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RESEARCH ARTICLE

On τ-closed n-multiply ω-composition formations with Boolean sublattices

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Abstract. In the universe of finite groups the description of τ-closed n-multiply ω-composition formations with Boolean sublattices of τ-closed n-multiply ω-composition subformations is obtained.

Introduction

Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a nonempty τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$. Then $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ denotes the sublattice of all τ-closed n-multiply ω-composition formations $\mathfrak{M}$ such that $\mathfrak{F} \cap \mathfrak{H} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$. In this paper we obtain the description of τ-closed n-multiply ω-composition formations $\mathfrak{F}$ such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ and the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is Boolean, where $\mathfrak{H}$ is a nonempty nilpotent saturated formation. This is the solution of the following problem [1]:

Problem (A.N. Skiba, L.A. Shemetkov). Describe τ-closed n-multiply ω-composition formations $\mathfrak{F} \not\subseteq \mathfrak{H}$ such that the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is Boolean, where $\mathfrak{H}$ is a nonempty nilpotent saturated formation.

The solution of that problem in the case, when τ is the trivial subgroup functor (i.e., $\tau(G) = \{G\}$ for all groups G) and $\mathfrak{H} = \mathfrak{N}$ is the formation of all nilpotent groups is obtained in paper [2]. In the other case, when τ is the trivial subgroup functor, $n = 1$ and $\mathfrak{H} = \mathfrak{N}$, the solution of that problem was given in [3].

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1. Preliminaries

All groups considered in this paper are finite. We use the terminology of [1, 4, 5, 6]. Here, we recall some definitions and notation.

Let ω be a nonempty set of prime numbers. Every function of the form $f : \omega \cup \{\omega'\} \to \{\text{formations of groups}\}$ is called an ω-composition satellite. Following Doerk and Hawkes [7] we use $C^p(G)$ to denote the intersection of all centralizers of abelian chief p-factors of G (we note that $C^p(G) = G$ if G has no such chief factors). If $\mathfrak{X}$ is a set of groups, then we use $Com^+(\mathfrak{X})$ to denote the class of all abelian simple groups A such that $A \simeq H/K$ for some composition factor H/K of a group $G \in \mathfrak{X}$. We write $Com^+(G)$ for the set $Com^+(\{G\})$.

Let f be an ω-composition satellite. Then following [1] we put

$$CF_\omega(f) = \bigl\{G \mid G/R_\omega(G) \in f(\omega') \\ \text{and } \ G/C^p(G) \in f(p) \text{ for all } p \in \pi(Com(G)) \cap \omega\bigr\}.$$

Here $R_\omega(G)$ denotes the largest normal soluble ω-subgroup of G. If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for some ω-composition satellite f, then $\mathfrak{F}$ is called an ω-composition formation, and f is its an ω-composition satellite.

Every group formation is considered as 0-multiply ω-composition formation. For $n \geq 1$, a formation $\mathfrak{F}$ is called n-multiply ω-composition formation, if $\mathfrak{F} = CF_\omega(f)$, where all values of an ω-composition satellite f are $(n-1)$-multiply ω-composition formations.

Let τ be a functor such that for any group G, $\tau(G)$ is a set of subgroups of G, and $G \in \tau(G)$. Following [5] we say that τ is a subgroup functor if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. A group $\mathfrak{F}$ is called τ-closed if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$. The set of all τ-closed n-multiply ω-composition formations is denoted by $c^\tau_{\omega_n}$. A satellite f is called a $c^\tau_{\omega_{n-1}}$-valued ω-composition sattelite if all values of f belong to $c^\tau_{\omega_{n-1}}$.

A τ-closed n-multiply ω-composition formation $\mathfrak{F}$ is called $\mathfrak{H}^\tau_{\omega_n}$-critical (or a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation) if $\mathfrak{F} \not\subseteq \mathfrak{H}$ but all proper τ-closed n-multiply ω-composition subformations of $\mathfrak{F}$ are contaned in $\mathfrak{H}$.

Let $\mathfrak{X}$ be a set of groups. Then $c^\tau_{\omega_n}\text{form}\mathfrak{X}$ is the τ-closed n-multiply ω-composition formation generated by $\mathfrak{X}$, i.e., $c^\tau_{\omega_n}\text{form}\mathfrak{X}$ is the intersection of all τ-closed n-multiply ω-composition formation containing $\mathfrak{X}$. If $\mathfrak{X} = \{G\}$, then $c^\tau_{\omega_n}\text{form}\mathfrak{X} = c^\tau_{\omega_n}\text{form}G$ is called an one-generated τ-closed n-multiply ω-composition formation.

