^k}C_6)$ since $\langle x^2, y \rangle \cong C_6$ and $\gamma\delta = \delta\gamma$. Therefore $C_L(y) \cong C_3^{6k}$. \square

Lemma 7. *Let T be the subgroup of L consisting of elements of the form $1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y}$ where $\check{y} = \sum_{i=0}^2 iy^i$ and $r, r_1 \in \mathbb{F}_{3^k}$. Then $T \cong C_3^{4k}$.*

Proof. Let $t_1 = 1 + (a_1 + b_1x^2)\widehat{y} + (rx + r_1x^3)\check{y} \in T$ and $t_2 = 1 + (a_2 +$

$b_2x^2)\widehat{y} + (sx + s_1x^3)\check{y} \in T$ where $a_i, b_j, r, r_1, s, s_1 \in \mathbb{F}_{3^k}$. Then

$$\begin{aligned}
t_1t_2 &= 1 + (a_2 + b_2x^2)\widehat{y} + (sx + s_1x^3)\check{y} + (a_1 + b_1x^2)\widehat{y} \\
&\quad + (a_1 + b_1x^2)\widehat{y}(a_2 + b_2x^2)\widehat{y} + (a_1 + b_1x^2)\widehat{y}(sx + s_1x^3)\check{y} \\
&\quad + (rx + r_1x^3)\check{y} + (rx + r_1x^3)\check{y}(a_2 + b_2x^2)\widehat{y} \\
&\quad + (rx + r_1x^3)\check{y}(sx + s_1x^3)\check{y} \\
&= 1 + ((a_1 + a_2) + (b_1 + b_2)x^2)\widehat{y} + ((r + s)x + (r_1 + s_1)x^3)\check{y} \\
&\quad + (rx + r_1x^3)\check{y}(sx + s_1x^3)\check{y} \\
&= 1 + ((a_1 + a_2 + \delta_1) + (b_1 + b_2 + \delta_2)x^2)\widehat{y} \\
&\quad + ((r + s)x + (r_1 + s_1)x^3)\check{y} \in T
\end{aligned}$$

where $\delta_1 = 2(rs_1 + sr_1)$ and $\delta_2 = 2(rs + r_1s_2)$. It can easily be shown that T is abelian. Therefore $T \cong C_3^{4k}$. \square

Lemma 8. $L \cong C_3^{6k} \rtimes C_3^{2k}$.

Proof. Let $t = 1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y} \in T$ and $c = \sum_{i=0}^2 (\alpha_i +$

$\alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y} \in C_L(y)$ where $\sum_{i=0}^2 \alpha_i = 1$, $\sum_{i=0}^2 \alpha_{i+3} = 0$ and

$\alpha_i, a, b, r, r_1 \in \mathbb{F}_{3^k}$. Then

$$\begin{aligned}
c^t &= (1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y})^2 \\
&\quad \times \left(\sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (\alpha_3x + \alpha_7x^3)\widehat{y} \right) \\
&\quad \times (1 + (a + bx^2)\widehat{y} + (rx + r_1x^3)\check{y}) \\
&= (1 - (a + bx^2)\widehat{y} - (rx + r_1x^3)\check{y} - (rx + r_1x^3)^2\widehat{y}) \\
&\quad \times \left(\sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + (a + bx^2)\widehat{y} \right. \\
&\quad \left. + \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i(rx + r_1x^3)\check{y} + (\alpha_3x + \alpha_7x^3)\widehat{y} \right) \\
&= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i(rx + r_1x^3)\check{y} \\
&\quad + (\alpha_3x + \alpha_7x^3)\widehat{y} - (rx + r_1x^3)\check{y} \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i
\end{aligned}$$

$$= \sum_{i=0}^2 (\alpha_i + \alpha_{i+4}x^2)y^i + ((\alpha_3 + \delta_1)x + (\alpha_7 + \delta_2)x^3)\widehat{y} \in C_L(y)$$

where $\delta_1 = r(\alpha_2 - \alpha_1) + r_1(\alpha_6 - \alpha_5)$, $\delta_2 = r(\alpha_6 - \alpha_5) + r_1(\alpha_2 - \alpha_1)$.

Clearly $c^t \in C_L(y)$ and T normalizes $C_L(y)$. Let

$$U = C_L(y) \cap T = \{1 + (a + bx^2)\widehat{y} \mid a, b \in \mathbb{F}_{3^k}\}.$$

Clearly $|C_L(y) \cap T| = 3^{2k}$ and by the second Isomorphism Theorem $|C_L(y)T| = 3^{3k}$. Therefore $L = C_L(y)T$. T is an elementary abelian 3-group ($Y \cong U \times V$). Clearly $V \cap C_L(y) = \{1\}$ and V normalizes $C_L(y)$. Therefore $L = C_L(y)V \cong C_3^{6k} \rtimes C_3^{2k}$. \square

Recall that $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) \cong L \rtimes \mathcal{U}(\mathbb{F}_{3^k}C_4)$. Now

$$\mathbb{F}_{3^k}C_4 \cong \begin{cases} \mathbb{F}_{3^k}^4 & \text{if } 4 \mid (3^k - 1) \\ \mathbb{F}_{3^k}^2 \oplus \mathbb{F}_{3^{2k}} & \text{if } 4 \nmid (3^k - 1). \end{cases}$$

$\mathbb{F}_{3^k}C_4 \cong \mathbb{F}_{3^k}^2 \oplus \mathbb{F}_{3^{2k}}$. Thus $\mathcal{U}(\mathbb{F}_{3^k}C_4) \cong C_{3^k-1}^4$ if $4 \mid (3^k - 1)$ or $\mathcal{U}(\mathbb{F}_{3^k}C_4) \cong C_{3^k-1}^2 \times C_{3^{2k-1}}$ if $4 \nmid (3^k - 1)$. Therefore

$$\begin{aligned} \mathcal{U}(\mathbb{F}_{3^k}(C_3 \rtimes C_4)) &\cong \\ &\cong \begin{cases} [(C_3^{6k} \rtimes C_3^{2k}) \rtimes C_{3^k-1}^3] \times \mathcal{U}(\mathbb{F}_{3^k}) & \text{if } 4 \mid (3^k - 1) \\ [(C_3^{6k} \rtimes C_3^{2k}) \rtimes (C_{3^k-1} \times C_{3^{2k-1}})] \times \mathcal{U}(\mathbb{F}_{3^k}) & \text{if } 4 \nmid (3^k - 1). \end{cases} \end{aligned}$$

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