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Free semigroups in wreath powers of transformation semigroups

Andriy Oliynyk

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Abstract. It is established a criterion when the infinite wreath power of a finite transformation semigroup contains a free subsemigroup. It is shown that the infinite wreath power of a transformation semigroup either contains no free non-commutative subsemigroups or most of its finitely generated subsemigroups are free.

1. Introduction

It is well-known that inverse limits of wreath products of permutation groups are rich of free subgroups ([1], [2]). On the other hand the analogous theorem for free subsemigroups in inverse limits of wreath products of transformation semigroups is not true in general as this inverse limit do not certainly contain such subsemigroups. Nevertheless, in the category sense of Baire in some topological semigroups most finitely generated subsemigroups
infinite wreath powers of transformation semigroups as inverse limits of their finite wreath powers.

The work is organized as follows. In Section 2 we recall main definitions and notations concerning wreath powers of transformation semigroups. In Section 3 we prove a criterion in terms of finite transformation semigroup when its infinite wreath power contains free subsemigroups. In Section 4 we prove that if the infinite wreath power of a transformation semigroup contain free non-commutative subsemigroups then most of their finitely generated subsemigroups are free.

2. Wreath powers

For details on this section see [6] and [7].

Let X be a non-empty set. As usual, denote by $\mathcal{T}_X$ the full transformation semigroup of X. In this note a transformation semigroup (T,X) is a subsemigroup T of the $\mathcal{T}_X$ acting on X. We will use the right actions of transformations. The set of idempotents of a semigroup S will be denoted by $E(S)$. For subsets $A,B\subseteq S$ let $AB=\{ab|a\in A,b\in B\}$. Analogously, $A^n=\{a^n|a\in A\}$ for $n\geq 1$.

Denote by $X^{(n)}$ the n-th cartesian power of X, $n\geq 1$, and by X^ω its countable cartesian power. Let $X^{[n]}=\cup_{i=0}^n X^{(i)}$, $n\geq 0$. Consider a free monoid X^* with a basis X. Then X^* is naturally identified with the set $\cup_{i=0}^\infty X^{(i)}$, where $X^{(0)}=\{\Lambda\}$ and Λ is the identity of X^*, the empty word.

Being an associative operation, the wreath product of transformation semigroups can be defined on arbitrary finite number of semigroups. Let $W^n(T,X)$ denotes the wreath product of n copies of the transformation semigroup (T,X) acting of the set $X^{(n)}$, $n\geq 1$. Each element $\bar{t}$ of the semigroup $W^n(T,X)$ can be written in the form

$$\bar{t}=[t_1;t_2(x_1);\ldots;t_n(x_1,\ldots,x_{n-1})],$$

where $t_1\in T$, $t_2(x_1):X\to T,\ldots,t_n(x_1,\ldots x_{n-1}):X^{(n-1)}\to T$. Such an element acts on a point $(a_1,a_2,\ldots,a_n)\in X^{(n)}$ by the rule

$$(a_1,a_2,\ldots,a_n)^{[t_1;t_2(x_1);\ldots;t_n(x_1,\ldots,x_{n-1})]}=(a_1^{t_1},a_2^{t_2(a_1)},\ldots,a_n^{t_n(a_1,\ldots,a_{n-1})}).$$

The natural projection $\pi_n:W^n(T,X)\to W^{n-1}(T,X)$ which erases the last coordinate is an epimorphism, $n\geq 2$. Hence, we obtain an inverse limit

$$W=\varprojlim(W^n(T,X),\pi_n)$$

acting on the set

$$\prod_{n=1}^{\infty}X^{(n)}.$$

The subset

$$\{((x_1),(x_1,x_2),\ldots,(x_1,x_2,\ldots,x_n),\ldots)|x_i\in X,i\geq 1\},$$

which is naturally identified with X^ω, is invariant under this action. We call the transformation semigroup (W,Ω) the infinite wreath power of (T,X), and denote it by $W^\infty(T,X)$.