For any set $\{\mathfrak{F}_i \mid i \in I\}$ of τ-closed n-multiply ω-composition formations, we put

$$\vee^{\tau}_{\omega_n}(\mathfrak{F}_i \mid i \in I) = c^{\tau}_{\omega_n}\mathrm{form}(\bigcup_{i\in I}\mathfrak{F}_i).$$

In particular, $\mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{H} = c^{\tau}_{\omega_n}\mathrm{form}\,(\mathfrak{M} \cup \mathfrak{H})$.

A lattice is called modular if and only if its elements satisfy the modular identity: if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

A lattice is called distributive if and only if it satisfies the following:

*L*1. $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \wedge z) \wedge (z \wedge x)$,

*L*2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

*L*3. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

An element which covers 0 in a partly ordered set P (i.e., a minimal element in the subset of P obtained by excluding 0) is called an atom.

By a complement of an element x of a lattice L with 0 and 1 is meant an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$; L is called complemented if all its elements have complements.

In the paper we consider only subgroup functors τ such that the set $\tau(G)$ is contained in the set of all subnormal subgroups of G, for any group G.

Lemma 1 ([8], Theorem 2). *The lattice $c^{\tau}_{\omega_n}$ of all τ-closed n-multiply ω-composition formations is modular for any non-negative number n.*

Lemma 2 ([6, p. 65]). *Any sublattice of a modular lattice is modular.*

Lemma 3 ([6, p. 73], Theorem 6). *Let L be any modular lattice, and let u and v be any two elements of L. Then the correspondences $x \to u \vee x$ and $y \to v \wedge y$ are inverse isomorphisms between $[u \wedge v, v]$ and $[u, u \vee v]$. Moreover, they carry quotients in these intervals into transposed quotients.*

Lemma 4 ([9], Theorem 2). *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ ($n \geq 1$). Then $\mathfrak{H}^{\tau}_{\omega_n}$-defect of the formation $\mathfrak{F}$ is equal 1 if and only if $\mathfrak{F} = \mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{K}$, where $\mathfrak{M} \subseteq \mathfrak{H}$, $\mathfrak{K}$ is a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation, besides:*

1) if τ-closed n-multiply ω-composition subformation $\mathfrak{F}_1$ of $\mathfrak{F}$, such that $\mathfrak{F}_1 \subseteq \mathfrak{H}$, then $\mathfrak{F}_1 \subseteq \mathfrak{M} \vee^{\tau}_{\omega_n} (\mathfrak{K} \cap \mathfrak{H})$;

2) if τ-closed n-multiply ω-composition subformation $\mathfrak{F}_1$ of $\mathfrak{F}$, such that $\mathfrak{F}_1 \not\subseteq \mathfrak{H}$, then $\mathfrak{F}_1 = \mathfrak{K} \vee^{\tau}_{\omega_n} (\mathfrak{F}_1 \cap \mathfrak{H})$.

Lemma 5 ([9], Lemma 17). *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, $\mathfrak{M} \subseteq \mathfrak{H}$, and let $\Omega = \{\mathfrak{K}_i \mid i \in I\}$ be some set of minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formations. If $\mathfrak{K}$ is some τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation of $\mathfrak{M} \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid i \in I)$, then $\mathfrak{K} \in \Omega$.*

Lemma 6 ([9], Theorem 4). *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be an one-generated τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ $(n \geq 1)$. If $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is a complemented lattice and $\mathfrak{M}$ is an element of $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$, then*

$$\mathfrak{M} = \vee^{\tau}_{\omega_n}((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid i \in I),$$

where $\{\mathfrak{K}_i \mid i \in I\}$ is the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{M}$.

Lemma 7 ([9], Theorem 1). *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ $(n \geq 1)$. Then $\mathfrak{F}$ has at least one a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformation.*

Lemma 8 ([9], Lemma 24). *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ $(n \geq 1)$. Then $\mathfrak{M}$ is an atom of the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ if and only if $\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}$, where $\mathfrak{K}$ is a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformation of $\mathfrak{F}$.*

Lemma 9 ([10, p. 50], Lemma 9). *The following inequalities hold in any lattice:*

1) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$;
2) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$;
3) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$;
4) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$.

