Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 10 **(2010)**. Number 2. pp. 51 – 63

On modules over group rings of soluble groups with commutative ring of scalars

O. Yu. Dashkova

Communicated by L. A. Kurdachenko

Abstract. The author studies an $\mathbf{R}G$-module A such that $\mathbf{R}$ is a commutative ring, $A/C_A(G)$ is not a Noetherian $\mathbf{R}$-module, $C_G(A) = 1, G$ is a soluble group. The system of all subgroups $H \leq G$, for which the quotient modules $A/C_A(H)$ are not Noetherian $\mathbf{R}$-modules, satisfies the maximal condition. This condition is called the condition $max - nnd$. The structure of the group G is described.

Let A be a vector space over a field F. The subgroups of the group $GL(F, A)$ of all automorphisms of A are called linear groups. If A has a finite dimension over F then $GL(F, A)$ can be identified with the group of non-singular $n \times n$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in mathematics and have been well-studied. When A is infinite dimensional over F, the situation is totally different. The study of infinite dimensional linear groups requires some additional restrictions. In [1] it was introduced the definition of a central dimension of an infinite dimensional linear group. Let H be a subgroup of $GL(F, A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $dim_F H$ to be $dim_F(A/C_A(H))$. The subgroup H is said to have a finite central dimension if $dim_F H$ is finite and H has an infinite central dimension otherwise. Let $G \leq GL(F, A)$. In [1] it was defined $\mathbf{L_{id}(G)}$ as a set of all subgroups of G of infinite central dimension. In order to study infinite dimensional linear groups that are close to finite dimensional, it is natural to start making $\mathbf{L_{id}(G)}$ "very small". In [1] locally soluble infinite dimensional linear groups in

2000 Mathematics Subject Classification: *20F16; 20H25.*
Key words and phrases: *a maximal condition on subgroups, a Noetherian module, a soluble group.*

which the set $\mathbf{L_{id}(G)}$ satisfies the minimal condition have been described. The soluble linear groups in which the set $\mathbf{L_{id}(G)}$ satisfies the maximal condition have been studied in [2].

If $G \leq GL(F, A)$ then A can be considered as a FG-module. The natural generalization of this case is a consideration of the case when A is an $\mathbf{R}G$-module, $\mathbf{R}$ is a ring, the structure of which is similar to a structure of a field. The generalization of the notion of a central dimension of a linear group is the notion of a cocentralizer of a subgroup. This notion was introduced in [3]. Let A be an $\mathbf{R}G$-module where $\mathbf{R}$ is a ring, G is a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of H in the module A.

Sufficiently broad classes of modules over group rings are the classes of Noetherian modules over group rings and Artinian modules over group rings. Remind that a module is called an Artinian module if a partially ordered set of all submodules of this module satisfies the minimal condition. A module is called a Noetherian module if a partially ordered set of all submodules of this module satisfies the maximal condition. It should be noted that many problems of Algebra require the investigation of some specific Noetherian modules over group rings and Artinian modules over group rings as well modules over group rings which are not Noetherian or Artinian but which are similar to Noetherian modules or Artinian modules in some sense.

In [4] it was studied an $\mathbf{R}G$-module A such that $\mathbf{R}$ is a Dedekind domain and the cocentralizer of a group G in the module A is not an Artinian $\mathbf{R}$-module. It is defined $L_{nad}(G)$ as a system of all subgroups of a group G for which the cocentralizers in the module A are not Artinian $\mathbf{R}$-modules. It is introduced on $L_{nad}(G)$ the order with respect to the usual inclusion of subgroups. If $L_{nad}(G)$ satisfies the minimal condition as ordered set, we say that the group G satisfies the minimal condition on subgroups for which the cocentralizers in the module A are not Artinian $\mathbf{R}$-modules or simply that the group G satisfies the condition $min - nnd$. In [4] it is proved that a locally soluble group with the condition $min-nnd$ is soluble, and the structure of this group is described. The dual case when $L_{nad}(G)$ satisfies the maximal condition as ordered set it is studied in [5]. In [5] it is described the structure of a soluble group G.