Each element of the infinite wreath power $W^\infty(T,X)$ can be viewed as an infinite sequence

$$\bar{t}=[t_1;t_2(x_1);t_3(x_1,x_2);\ldots], \tag{1}$$

where $t_1 \in T$, $t_2(x) : X \to T, t_3(x_1, x_2) : X^2 \to T, \dots$. The action of the semigroup $W^\infty(T, X)$ on the set X^ω (or $X^{(n)}, n \geq 1$,) is given by the rule

$$u^{\bar{t}} = a_1^{t_1} a_2^{t_2(a_1)} a_3^{t_3(a_1,a_2)} \dots \tag{2}$$

for arbitrary $u = a_1 a_2 a_3 \dots \in X^\omega$. For $\bar{t}, \bar{s} \in W^\infty(T, X)$, where $\bar{t}$ is as above, and

$$\bar{s} = [s_1; s_2(x_1); s_3(x_1, x_2); \dots],$$

their product $\bar{t}\bar{s}$ has the form

$$\bar{t}\bar{s} = [t_1 s_1; t_2(x_1) s_2(x_1^{t_1}); t_3(x_1, x_2) s_3(x_1^{t_1}, x_2^{t_2(x_1)}); \dots]. \tag{3}$$

One may regard the infinite wreath power $W^\infty(T, X)$ of (T, X) as a set of infinite tuples of the form (1) with multiplication rule (3), acting on X^ω by (2).

We may use the following way to describe elements of wreath powers. Let $\bar{t} \in W^\infty(T, X)$ be of the form (1). For each $u \in X^*$ let us define an element $\bar{t}_u \in T$ by the rule

$$\bar{t}_u = \begin{cases} t_1, & \text{if } u = \Lambda \\ t_{|u|+1}(a_1, \dots, a_{|u|}), & \text{if } u = a_1 \dots a_{|u|} \end{cases},$$

where the length of the word u is denoted by $|u|$. Then the element $\mathcal{P}(\bar{t}) = \{\bar{t}_u, u \in X^*\}$ of the cartesian power T^{X^*} is called the portrait of $\bar{t}$. The correspondence $\mathcal{P}$ between sets $W^\infty(T, X)$ and T^{X^*} is one-to-one. This means that the infinite wreath power $W^\infty(T, X)$ can be regarded as the set of portraits T^{X^*}. For two portraits

$$\mathcal{P}(\bar{t}) = \{\bar{t}_u, u \in X^*\}, \quad \mathcal{P}(\bar{s}) = \{\bar{s}_u, u \in X^*\}$$

by (3) their product is computed as

$$\mathcal{P}(\bar{t} \cdot \bar{s}) = \{\bar{t}_u \cdot \bar{s}_{u^{\bar{t}}}, u \in X^*\}.$$

In the same way portraits of elements of finite wreath power $W^n(T, X)$ as elements of the cartesian power $T^{X^{[n-1]}}$ are defined, $n \geq 1$.

The following proposition immediately follows from the definitions above.

Proposition 1. *Let (T, X) be a transformation semigroup.*

1. *For arbitrary $k \geq 1$ the infinite wreath power $W^\infty(T, X)$ admits a decomposition $W^k(T, X) \wr W^\infty(T, X)$.*

2. *For arbitrary sequence $k_1, k_2, \ldots$ of positive integers the infinite wreath power $W^\infty(T,X)$ contains as a subsemigroup the cartesian product*

$$\prod_{i=1}^{\infty} W^{k_i}(T,X).$$

Proof. (1) Let $k \geq 1$ be fixed. For a tuple

$$\bar{t} = [t_1; t_2(x_1); t_3(x_1,x_2); \ldots] \in W^\infty(T,X)$$

define a tuple

$$p_k(\bar{t}) \in W^k(T,X)$$

and a mapping

$$s_k(\bar{t}) : X^k \to W^\infty(T,X)$$

as follows

$$p_k(\bar{t}) = [t_1; \ldots t_k(x_1,\ldots,x_{k-1})],$$

$$s_k(\bar{t})(a_1,\ldots,a_k) = [t_{k+1}(a_1,\ldots,a_k), t_{k+2}(a_1,\ldots,a_k,x_1),\ldots],$$

where $a_1,\ldots,a_k \in X$. Then the rule

$$\bar{t} \mapsto [p_k(\bar{t}); s_k(\bar{t})]$$

is a required isomorphism.