The first three are called distributive inequalities, and the last is the modular inequality.

2. The Main result

Lemma 10. *Let $\mathfrak{F} \cap \mathfrak{H} \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M} \subseteq \mathfrak{F}$, where $\mathfrak{M}_1$, $\mathfrak{M}$, $\mathfrak{F}$ are τ-closed n-multiply ω-composition formations, and let $\mathfrak{H}$ be a nonempty nilpotent saturated formation $(n \geq 1)$. If $\mathfrak{H}_1$ is a complement of $\mathfrak{M}_1$ in the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$, then $\mathfrak{M} \cap \mathfrak{H}_1$ is a complement of $\mathfrak{M}_1$ in the lattice $\mathfrak{M}/^{\tau}_{\omega_n}\mathfrak{M} \cap \mathfrak{H}$.*

Proof. By the conditions of the lemma we have that $\mathfrak{M}_1 \cap \mathfrak{H}_1 = \mathfrak{F} \cap \mathfrak{H}$ and $\mathfrak{M}_1 \vee^{\tau}_{\omega_n} \mathfrak{H} = \mathfrak{F}$. It follows from Lemmas 1 and 2 that $\mathfrak{M} = \mathfrak{M} \cap (\mathfrak{M}_1 \vee^{\tau}_{\omega_n} \mathfrak{H}_1) = \mathfrak{M}_1 \vee^{\tau}_{\omega_n} (\mathfrak{M} \cap \mathfrak{H}_1)$. Moreover, it is clear that $(\mathfrak{M} \cap \mathfrak{H}_1) \cap \mathfrak{M}_1 = \mathfrak{M} \cap \mathfrak{H}$. Hence $\mathfrak{M} \cap \mathfrak{H}_1$ is a complement of $\mathfrak{M}_1$ in the lattice $\mathfrak{M}/^{\tau}_{\omega_n} \mathfrak{M} \cap \mathfrak{H}$. □

Lemma 11. *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, let $\mathfrak{F}$ and $\mathfrak{M}$ be τ-closed n-multiply ω-composition formations such that $\mathfrak{M} \subseteq \mathfrak{F}$ ($n \geq 1$). If $\mathfrak{F}/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H}$ is a complemented lattice, then $\mathfrak{M}/^{\tau}_{\omega_n} \mathfrak{M} \cap \mathfrak{H}$ is also a complemented lattice.*

Proof. From Lemmas 1 and 2 we see that the lattice $\mathfrak{F}/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H}$ is modular. By Lemma 3 we have the following isomorphism of lattices:

$$((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{M})/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H} \simeq \mathfrak{M}/^{\tau}_{\omega_n} (\mathfrak{M} \cap \mathfrak{F} \cap \mathfrak{H}) = \mathfrak{M}/^{\tau}_{\omega_n} \mathfrak{M} \cap \mathfrak{H}.$$

Since $((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{M})/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H}$ is a sublattice of $\mathfrak{F}/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H}$, then by Lemma 10, it follows that $(\mathfrak{F} \cap \mathfrak{H} \vee^{\tau}_{\omega_n} \mathfrak{M})/^{\tau}_{\omega_n} \mathfrak{F} \cap \mathfrak{H}$ is a complemented lattice. Consequently, $\mathfrak{M}/^{\tau}_{\omega_n} \mathfrak{M} \cap \mathfrak{H}$ is also a complemented lattice. □

Lemma 12. *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ ($n \geq 1$). Denote by Ω the set of all τ-closed n-multiply ω-composition formations $\mathfrak{F}_i$ of $\mathfrak{F}$ ($i \in I$) such that $\mathfrak{F}_i \not\subseteq \mathfrak{H}$ and $\mathfrak{F} \cap \mathfrak{H}$ is a maximal τ-closed n-multiply ω-composition subformation of $\mathfrak{F}_i$. Put*

$$\mathfrak{R} = c^{\tau}_{\omega_n}\text{form}\,(\bigcup_{i \in I} \mathfrak{F}_i),$$

where $\mathfrak{F}_i \in \Omega$. If $\mathfrak{M}$ is a τ-closed n-multiply ω-composition subformation of $\mathfrak{R}$ with maximal subformation $\mathfrak{F} \cap \mathfrak{H}$ and $\mathfrak{M} \not\subseteq \mathfrak{H}$, then $\mathfrak{M} \in \Omega$.