In this paper it is considered an $\mathbf{R}G$-module A such that the cocentralizer of a group G in the module A is not a Noetherian $\mathbf{R}$-module. Let $L_{nnd}(G)$ be a system of all subgroups of a group G for which the cocentralizers in the module A are not Noetherian $\mathbf{R}$-modules. Introduce on $L_{nnd}(G)$ the order with respect to the usual inclusion of subgroups. If $L_{nnd}(G)$ satisfies the maximal condition as ordered set, we shall say that the group G satisfies the maximal condition on subgroups for which the cocentralizers in the module A are not Noetherian $\mathbf{R}$-modules or simply that the group G satisfies the condition $max - nnd$. In this paper we study soluble groups with the condition $max - nnd$ and generalize the results on soluble infinite dimensional linear groups with the condition $max - id$ [2]. It is considered the case where $\mathbf{R}$ is a commutative ring. The analogous problem for $\mathbf{R}G$-module A where $\mathbf{R}$ is a ring of integer numbers it was investigated in [6].

Later on it is considered an $\mathbf{R}G$-module A such that $C_G(A) = 1$, $\mathbf{R}$ is a commutative ring.

Lemma 1. *Let A be an $\mathbf{R}G$-module.*

(i) *If $K \leq H \leq G$, and the cocentralizer of a subgroup H in the module A is a Noetherian $\mathbf{R}$-module, then the cocentralizer of a subgroup K in the module A is a Noetherian $\mathbf{R}$-module also.*

(ii) *If $U, V \leq G$ such that its cocentralizers in the module A are Noetherian $\mathbf{R}$-modules, then the cocentralizer of a subgroup $< U, V >$ in the module A is a Noetherian $\mathbf{R}$-module also.*

Corollary 2. *Let A be an $\mathbf{R}G$-module and suppose that a group G satisfies the condition $max - nnd$. Let $ND(G)$ be a set of all elements $x \in G$, such that the cocentralizer of a group $< x >$ in the module A is a Noetherian $\mathbf{R}$-module. Then $ND(G)$ is a normal subgroup of G.*

Proof. By Lemma 1 $ND(G)$ is a subgroup of G. Since $C_A(x^g) = C_A(x)g$ for all $x, g \in G$, then $ND(G)$ is a normal subgroup of G. The corollary is proved. □

Lemma 3. *Let A be an $\mathbf{R}G$-module and suppose that G satisfies the condition $max - nnd$.*

(i) *If H is a subgroup of G then H satisfies $max - nnd$.*

(ii) *If $H_1 < H_2 < ... < H_n < ...$ is an infinite ascending chain of subgroups then the cocentralizer of every subgroup H_n in the module A is a Noetherian $\mathbf{R}$-module.*

(iii) *If the cocentralizer of a subgroup H in the module A is not a Noetherian $\mathbf{R}$-module then an ordered set by inclusion $\mathbf{L}[H, G]$ of all subgroups including H satisfies the condition $max - nnd$.*

The proof of this lemma is analogous to the proof of Lemma 1.2 [2].

Corollary 4. *Let A be an $\mathbf{R}G$-module and suppose that G satisfies the condition $max - nnd$. Then either the cocentralizer of every finitely generated subgroup of G in the module A is a Noetherian $\mathbf{R}$-module or G is a finitely generated group. In particular, if G is not a finitely generated group then $G = ND(G)$.*

Proof. Suppose that a group G is not finitely generated. Let L be a finitely generated subgroup of a group G. Then $G \backslash L \neq \emptyset$. Let $a_1 \in G \backslash L$, $L_1 =< L, a_1 >$. Since a subgroup L_1 is finitely generated, $G \backslash L_1 \neq \emptyset$. Using the similar arguments, it is constructed a strictly ascending series $L < L_1 < ... < L_n < ...$ of finitely generated subgroups. By Lemma 3 the cocentralizer of every subgroup L_n in the module A is a Noetherian $\mathbf{R}$-module. In particular, the cocentralizer of a subgroup L in the module A is a Noetherian $\mathbf{R}$-module. The corollary is proved. □

Corollary 5. *Let A be an $\mathbf{R}G$-module and suppose that G satisfies the condition $max - nnd$. Let K and H be subgroups of a group G and K be a normal subgroup of H. If the quotient group H/K does not satisfy the maximal condition then the cocentralizer of a subgroup K in the module A is a Noetherian $\mathbf{R}$-module.*

Lemma 6. *Let A be an $\mathbf{R}G$-module and suppose that G satisfies the condition $max - nnd$. Let H and Q be subgroups of a group G satisfying the following conditions:*

(i) *Q is a normal subgroup of H.*

(ii) *$H/Q = B/Q \times C/Q$.*

If the subgroups B/Q and C/Q do not satisfy the condition $max - nnd$ then the cocentralizer of a subgroup H in the module A is a Noetherian $\mathbf{R}$-module.