(2) Let $k_i, i \geq 1$, be a fixed sequence of positive integers. Let $k_0 = 0$. Each element s of the cartesian product $\prod_{i=1}^{\infty} W^{k_i}(T,X)$ has the form

$$s = (\bar{s}(1), \bar{s}(2), \ldots),$$

where $\bar{s}(i) \in W^{k_i}(T,X), i \geq 1$. Define an element $\varphi(s) = \bar{s} \in W^\infty(T,X)$. To do this it is sufficient to describe its portrait $\mathcal{P}(\bar{s})$. For any $w \in X^*$ there exist $i \geq 1$ such that

$$k_0 + k_1 + \ldots + k_{i-1} \leq |w| < k_0 + k_1 + \ldots + k_i.$$

Denote by w_1 the word obtained by deleting first $k_0 + k_1 + \ldots + k_{i-1}$ coordinates of w. Then we define the transformation $\bar{s}_w$ to be equal to the transformation $\bar{s}(i)_{w_1}$. It is easily verified that φ gives a required isomorphic embedding. $\square$

3. Existence of free subsemigroups

We start with some simple statements about finite transformation semigroups.

Lemma 1. *For a finite semigroup T the following conditions are equivalent.*

1. *The semigroup T is not nilpotent.*
2. *Not every element of T is nilpotent.*
3. *The semigroup T contains an idempotent which is not a zero element of T.*

Proof. The eqivalence of the first two conditions is well known (see, for instance, [8, Proposition 8.1.2]).

Let T be not nilpotent. Each element of T in some power is an idempotent. If all these idempotents are equal to the same idempotent then this idempotent is not a zero of T for (T, X) is not nilpotent. In other case there are at least two different idempotents in T and at least one of then is not a zero. In both cases there exist an idempotent in T which is not a zero element.

Let T contains an idempotent e which is not a zero element. If e is the unique idempotent of T then T does not contain a zero and is not nilpotent. If T contains other idempotents then each power of e equals e and is not equal to none of them. Therefore e is not nilpotent even if T contains a zero. The proof is complete. $\square$

Lemma 2. *Let in finite semigroup T each idempotent is a left zero. Then $T \cdot E(T) = E(T)$ and $T^{|T|} = E(T)$.*

Proof. Let $t \in T$, $e \in E(T)$. Then $(te)^2 = t(ete) = te$ and we obtain the inclusion $T \cdot E(T) \subseteq E(T)$. The inverse inclusion is always true. Therefore $T \cdot E(T) = E(T)$.

Assume that for some $s_1, \ldots, s_k \in T$ the product $s_1 \ldots s_k$ is not an idempotent. If for some i, $1 \le i \le k$, the element s_i is an idempotent then $s_1 \ldots s_k = s_1 \ldots s_i$ for s_i is a left zero. Since $T \cdot E(T) \subseteq E(T)$ it would implies $s_1 \ldots s_i \in E(T)$ which contradicts with the assumption. Then elements $s_1, \ldots, s_k$ are not idempotents and each product $t_i = s_1 \ldots s_i$, $1 \le i \le k$, is not an idempotent as well for idempotents are left zeroes in T. Assume that among elements $t_1, \ldots, t_k$ there are equal ones. Let $t_i = t_j$, where $i < j$, and $t = s_{i+1} \ldots s_j$. Then for arbitrary $r \ge 1$ the equalities $t_i t^r = t_j t^{r-1} = t_i t^{r-1} = \ldots = t_i$ hold. Since T is finite for some $r_0 \ge 1$ the element t^{r_0} is an idempotent. Then $t_i = t_i t^{r_0} \in E(T)$. This is a contradiction. Hence, elements $t_1, \ldots, t_k$ are pairwise disjoint. Recall that they are not idempotents. This means that $k < |T|$. Then $T^{|T|} \subseteq E(T)$. The inverse inclusion always holds. Therefore $T^{|T|} = E(T)$ and the proof is complete. □