Proof. Following Lemma 4, for every $i \in I$ we have $\mathfrak{F}_i = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i$, where $\mathfrak{K}_i$ is a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation. Then

$$\mathfrak{R} = c^{\tau}_{\omega_n}\text{form}\,(\bigcup_{i \in I} \mathfrak{F}_i) = c^{\tau}_{\omega_n}\text{form}\,(\bigcup_{i \in I}((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i)) =$$

$$= c^{\tau}_{\omega_n}\text{form}\,((\mathfrak{F} \cap \mathfrak{H}) \cup (\vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid i \in I)) = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid i \in I).$$

From Lemma 4 we have $\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}$, where $\mathfrak{K}$ is a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation. Consequently, by Lemma 5 we get $\mathfrak{K} \in \{\mathfrak{K}_i \mid i \in I\}$, i.e. $\mathfrak{K} = \mathfrak{K}_i$ for some $i \in I$. Thus,

$$\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K} \in \{(\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid i \in I\} = \Omega.$$

□

Theorem 1. *Let $\mathfrak{H}$ be a nonempty nilpotent saturated formation, and let $\mathfrak{F}$ be a τ-closed n-multiply ω-composition formation such that $\mathfrak{F} \not\subseteq \mathfrak{H}$ $(n \geq 1)$. Then the following statements are equivalent:*

1) $\mathfrak{F}/^\tau_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is a complemented lattice;

2) $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{H}) \vee^\tau_{\omega_n} (\vee^\tau_{\omega_n}\mathfrak{K}_i \mid i \in I)$, where $\{\mathfrak{K}_i \mid i \in I\}$ is the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformation of $\mathfrak{F}$;

3) $\mathfrak{F}/^\tau_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is a Boolean lattice.

Proof. 1)⇒2). Let $\mathfrak{M}$ be an one-generated τ-closed n-multiply ω-composition subformation of $\mathfrak{F}$. Then by Lemma 11 the lattice $\mathfrak{M}/^\tau_{\omega_n}\mathfrak{M} \cap \mathfrak{H}$ is complemented. From Lemma 6 we have:

$$\mathfrak{M} = \vee^\tau_{\omega_n}((\mathfrak{M} \cap \mathfrak{H}) \vee^\tau_{\omega_n} \mathfrak{K}_i \mid i \in I) = (\mathfrak{M} \cap \mathfrak{H}) \vee^\tau_{\omega_n} (\vee^\tau_{\omega_n}\mathfrak{K}_j \mid j \in J),$$

where $\{\mathfrak{K}_j \mid j \in J \subseteq I\}$ is the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{M}$. Obviously, any τ-closed n-multiply ω-composition formation is a join (in the lattice $c^\tau_{\omega_n}$) of all proper one-generated τ-closed n-multiply ω-composition subformations, i.e.,

$$\mathfrak{F} = \vee^\tau_{\omega_n}(c^\tau_{\omega_n}\mathrm{form}\, G \mid G \in \mathfrak{F}) = \vee^\tau_{\omega_n}(\mathfrak{M}_i \mid i \in I).$$

Then

$$\mathfrak{F} = \vee^\tau_{\omega_n}(\mathfrak{M}_i \mid i \in I) = \vee^\tau_{\omega_n}((\mathfrak{M}_i \cap \mathfrak{H}) \vee^\tau_{\omega_n} (\vee^\tau_{\omega_n}\mathfrak{K}_j \mid j \in J_i) \mid i \in I) =$$

$$= \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I) \vee^\tau_{\omega_n} (\vee^\tau_{\omega_n}\mathfrak{K}_i \mid i \in I).$$

For every $i \in I$ we have $\mathfrak{M}_i \subseteq \mathfrak{F}$. Then $\mathfrak{M}_i \cap \mathfrak{H} \subseteq \mathfrak{F} \cap \mathfrak{H}$. Therefore, $\vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I) \subseteq \mathfrak{F} \cap \mathfrak{H}$. Suppose that $\mathfrak{F} \cap \mathfrak{H} \not\subseteq \vee^\tau_{\omega_n}((\mathfrak{M}_i) \cap \mathfrak{H} \mid i \in I)$, and let $G \in (\mathfrak{F} \cap \mathfrak{H}) \setminus \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I)$. Since

$$c^\tau_{\omega_n}\mathrm{form}\, G = (c^\tau_{\omega_n}\mathrm{form}\, G) \cap \mathfrak{H} \subseteq \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I),$$

we have $G \in \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I)$. A contradiction. Hence, we get

$$\mathfrak{F} \cap \mathfrak{H} \subseteq \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I) \text{ and } \mathfrak{F} \cap \mathfrak{H} = \vee^\tau_{\omega_n}(\mathfrak{M}_i \cap \mathfrak{H} \mid i \in I).$$

Thus, $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{H}) \vee^\tau_{\omega_n} (\vee^\tau_{\omega_n}\mathfrak{K}_i \mid i \in I)$, i.e., 2) is true.