Proof. Since $H/B \simeq C/Q$ then the quotient group H/B does not satisfy the condition $max - nnd$. Corollary 5 yields that the cocentralizer of a subgroup B in the module A is a Noetherian $\mathbf{R}$-module. By the same reasons the cocentralizer of a subgroup C in the module A is a Noetherian $\mathbf{R}$-module. Since $H = BC$, by Lemma 1 the cocentralizer of a subgroup H in the module A is a Noetherian $\mathbf{R}$-module. The lemma is proved. □

Corollary 7. *Let A be an $\mathbf{R}G$-module and suppose that G satisfies the condition $max - nnd$. Let H and Q be subgroups of a group G satisfying the following conditions:*

(i) Q is a normal subgroup of H.

(ii) $H/Q = Dr_{\lambda \in \Lambda}(L_\lambda/Q)$, where $L_\lambda \neq Q$ for every $\lambda \in \Lambda$.

If the set Λ is infinite then the cocentralizer of a subgroup H in the module A is a Noetherian **R***-module.*

Proof. There are two infinite subsets Γ and Δ such that $\Gamma \cup \Delta = \Lambda$, $\Gamma \cap \Delta = \emptyset$. It follows that Γ (respectively Δ) has an infinite ascending chain

$$\Gamma_1 \subset \Gamma_2 \subset ... \subset \Gamma_n \subset ...,$$

and respectively

$$\Delta_1 \subset \Delta_2 \subset ... \subset \Delta_n \subset ...$$

of infinite subsets. Let $U/Q = Dr_{\lambda \in \Gamma}(L_\lambda/Q), V/Q = Dr_{\lambda \in \Delta}(L_\lambda/Q)$. Then there are two ascending chains of subgroups

$$< L_\lambda | \lambda \in \Delta_1 > \, < \, < L_\lambda | \lambda \in \Delta_2 > \, < ... < \, < L_\lambda | \lambda \in \Delta_n > \, < ...$$

and

$$< L_\lambda | \lambda \in \Gamma_1 > \, < \, < L_\lambda | \lambda \in \Gamma_2 > \, < ... < \, < L_\lambda | \lambda \in \Gamma_n > \, < ... \ .$$

It follows that the quotient groups U/Q and V/Q do not satisfy the condition $max - nnd$. Since $H/Q = U/Q \times V/Q$, by Lemma 6 the cocentralizer of a subgroup H in the module A is a Noetherian **R**-module. Corollary is proved. □

Lemma 8. *Let A be an* **R***G-module and suppose that G is a soluble group and G satisfies the condition $max - nnd$. Then the quotient group $G/ND(G)$ is a polycyclic group.*

Proof. Let $< 1 >= D_0 \leq D_1 \leq ... \leq D_n = G$ be the derived series of G. If G is not a finitely generated group then by Corollary 4 $G = ND(G)$. Suppose that G is a finitely generated group. Then the quotient group G/D_{n-1} is finitely generated. If the quotient groups D_{j+1}/D_j are finitely generated for every j, $j = 0, 1, ..., n-1$, then G is a polycyclic group. Therefore it is possible to assume that there is the number $m \in \{1, 2, ..., n-1\}$, such that the quotient group D_m/D_{m-1} is not finitely generated but the quotient groups D_{j+1}/D_j is finitely generated for each $j = m, ..., n-1$. In particular, D_m is not a finitely generated group. By Corollary 4 $D_m \leq ND(G)$. The lemma is proved. □

Later on it is considered an **R**G-module A such that the cocentralizer of the group G in the module A is not a Noetherian **R**-module.

Lemma 9. *Let A be an* **R***G-module and suppose that the group G satisfies the condition $max - nnd$. If $G \neq [G, G] = D$, then either the quotient group $G_{ab} = G/D$ is finitely generated or it includes a finitely generated subgroup S/D such that G/S is a Prüfer p-group for some prime p.*

Proof. Suppose that the quotient group G/D is not finitely generated. Let T/D be a periodic part of G/D. Choose in $Q = G/T$ a maximal $\mathbf{Z}$-independent set of elements $\{u_\lambda | \lambda \in \Lambda\}$. Then the subgroup $U =< u_\lambda | \lambda \in \Lambda >$ is free abelian, $U = Dr_{\lambda \in \Lambda} < u_\lambda >$, and the quotient group Q/U is periodic.