Lemma 3. *For a finite transformation semigroup (T, X) the following conditions are equivalent.*

1. *The semigroup (T, X) contains an idempotent which is not a left zero element.*

2. *There exist $e, s \in T$ and $x \in X$ such that e is an idempotent and*

$$x^e = x \neq x^s.$$

Proof. Let $e \in T$ be an idempotent such that $es \neq e$ for some $s \in T$. It implies that there exist $y \in X$ such that $y^{es} \neq y^e$. Take $x = y^e$. Then $x^e = y^{e^2} = y^e = x$ and $x^s = y^{es} \neq y^e = x$.

From the other hand, let $e, s \in T$ and $x \in X$ be such that $e^2 = e$ and $x^e = x \neq x^s$. Then $x^{es} = x^s \neq x = x^e$ and $es \neq e$. Hence the idempotent e is not a left zero element. □

If a semigroup (T, X) satisfy conditions of Lemma 3 then it is neither nilpotent by Lemma 1, nor contains a left zero.

Denote by F_k a free semigroup with basis $\{y_1, \ldots, y_k\}$, $k \geq 1$. Let $v \in F_k$ be a word of length $n \geq 1$. Then

$$v = z_1 \ldots z_n \qquad \text{for some } z_1, \ldots, z_n \in \{y_1, \ldots, y_k\}.$$

Define a mapping $\chi_v : \{1, \ldots n\} \to \{1, \ldots k\}$ by the condition

$$\chi_v(i) = j \text{ iff } z_i = y_j, \quad 1 \leq i \leq n.$$

Let $v_i = z_1 \ldots z_i, 1 \leq i \leq n$. In particular, $v_n = v$. For a k-tuple $(s_1, \ldots, s_k)$ over a semigroup S let us denote by $v_i(s_1, \ldots, s_k)$ the value of the word v_i on this tuple. One obtains this value substituting s_j instead of y_j, $1 \leq j \leq k$, and calculating the product in S.

Lemma 4. *Let a finite semigroup (T, X) contains an idempotent which is not a left zero element. For arbitrary word $v \in F_k$ of length n there exist a tuple $(\bar{t}(1), \ldots, \bar{t}(k))$ over the nth wreath power $W^n(T, X)$ of (T, X) such that for arbitrary word $u \in F_k$ of length $\leq n$, $u \neq v$, the inequality*

$$u(\bar{t}(1), \ldots, \bar{t}(k)) \neq v(\bar{t}(1), \ldots, \bar{t}(k))$$

holds.

Proof. By Lemma 3 we can fix $e, s \in T$ and $x, y \in X$ such that $e \in E(T)$, $x^e = x$, $x^s = y$ and $x \neq y$.

To define a required tuple $(\bar{t}(1), \ldots, \bar{t}(k))$ over $W^n(T, X)$ we describe portraits of $\bar{t}(1), \ldots, \bar{t}(k) \in W^n(T, X)$. Let

$$\bar{t}(i)_w = \begin{cases} s, & \text{if } \chi_v(1) = i \text{ and } |w| = 0 \\ s, & \text{if } \chi_v(|w|+1) = i \text{ and } w = w_1 y \text{ for some } w_1 \\ e & \text{otherwise} \end{cases}$$

for arbitrary $w \in X^{[n-1]}$ and $1 \leq i \leq k$.

To avoid confusion in notations let us denote by ω_k the word $\underbrace{x \ldots x}_{k \text{ times}}$, $k \geq 1$, and let ω_0 denotes the empty word Λ. We will show by induction on i that

$$\omega_n^{v_i(\bar{t}(1), \ldots, \bar{t}(k))} = w_i y \omega_{n-i}$$

for some $w_i \in X^{(i-1)}$, $i \leq n$.