2)⇒1). First we shall show that every element of $\mathfrak{F}/^\tau_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is a join of atoms which are contained in that element.

Let $\mathfrak{M}$ be an element of $\mathfrak{F}/^\tau_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$. Then $\mathfrak{M}$ is τ-closed n-multiply ω-composition formation. Let $\psi = \{\mathfrak{K}_i \mid i \in I\}$ be the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{F}$, $\psi_1 = \{\mathfrak{K}_i \mid i \in I_1\}$ be a set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{F}$ such that $\mathfrak{K}_i \subseteq \mathfrak{M}$ for all $i \in I_1 \subseteq I$, and let ψ_2 is a complement to ψ_1 in ψ. Then $\mathfrak{R}_i = c^{\tau}_{\omega_n}\mathrm{form}\,(\psi_i)$ is the τ-closed n-multiply ω-composition formation generated by ψ_i, where $i = 1, 2$. Since the lattice $c^{\tau}_{\omega_n}$ is modular, we have:

$$\mathfrak{M} = \mathfrak{M} \cap \mathfrak{F} = \mathfrak{M} \cap ((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n}\mathfrak{K}_i \mid i \in I)) =$$
$$= (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\mathfrak{M} \cap (\vee^{\tau}_{\omega_n}\mathfrak{K}_i \mid i \in I)) =$$
$$= (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\mathfrak{M} \cap (\mathfrak{R}_1 \vee^{\tau}_{\omega_n} \mathfrak{R}_2)) = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{R}_1 \vee^{\tau}_{\omega_n} (\mathfrak{M} \cap \mathfrak{R}_2).$$

Assume that $\mathfrak{M} \cap \mathfrak{R}_2 \not\subseteq \mathfrak{F} \cap \mathfrak{H}$. Then by Lemma 7 $\mathfrak{M} \cap \mathfrak{R}_2$ has a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation $\mathfrak{K}_i$ for some $i \in I$. Hence, by Lemma 5 we get $\mathfrak{K}_i \in \psi_1 \cap \psi_2 = \varnothing$. A contradiction. Therefore, $\mathfrak{M} \cap \mathfrak{R}_2 \subseteq \mathfrak{F} \cap \mathfrak{H}$. Thus,

$$\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{R}_1 = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n}\mathfrak{K}_i \mid \mathfrak{K}_i \in \psi_1) =$$
$$= \vee^{\tau}_{\omega_n}((\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i \mid \mathfrak{K}_i \in \psi_1).$$

From Lemma 6 we see that every element $\mathfrak{M}$ of the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is a join of an atoms which are contained in $\mathfrak{M}$.

Now we shall show that every element $\mathfrak{M}$ of the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ has a complement. If $\mathfrak{M} = \mathfrak{F}$, then $\mathfrak{F} \cap \mathfrak{H}$ is complement to $\mathfrak{M}$ in $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$. Therefore, we can assume that $\mathfrak{M} \neq \mathfrak{F}$. Denote by Σ the set of all atoms of $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$, and denote by Ω_1 a set of all atoms of $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ contained in $\mathfrak{M}$. If $\Sigma = \Omega_1$, then

$$\mathfrak{M} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n}\mathfrak{K}_i \mid i \in I) = \mathfrak{F}.$$