Assume that the set Λ is infinite. Since a free abelian group is projective (theorem 14.6 [7]), U includes the subgroup Y such that U/Y is a direct product of countable many copies of Prüfer p-groups. Then $Q/Y = U/Y \times W/Y$ for some subgroup W (theorem 21.2 [7]). Thus the quotient group Q/W is a direct product of countable many copies of Prüfer p-groups. By Corollary 7 the cocentralizer of a group G in the module A is a Noetherian $\mathbf{R}$-module. This contradiction shows that Λ is finite. Therefore the rank $r_0(Q)$ is finite.

Let V/D be a preimage of the subgroup U in G/D. Then by the choice of the subgroup U the quotient group $(G/D)/(V/D) \simeq G/V$ is periodic. If the set $\pi(G/V)$ is infinite then by Corollary 7 the cocentralizer of a group G in the module A is a Noetherian $\mathbf{R}$-module. Therefore the set $\pi(G/V)$ is finite. Let S_p/V be a Sylow p-subgroup of the quotient group G/V for every prime p and let

$$\Psi = \{p | p \in \pi(G/V), |S_p/V| = \infty\}.$$

By Lemma 6 $\Psi = \{p\}$. Let $P/V = Dr_{q \neq p} S_p/V$. Then the quotient group $G/P \simeq S_p/V$ is infinite and the quotient group P/V is finite. Suppose that the quotient group $(G/P)/(G/P)^p$ is infinite. Therefore by Corollary 7 the cocentralizer of a group G in the module A is a Noetherian $\mathbf{R}$-module. Contradiction. Consequently the quotient group $(G/P)/(G/P)^p$ is finite. It follows that $S_p/V = C_p/V \times K_p/V$ where K_p/V is finite and C_p/V is a divisible p-group. By Lemma 6 C_p/V is a Prüfer p-group. Hence the subgroup $S/V = P/V \times K_p/V$ is finite. Therefore S/D is a finitely generated group and the quotient group G/S is a Prüfer p-group. The lemma is proved. □

Corollary 10. *Let A be an $\mathbf{R}G$-module and suppose that a group G is abelian and G satisfies the condition $max - nnd$. If the cocentralizer of a group G in the module A is not a Noetherian $\mathbf{R}$-module then either a group G is finitely generated or it includes a finitely generated subgroup S such that G/S is a Prüfer p-group for some prime p.*

Corollary 11. *Let A be an $\mathbf{R}G$-module and suppose that G is an infinite abelian group and G satisfies the condition $max - nnd$. If the cocentralizer of a group G in the module A is not a Noetherian $\mathbf{R}$-module and the cocentralizer of every proper subgroup of a group G in the module A is a Noetherian $\mathbf{R}$-module then G is a Prüfer p-group for some prime p.*

Proof. Suppose the contrary. By Lemma 9 G includes a finitely generated subgroup S such that G/S is a Prüfer p-group for some prime p. Assume that the subgroup S is finite. Then $G = M \times K$, where M is a Prüfer p-group for some prime p, K is a non-trivial finite subgroup (theorem 21.2 [7]). The cocentralizers of the proper subgroup M and the proper subgroup K in the module A are Noetherian $\mathbf{R}$-modules. Then the cocentralizer of the group G in the module A is a Noetherian $\mathbf{R}$-module by Lemma 1. Contradiction. If the finitely generated subgroup S is infinite then S is a non-periodic subgroup. Therefore $S \neq S^q$ for some prime q, $q \neq p$. Then $G/S^q = H/S^q \times L/S^q$ where H/S^q is a Prüfer p-group for prime p, L/S^q is a non-trivial finite group (theorem 21.2 [7]). It followes that $G =< H, L >$. The cocentralizers of the proper subgroup H and the proper subgroup L in the module A are Noetherian $\mathbf{R}$-modules. Then the cocentralizer of the group G in the module A is a Noetherian $\mathbf{R}$-module by Lemma 1. Contradiction. Therefore the group G is a Prüfer p-group for some prime p. The corollary is proved. □