The case $i = 1$. Then $v_1(\bar{t}(1), \ldots, \bar{t}(k)) = t(\chi_v(1))$. By the definition of $t(\chi_v(1))$ we obtain $t(\chi_v(1))_{\omega_0} = s$ and $t(\chi_v(1))_{\omega_l} = e$ for $1 \leq l \leq n-1$. This means that

$$\omega_n^{v_1(\bar{t}(1), \ldots, \bar{t}(k))} = \omega_n^{t(\chi_v(1))} = x^s x^e \ldots x^e = y \omega_{n-1}.$$

Assume that for some i, $1 \leq i < n$, our claim is valid. Then

$$v_{i+1}(\bar{t}(1), \ldots, \bar{t}(k)) = v_i(\bar{t}(1), \ldots, \bar{t}(k)) \cdot t(\chi_v(i+1)).$$

By the definition of $t(\chi_v(i+1))$ we obtain

$$t(\chi_v(i+1))_{w_i y} = s \text{ and } t(\chi_v(i+1))_{w_i y \omega_l} = e \text{ for } 1 \leq l \leq n-i.$$

Then we have

$$\omega_n^{v_{i+1}(\bar{t}(1), \ldots, \bar{t}(k))} = (w_i y \omega_{n-i})^{t(\chi_v(i+1))} = \\ (w_i y)^{t(\chi_v(i+1))} x^s x^e \ldots x^e = w_{i+1} y \omega_{n-i-1},$$

where $w_{i+1} = (w_i y)^{t(\chi_v(i+1))} \in X^{(i+1)}$, and the proof for $i+1$ is complete. In particular, it implies, that the last letter of the word

$$\omega_n^{v(\bar{t}(1), \ldots, \bar{t}(k))}$$

equals y.

Consider now a word $u \in F_k$ of length $\leq n$ such that $u \neq v$. It is sufficient to show that the last letter of the word

$$\omega_n^{u(\bar{t}(1), \ldots, \bar{t}(k))}$$

equals x. The definition of the elements $t(1), \ldots, t(k)$ implies that for arbitrary i, j, $1 \le i \le n$, $1 \le j \le k$, and the word $w \in X^{(n-i)}$ such that the last letter of w equals y (or $w = \Lambda$ if $i = n$) equality

$$(w\omega_i)^{t(j)} = \begin{cases} w^{t(j)} y \omega_{i-1}, & \text{if } j = \chi_v(n-i+1) \\ w^{t(j)} \omega_i & \text{otherwise} \end{cases}$$

holds. This means that to change the last letter of ω_n by the product of elements of the set $\{t(1), \ldots t(k)\}$ one has to take at least n multipliers. If $|u| < n$, then the element $u(\bar{t}(1), \ldots, \bar{t}(k))$ is a product of $< n$ multipliers and does not change the last letter of ω_n. If $|u| = n$, but $u \ne v$ then take the smallest number j such that jth letters of u and v are different. Then $\chi_u(j) \ne \chi_v(j)$ and the last $n - j + 1$ letters of the word

$$\omega_n^{v_j(\bar{t}(1), \ldots, \bar{t}(k))}$$

equal x. This implies that the rest $n - j$ multipliers of $u(\bar{t}(1), \ldots, \bar{t}(k))$ does not change the last letter of this word. The proof is complete. □

Theorem 1. *The infinite wreath power of a finite transformation semigroup (T, X) contains a free subsemigroup of arbitrary finite rank if and only if the semigroup (T, X) contains an idempotent which is not a left zero element.*

Proof. Let the semigroup (T, X) contains an idempotent which is not a left zero element. Fix arbitrary words u, v of the free semigroup F_k of rank k, $u \ne v$. Let $n = \max\{|u|, |v|\}$. By Lemma 4 there exist k elements $\bar{t}(1), \ldots, \bar{t}(k) \in W^n(T, X)$ such that

$$u(\bar{t}(1), \ldots, \bar{t}(k)) \ne v(\bar{t}(1), \ldots, \bar{t}(k)).$$

Hence for arbitrary increasing sequence $n_1, n_2, \ldots$ of positive integers the cartesian product

$$\prod_{i=1}^{\infty} W^{n_i}(T, X)$$

contains a free subsemigroup of rank k. The required statement now follows from part (2) of Proposition 1.