A contradiction. Therefore, $\Sigma \neq \Omega_1$. Let Ω_2 be a complement of Ω_1 in Σ, and let $\mathfrak{K} = c^{\tau}_{\omega_n}\mathrm{form}\,(\Omega_2)$. We prove that $\mathfrak{K}$ is a complement of $\mathfrak{M}$ in $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$. Since by the condition $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} (\vee^{\tau}_{\omega_n}\mathfrak{K}_i \mid i \in I)$, then by Lemma 8 we have $\mathfrak{F} = \mathfrak{M} \vee^{\tau}_{\omega_n} \mathfrak{K}$. Let $\mathfrak{R} = \mathfrak{M} \cap \mathfrak{K}$. Since $\mathfrak{M}$ and $\mathfrak{K}$ are elements of the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$, we have $\mathfrak{F} \cap \mathfrak{H} \subseteq \mathfrak{R}$. Assume that $\mathfrak{R} \not\subseteq \mathfrak{F} \cap \mathfrak{H}$. According to Lemma 7, the formation $\mathfrak{R}$ has a minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-formation $\mathfrak{K}_i$ for some $i \in I$. Hence, $(\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i \subseteq \mathfrak{R}$. By Lemma 4, $(\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i$ has a maximal τ-closed n-multiply ω-composition subformation which is contained in $\mathfrak{H}$. Now by Lemma 12 and applying the result of previous paragraph, we have $(\mathfrak{F} \cap \mathfrak{H}) \vee^{\tau}_{\omega_n} \mathfrak{K}_i \in \Omega_1 \cap \Omega_2 = \varnothing$. A contradiction. Hence, $\mathfrak{M} \cap \mathfrak{K} = \mathfrak{F} \cap \mathfrak{H}$. Thus, $\mathfrak{K}$ is a complement of $\mathfrak{M}$ in $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$. So, the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ is complemented.

3)⇒1). This case is obvious.

1)⇒3). We need to show that for any elements $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ of the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F} \cap \mathfrak{H}$ the following equality is true:

$$\mathfrak{M}_1 \cap (\mathfrak{M}_2 \vee^{\tau}_{\omega_n} \mathfrak{M}_3) = (\mathfrak{M}_1 \cap \mathfrak{M}_2) \vee^{\tau}_{\omega_n} (\mathfrak{M}_1 \cap \mathfrak{M}_3). \qquad (*)$$

According to Lemma 9, for the lattice $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F}\cap\mathfrak{H}$ the following inclusion holds:

$$(\mathfrak{M}_1\cap\mathfrak{M}_2)\vee^{\tau}_{\omega_n}(\mathfrak{M}_1\cap\mathfrak{M}_3)\subseteq\mathfrak{M}_1\cap(\mathfrak{M}_2\vee^{\tau}_{\omega_n}\mathfrak{M}_3).$$

Put $\mathfrak{X}=\mathfrak{M}_1\cap(\mathfrak{M}_2\vee^{\tau}_{\omega_n}\mathfrak{M}_3)$ and $\mathfrak{Y}=(\mathfrak{M}_1\cap\mathfrak{M}_2)\vee^{\tau}_{\omega_n}(\mathfrak{M}_1\cap\mathfrak{M}_3)$. We show $\mathfrak{X}\subseteq\mathfrak{Y}$. By Lemma 11 for every element $\mathfrak{M}$ of $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F}\cap\mathfrak{H}$ a lattice $\mathfrak{M}/^{\tau}_{\omega_n}\mathfrak{M}\cap\mathfrak{H}$ is complemented. Since 1)⇒2) is true, it follows that

$$\mathfrak{M}=(\mathfrak{M}\cap\mathfrak{H})\vee^{\tau}_{\omega_n}(\vee^{\tau}_{\omega_n}\mathfrak{K}_i\mid i\in I),\qquad(**)$$

where $\{\mathfrak{K}_i\mid i\in I\}$ is the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{M}$. Since $\mathfrak{X}$ and $\mathfrak{Y}$ are elements of $\mathfrak{F}/^{\tau}_{\omega_n}\mathfrak{F}\cap\mathfrak{H}$, $(**)$ is true. Therefore, now we need to show that every minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformation $\mathfrak{K}$ of $\mathfrak{X}$ is contained in $\mathfrak{Y}$. Denote by ψ_j the set of all minimal τ-closed n-multiply ω-composition non-$\mathfrak{H}$-subformations of $\mathfrak{M}_j$, where $j=1,2,3$. Clearly, $\mathfrak{K}\subseteq\mathfrak{M}_1$. From $(**)$ for $\mathfrak{M}_1$ we have $\mathfrak{K}\in\psi_1$. Besides, we have

$$\mathfrak{K}\subseteq\Big((\mathfrak{M}_2\cap\mathfrak{H})\vee^{\tau}_{\omega_n}\Big(c^{\tau}_{\omega_n}\mathrm{form}\Big(\bigcup_{\mathfrak{K}_i\in\psi_2}\mathfrak{K}_i\Big)\Big)\Big)\vee^{\tau}_{\omega_n}$$