Lemma 12. *Let A be an $\mathbf{R}G$-module and suppose that $G = ND(G)$. Then the cocentralizer of every subgroup of finite index of G is not a Noetherian $\mathbf{R}$-module.*

Proof. Suppose the contrary. Assume that H is a subgroup of finite index of G and the cocentralizer of a subgroup H in the module A is a Noetherian $\mathbf{R}$-module. Let $K = Core_G(H)$. Then K is a normal subgroup of finite index of G. By Lemma 1 the cocentralizer of a subgroup K in the module A is a Noetherian $\mathbf{R}$-module. There is a finitely generated subgroup L such that $G = LK$. Since $G = ND(G)$, the cocentralizer of a subgroup L in the module A is a Noetherian $\mathbf{R}$-module. By Lemma 1 the cocentralizer of a group G in the module A is a Noetherian $\mathbf{R}$-module. Contradiction. The lemma is proved. □

Lemma 13. *Let A be an $\mathbf{R}G$-module. Suppose that a group G is soluble and satisfies the condition $max - nnd$. If the quotient group $G/[G,G]$ is not finitely generated then the group G has the series of the normal subgroups $B \leq N \leq L \leq F \leq Q \leq G$ such that the subgroup B is abelian, the quotient group N/R is locally nilpotent, the quotient group L/N is nilpotent, the quotient group F/L is abelian, the quotient group Q/F is a Prüfer p-group for some prime p, the quotient group G/Q is finite, and the cocentralizer of a subgroup F in the module A is a Noetherian $\mathbf{R}$-module.*

Proof. By Lemma 9 a group G contains the normal subgroup M such that the quotient group G/M is a Prüfer q-group for some prime q. By Corollary 4 $G = ND(G)$ and by Lemma 3 the cocentralizer of a subgroup

M in the module A is a Noetherian $\mathbf{R}$-module. Let $C = C_A(M)$. Then A/C is a Noetherian $\mathbf{R}$-module. Furthermore, $M \leq C_G(C)$. If $G = C_G(C)$ then $C \leq C_A(G)$. Therefore $A/C_A(G)$ is a Noetherian $\mathbf{R}$-module. Contradiction. It followes that $G \neq C_G(C)$. Therefore $G/C_G(C)$ is a Prüfer q-group. Since M is a normal subgroup of G then M is an $\mathbf{R}G$-submodule of the module A. By the choice of a subgroup C the quotient module A/C is a Noetherian $\mathbf{R}$-module. The module A has the finite series of $\mathbf{R}G$-submodules

$$< 0 >= C_0 \leq C_1 = C \leq C_2 = A,$$

where C_2/C_1 is a finitely generated $\mathbf{R}$-module.

By theorem 13.5 [8] the quotient group $S = G/C_G(C_2/C_1)$ contains the normal locally nilpotent subgroup N_0 such that the quotient group S/N_0 is embedded in the Cartesian product $\overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}G_\alpha$ of finite dimensional linear groups G_α of degree $f \leq n$ where n depends on the number of generating elements of an $\mathbf{R}$-module C_2/C_1 only. Since the group G is soluble then the quotient group S is soluble too. Therefore the projection H_α of the quotient group S/N_0 on each subgroup G_α is a soluble finite dimensional linear group of degree $\leq n$. By theorem 3.6 [8] each group H_α contains the normal subgroup K_α such that $|H_\alpha : K_\alpha| \leq \mu(n)$, the subgroup K_α is triangulable and K_α contains the nilpotent subgroup M_α of the step $\leq n-1$ such that M_α is a normal subgroup of G_α and the quotient group K_α/M_α is abelian. Therefore the group $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}H_\alpha$ contains the normal nilpotent subgroup $M = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}M_\alpha$ of step $\leq n-1$, the quotient group H/M has the normal abelian subgroup K/M where $K = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}K_\alpha$ and the quotient group $(H/M)/(K/M)$ is a locally finite group of the period $\leq \mu(n)!$. Since the group S is embedded in the Cartesian product $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}}H_\alpha$ then the group S has the the series of the normal subgroups $N_0 \leq L_0 \leq F_0 \leq S$ such that the subgroup N_0 is locally nilpotent, the quotient group L_0/N_0 is nilpotent, the quotient group F_0/L_0 is abelian and the quotient group S/F_0 is a locally finite group of the finite period.