From the other hand, assume that all idempotents of the semigroup (T, X) are left zero elements. By Lemma 2 it implies the equality $T^{|T|} = E(T)$. From the multiplication rule of elements of wreath powers written in terms of their portraits it immediately follows that $W^\infty(T, X)^{|T|} \subset W^\infty(E(T), X)$. Since idempotents of T are left zeros each element of the last wreath power is an idempotent. This imply that each element of $W^\infty(T, X)$ generates finite subsemigroup. Therefore $W^\infty(T, X)$ contains no free subsemigroups. □

4. Most subsemigroups are free

Let X be a set, $|X| > 1$. The wreath power $W^\infty(\mathcal{T}(X), X)$ becomes a complete metric semigroup via the metric

$$d(\bar{t}, \bar{s}) = \begin{cases} 0, & \text{if } \bar{t} = \bar{s} \\ 2^{-\kappa(\bar{t}, \bar{s})} & \text{otherwise,} \end{cases}$$

where $\kappa(\bar{t}, \bar{s})$ is the least length of words $w \in X^*$ such that $\bar{t}_w \neq \bar{s}_w$, $\bar{t}, \bar{s} \in W^\infty(\mathcal{T}(X), X)$. It was established in [3] for finite X and for infinite X the proof is the same.

For arbitrary transformation semigroup (T, X) the wreath power $W^\infty(T, X)$ is a closed subsemigroup in $W^\infty(\mathcal{T}(X), X)$ and hence is a metric semigroup as well. For any $k \geq 1$ the product topology on $(W^\infty(T, X))^k$ can be defined by a metric d_k such that

$$d_k(\mathbf{t}, \mathbf{s}) = \max\{d(\bar{t}_i, \bar{s}_i) : 1 \leq i \leq k\},$$

where $\mathbf{t} = (\bar{t}_1, \ldots, \bar{t}_k), \mathbf{s} = (\bar{s}_1, \ldots, \bar{s}_k) \in (W^\infty(T, X))^k$. Therefore the metric space $(W^\infty(T, X))^k$ is complete.

Define a subset $\mathcal{F}_k \subset (W^\infty(T, X))^k$ as a set of elements $(\bar{t}_1, \ldots, \bar{t}_k) \in (W^\infty(T, X))^k$ such that the subsemigroup $\langle \bar{t}_1, \ldots, \bar{t}_k \rangle$ of $W^\infty(T, X)$ is free of rank k.

Theorem 2. *Let (T, X) be a transformation semigroup, $k \geq 1$. If the wreath power $W^\infty(T, X)$ contains a free subsemigroup of rank k then the subset $\mathcal{F}_k$ is dense in $(W^\infty(T, X))^k$.*

Proof. Fix arbitrary $\mathbf{t} = (\bar{t}_1, \ldots, \bar{t}_k) \in (W^\infty(T, X))^k$ and positive integer m. Let elements $\bar{f}_1, \ldots, \bar{f}_k \in W^\infty(T, X)$ generate a free semigroup of rank k. It is sufficient to construct $\mathbf{s} = (\bar{s}_1, \ldots, \bar{s}_k) \in \mathcal{F}_k$ such that $d_k(\mathbf{t}, \mathbf{s}) < 2^{-m}$.

We will use notations from the proof of Proposition 1. For arbitrary i, $1 \leq i \leq k$, let an element $\bar{s}_i \in W^\infty(T, X)$ satisfy equalities

$$p_m(\bar{s}_i) = p_m(\bar{t}_i)$$

and

$$s_k(\bar{s}_i)(a_1, \ldots, a_m) = \bar{f}_i, \quad a_1, \ldots, a_k \in X.$$

It follows from the first equality that $\kappa(\bar{t}, \bar{s}) > m$ and $d_k(\mathbf{t}, \mathbf{s}) < 2^{-m}$. For arbitrary word $v \in F_k$ denote the value $v(\bar{s}_1, \ldots, \bar{s}_k)$ by $v(\mathbf{s})$. Then the other equalities imply

$$s_k(v(\mathbf{s}))(a_1, \ldots, a_m) = v(\bar{f}_1, \ldots, \bar{f}_k), \quad a_1, \ldots, a_k \in X.$$

This means that the semigroup generated by $\bar{s}_1, \ldots, \bar{s}_k$ is free of rank k and $\mathbf{s} \in \mathcal{F}_k$. $\square$

Recall that a subset of a topological space is called nowhere dense if its closure has empty interior. A countable union of nowhere dense subsets is called meagre. The complement of a meagre set is called co-meagre.