$$\vee^{\tau}_{\omega_n}\Big((\mathfrak{M}_3\cap\mathfrak{H})\vee^{\tau}_{\omega_n}\Big(c^{\tau}_{\omega_n}\mathrm{form}\Big(\bigcup_{\mathfrak{K}_i\in\psi_3}\mathfrak{K}_i\Big)\Big)\Big)=$$

$$=((\mathfrak{M}_2\vee^{\tau}_{\omega_n}\mathfrak{M}_3)\cap\mathfrak{H})\vee^{\tau}_{\omega_n}\Big(c^{\tau}_{\omega_n}\mathrm{form}\Big(\bigcup_{\mathfrak{K}_i\in\psi_2\cup\psi_3}\mathfrak{K}_i\Big)\Big).$$

From Lemma 12 we get $\mathfrak{K}\in\psi_2\cup\psi_3$. Then either $\mathfrak{K}\subseteq\mathfrak{M}_1\cap\mathfrak{M}_2$ or $\mathfrak{K}\subseteq\mathfrak{M}_1\cap\mathfrak{M}_3$. Therefore, $\mathfrak{K}\subseteq\mathfrak{Y}$, i.e., equality $(*)$ is true. So, 3) is true.

□

Recall that $L^{\tau}_{\omega_n}(\mathfrak{F})$ denotes the lattice of all τ-closed n-multiply ω-composition subformations of $\mathfrak{F}$. In the case $\mathfrak{H}=(1)$ from theorem 1 we obtain

Corollary 1. *Let $\mathfrak{F}$ be a non-identity τ-closed n-multiply ω-composition formation ($n\geq 1$). Then the following statements are equivalent:*

1) $L^{\tau}_{\omega_n}(\mathfrak{F})$ is a complemented lattice;

2) $\mathfrak{F}=\otimes_{i\in I}\mathfrak{F}_i$, where $\{\mathfrak{F}_i\mid i\in I\}$ is the set of all atoms of the lattice $L^{\tau}_{\omega_n}(\mathfrak{F})$;

3) $L^{\tau}_{\omega_n}(\mathfrak{F})$ is a Boolean lattice.

In the case when τ is trivial subgroup functor, $n=1$, $\omega=\mathbb{P}$ and $\mathfrak{H}=(1)$ from theorem 1 we get

Corollary 2 ([11], Theorem 2.3). *Let $\mathfrak{F}$ be a non-identity composition formation. Then the following statements are equivalent:*

1) $L_c(\mathfrak{F})$ is a complemented lattice;

2) for any group $G \in \mathfrak{F}$, we have $G = A \times A_1 \times \cdots \times A_t$, where A is nilpotent and G, $A_1, \ldots, A_t$ are simple non-abelian groups.

Corollary 3 ([2], Theorem 6). *Let $\mathfrak{F}$ be a non-nilpotent n-multiply ω-composition formation ($n \geq 1$). Then the following statements are equivalent:*

1) $\mathfrak{F}/^{\omega}_{n}\mathfrak{F} \cap \mathfrak{N}$ is a complemented lattice;

2) $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee^{\omega}_{n} (\vee^{\omega}_{n}\mathfrak{K}_i \mid i \in I)$, where $\{\mathfrak{K}_i \mid i \in I\}$ is the set of all minimal n-multiply ω-composition non-nilpotent subformations of $\mathfrak{F}$;

3) $\mathfrak{F}/^{\omega}_{n}\mathfrak{F} \cap \mathfrak{N}$ is a Boolean lattice.

Corollary 4 ([3], Theorem 1). *Let $\mathfrak{F}$ be a non-nilpotent ω-composition formation. Then the following statements are equivalent:*

1) $\mathfrak{F}/^{\omega}\mathfrak{F} \cap \mathfrak{N}$ is a complemented lattice;

2) $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{N}) \vee^{\omega} (\vee^{\omega}\mathfrak{K}_i \mid i \in I)$, where $\{\mathfrak{K}_i \mid i \in I\}$ is the set of all minimal ω-composition non-nilpotent subformations of $\mathfrak{F}$;

3) $\mathfrak{F}/^{\omega}\mathfrak{F} \cap \mathfrak{N}$ is a Boolean lattice.

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