Let

$$B = C_G(C_1) \cap C_G(C_2/C_1).$$

Each element of the group G acts trivially in every factor $C_{j+1}/C_j, j = 0, 1$. It follows that the subgroup B is abelian. By Remak's theorem

$$G/B \leq G/C_G(C_1) \times G/C_G(C_2/C_1).$$

Since $G/C_G(C_1) \simeq C_{q^\infty}$ then the group G has the series of normal subgroups $B \leq N \leq L \leq F \leq Q \leq G$ such that the subgroup B is abelian, the quotient group N/B is locally nilpotent, the quotient group L/N is nilpotent, the quotient group F/L is abelian, the quotient group Q/F is a Prüfer p-group for some prime p, the quotient group G/Q is a locally finite group of the finite period.

At first we prove that the quotient group G/Q is finite. Since the group G is soluble then the quotient group G/Q is soluble too. Let $E = Q/Q \leq D_1/Q \leq ... \leq D_{k-1}/Q \leq D_k/Q = G/Q$ be the derived series of G/Q.

We shall prove that the quotient group $(G/Q)/(D_{k-1}/Q) \simeq G/D_{k-1}$ is finite. Since the quotient group G/D_{k-1} is a periodic abelian group then G/D_{k-1} is decomposed in the direct product of its primary components $G/D_{k-1} = G_{p_1}/D_{k-1} \times G_{p_2}/D_{k-1} \times ... \times G_{p_i}/D_{k-1} \times ...$. Suppose that there is an infinite set of non-trivial primary components G_{p_i}/D_{k-1} in this decomposition. Then by Corollary 7 the cocentralizer of the group G in the module A is a Noetherian **R**-module. Contradiction.

Therefore $G/D_{k-1} = G_{p_1}/D_{k-1} \times G_{p_2}/D_{k-1} \times ... \times G_{p_n}/D_{k-1}$ for some natural number n. We prove that the quotient group G_{p_i}/D_{k-1} is finite for each number $i = 1, ..., n$. Otherwise there is an infinite quotient group G_{p_l}/D_{k-1} for some $l = 1, 2, ..., n$. Since the period of the quotient group G/Q is finite then the quotient group G_{p_l}/D_{k-1} is decomposed in the direct product $G_{p_i}/D_{k-1} = J_1/D_{k-1} \times J_2/D_{k-1} \times ... \times J_m/D_{k-1} \times ...$ of infinite number of non-trivial subgroups J_m/D_{k-1}. Therefore $G/D_{k-1} = G_{p_1}/D_{k-1} \times G_{p_2}/D_{k-1} \times ... \times G_{p_{l-1}}/D_{k-1} \times G_{p_{l+1}}/D_{k-1} \times ... \times G_{p_n}/D_{k-1} \times J_1/D_{k-1} \times J_2/D_{k-1} \times ... \times J_m/D_{k-1} \times ...$. Then by Corollary 7 the cocentralizer of the group G in the module A is a Noetherian **R**-module. Contradiction. Therefore the quotient group G_{p_i}/D_{k-1} is finite for each number $i = 1, ..., n$. It followes that the quotient group G/D_{k-1} is finite. By Lemma 12 the cocentralizer of the subgroup D_{k-1} in the module A is not a Noetherian **R**-module. The quotient-group D_{k-1}/Q is a locally finite group of the finite period. We conduct the same arguments as in the case of the quotient group G/Q. We obtain that the quotient group D_{k-1}/D_{k-2} is finite. We continue our arguments. At the step with the number k we obtain that the quotient group D_1/Q is finite. Thus the quotient group G/Q is finite too. It follows that the quotient group G/F has the normal Prüfer p-group Q/F for some prime p such that the index $|(G/F) : (Q/F)|$ is finite. By Lemma 12 the cocentralizer of the subgroup Q in the module A is not a Noetherian **R**-module. Since the quotient group Q/F is a Prüfer p-group then the infinite ascending series $Q_1/F < Q_2/F < ... < Q_n/F < ...$ exists. Therefore $Q_1 < Q_2 < ... < Q_n < ...$ is an infinite ascending series of subgroups of the group G. By Lemma 3 the cocentralizer of each subgroup Q_n in the module A is a Noetherian **R**-module. Since $F \leq Q_1$, then by

Lemma 1(i) the cocentralizer of the subgroup F in the module A is a Noetherian $\mathbf{R}$-module. The lemma is proved. □

The following result simply follows from Lemma 13.