Theorem 3. *Let (T, X) be a transformation semigroup. Then exactly one of the following holds.*

1. *The infinite wreath power $W^\infty(T, X)$ contains no free non-commutative subsemigroups.*
2. *For each $k \geq 1$ the subset $\mathcal{F}_k$ is co-meagre and not meagre in $(W^\infty(T, X))^k$.*

Proof. Assume that $W^\infty(T, X)$ contains a free non-commutative subsemigroup. Then for each $k \geq 1$ it contains a free subsemigroup of rank k and the subset $\mathcal{F}_k$ is dense in $(W^\infty(T, X))^k$ by Theorem 2. Baire's Category Theorem implies that it is sufficient to prove that $\mathcal{F}_k$ is co-meagre.

For arbitrary words $u, v \in F_k$ denote by $\mathcal{F}_k(u, v)$ the subset of elements $(\bar{t}_1, \ldots, \bar{t}_k) \in (W^\infty(T, X))^k$ such that $u(\bar{t}_1, \ldots, \bar{t}_k) = v(\bar{t}_1, \ldots, \bar{t}_k)$. Since multiplication is continuous in $W^\infty(T, X)$ the subset $\mathcal{F}_k(u, v)$ is closed in $(W^\infty(T, X))^k$. It is easy to see that

$$(W^\infty(T, X))^k \setminus \mathcal{F}_k = \bigcup_{\substack{u,v \in F_k \\ u \neq v}} \mathcal{F}_k(u, v).$$

For arbitrary $u, v \in F_k$, $u \neq v$, the complement of the set $\mathcal{F}_k(u, v)$ in $(W^\infty(T, X))^k$ contains $\mathcal{F}_k$. This means that this complement is open and dense. Hence the set $\mathcal{F}_k(u, v)$ is nowhere dense. Therefore $(W^\infty(T, X))^k \setminus \mathcal{F}_k$ is meagre. The proof is complete. □

References

[1] Meenaxi Bhattacharjee, *The ubiquity of free subgroups in certain inverse limits of groups*, J. Algebra **172** (1995), 134–146.

[2] Miklós Abért, *Group laws and free subgroups in topological groups*, Bull. London Math. Soc. **37** № 4, (2005), 525–534.

[3] A. S. Oliinyk, On free semigroups of automaton transformations// Math. Notes **63** № 2 (1998), 215–224.

[4] W.Hołubowski *The ubiquity of free subsemigroups of infinite triangular matrices*, Semigroup Forum **66** (2003), 231-235.

[5] V.Doroshenko *Free subsemigroups in topological semigroups*, Semigroup Forum **79** (2009), 427-434.

[6] R. I. Grigorchuk, V. V. Nekrashevych and V. I. Sushchansky, Automata, dynamical systems and groups// Proc. V.A. Steklov Inst. Math. **231** (2000) 134–215.

[7] J. Rhodes and B. Steinberg, *The q-theory of finite semigroups*, Springer Monographs in Mathematics, Springer, New York, 2009.

[8] O. Ganyushkin and V. Mazorchuk, *Classical finite transformation semigroups: An Introduction*, Algebra and Applications, Vol. 9, Springer, London, 2009.

Contact information

A. Oliynyk

Department of Mechanics and Mathematics
Kyiv Taras Shevchenko University
Volodymyrska, 60
Kyiv 01033
E-Mail: olijnyk@univ.kiev.ua
URL: http://algebra.kiev.ua/oliynyk/

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