Theorem 1. *Let A be an $\mathbf{R}G$-module and suppose that G is a soluble group satisfying the condition $max - nnd$. If the quotient group $G/[G,G]$ is not finitely generated then G satisfies the following conditions:*

(1) A has the finite series of $\mathbf{R}G$-submodules

$$< 0 >= C_0 \leq C_1 = C \leq C_2 = A,$$

such that C_2/C_1 is a finitely generated $\mathbf{R}$-module and the quotient group $Q = G/C_G(C_1)$ is a Prüfer q-group for some prime q;

(2) $B = C_G(C_1) \cap C_G(C_2/C_1)$ is an abelian normal subgroup such that the cocentralizer of it in the module A is a Noetherian $\mathbf{R}$-module;

(3) the group G has the seies of the normal subgroups $B \leq N \leq L \leq F \leq Q \leq G$ such that the subgroup B is abelian, the quotient group N/R is locally nilpotent, the quotient group L/N is nilpotent, the quotient group F/L is abelian, the quotient group Q/F is a Prüfer p-group for some prime p, the quotient group G/Q is finite, and the cocentralizer of a subgroup F in the module A is a Noetherian $\mathbf{R}$-module.

The next natutal step is the consideration of the case when a group G is finitely generated.

Theorem 2. *Let A be an $\mathbf{R}G$-module and suppose that G is a finitely generated soluble group satisfying the condition $max - nnd$. If the cocentralizer of the subgroup $ND(G)$ in the module A is a Noetherian $\mathbf{R}$-module then G has the series of normal subgropups $B \leq N \leq L \leq G$ such that the subgroup B is abelian, the quotient group N/B is locally nilpotent, the quotient group L/N is nilpotent, the guotient group G/L is polycyclic.*

Proof. Let $C = C_A(ND(G))$. Since the cocentralizer of the subgroup $ND(G)$ in the module A is a Noetherian $\mathbf{R}$-module then A has the finite series of $\mathbf{R}G$-submodules

$$< 0 >= C_0 \leq C_1 = C \leq C_2 = A,$$

such that C_2/C_1 is a finitely generated $\mathbf{R}$-module.

By theorem 13.5 [8] the quotient group $S = G/C_G(C_2/C_1)$ contains the normal locally nilpotent subgroup N_0 such that the quotient group S/N_0 is embedded in the Cartesian product $\overline{\mathbf{\Pi}}_{\alpha \in \mathcal{A}} G_\alpha$ of finite dimensional linear groups G_α of degree $f \leq n$ where n depends on the number of generating elements of an $\mathbf{R}$-module C_2/C_1 only. Since the group G is soluble then the quotient group S is soluble too. Therefore the projection H_α of the group S on each subgroup G_α is a soluble finite dimensional linear group of degree $\leq n$. By theorem 3.6 [8] each group H_α contains the normal subgroup K_α such that $|H_\alpha : K_\alpha| \leq \mu(n)$, the subgroup K_α is triangulable and K_α contains the nilpotent subgroup M_α of step $\leq n-1$ such that M_α is a normal subgroup of G_α and the quotient group K_α/M_α is abelian. Therefore the group $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} H_\alpha$ contains the normal nilpotent subgroup $M = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} M_\alpha$ of the step $\leq n-1$, the quotient group H/M has the normal abelian subgroup K/M where $K = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} K_\alpha$ and the quotient group $(H/M)/(K/M)$ is a locally finite group of the period $\leq \mu(n)!$. Since the group S is embedded in the Cartesian product $H = \overline{\mathbf{\Pi}}_{\alpha\in\mathcal{A}} H_\alpha$ then the group S has the the series of the normal subgroups $N_0 \leq L_0 \leq F_0 \leq S$ such that the subgroup N_0 is locally nilpotent, the quotient group L_0/N_0 is nilpotent, the quotient group F_0/L_0 is abelian and the quotient group S/F_0 is a locally finite group of the finite period.

Let

$$B = C_G(C_1) \cap C_G(C_2/C_1).$$

Each element of the group G acts trivially in every factor C_{j+1}/C_j, $j = 0, 1$. It follows that the subgroup B is abelian. By Remak's theorem

$$G/B \leq G/C_G(C_1) \times G/C_G(C_2/C_1).$$

Since $ND(G) \leq C_G(C_1)$ then the quotient group $G/C_G(C_1)$ is polycyclic by Lemma 8.

The quotient group $S = G/C_G(C_2/C_1)$ has the series of the normal subgroups $N_0 \leq L_0 \leq F_0 \leq S$ such that the subgroup N_0 is locally nilpotent, the quotient group L_0/N_0 is nilpotent, the quotient group F_0/L_0 is abelian, the quotient group S/F_0 is a locally finite group of the finite period. Since the group G is finitely generated then the quotient group S is finitely generated. Therefore the quotient group S/F_0 is finite. It followes that the quotient group S/L_0 is an almost abelian group. Since the quotient group S/L_0 is finitely generated then S/L_0 is a polycyclic group. Therefore the group S has the series of normal subgroups $N_0 \leq L_0 \leq S$ such that the subgroup N_0 is locally nilpotent, the quotient group L_0/N_0 is nilpotent, the quotient group S/L_0 is a polycyclic group. Since the quotient group $G/C_G(C_1)$ is a polycyclic and the subgroup B is abelian then the group G has the series of normal subgroups $B \leq N \leq L \leq G$ such that the subgroup B is abelian, the quotient group N/B is locally nilpotent, the quotient group L/N is nilpotent, and G/L is a polycyclic group. The theorem is proved. □

Theorem 3. *Let A be an* $\mathbf{R}G$*-module and suppose that G is a finitely generated soluble group satisfying the condition $max - nnd$. If the co-*

centralizer of the subgroup $ND(G)$ in the module A is not a Noetherian $\mathbf{R}$-module then G contains the normal subgroup L satisfying the following conditions:

(1) *The quotient group G/L is polycyclic.*

(2) *$L \leq ND(G)$ and the cocentralizer of the subgroup L in the module A is not a Noetherian $\mathbf{R}$-module.*

(3) *The quotient group $L/[L, L]$ is not finitely generated.*

Proof. Let

$$< 1 >= D_0 \leq D_1 \leq ... \leq D_n = G$$

be the derived series of the group G. If the group G is polycyclic then the subgroup $ND(G)$ is polycyclic also. By Lemma 1 the cocentralizer of the subgroup $ND(G)$ in the module A is a Noetherian $\mathbf{R}$-module. Contradiction. Therefore there is the number $m \in \{1, ..., n-1\}$, such that the quotient group G/D_m is polycyclic and the quotient group D_m/D_{m-1} is not finitely generated. Let $L = D_m$. By Corollary 4 $L \leq ND(G)$. Suppose that the cocentralizer of the subgroup L in the module A is a Noetherian $\mathbf{R}$-module. Since the quotient group $ND(G)/L$ is finitely generated then by Lemma 1 the cocentralizer of a subgroup $ND(G)$ in the module A is a Noetherian $\mathbf{R}$-module also. Contradiction. The theorem is proved. □

References

[1] Dixon M.R., Evans M.J., Kurdachenko L.A. Linear groups with the minimal condition on subgroups of infinite central dimension. *Journal of Algebra.* – 2004.– V. 277. – P.172–186.

[2] Kurdachenko L.A., Subbotin I.Ya. Linear groups with the maximal condition on subgroups of infinite central dimension. *Publicacions Mat.* – 2006.– V. 50, no. 1. – P.103–131.

[3] Kurdachenko L.A. On groups with minimax classes of conjugate elements. *"Infinite groups and adjoining algebraic structures". Academy of Sciences of the Ukraine. Institute of Mathematics, Kyev.* – 1993. – P. 160–177 (in Russian).

[4] Dashkova O.Yu. On modules over group rings of locally soluble groups with rank restrictions on some systems of subgroups. *Asian-European Journal of Mathematics.* – 2010. – V. 3, no. 1. – P. 45–55.

[5] Dashkova O.Yu. On the one class of modules over group rings of soluble groups with the restrictions on some systems of subgroups. *Fundamental and Applied Mathematics.* – 2008. – V. 14, no. 7. – P. 111–119 (in Russian).

[6] Dashkova O.Yu. Modules over integral group rings of locally soluble groups with the restrictions on some systems of subgroups. *Reports of the National Academy of Sciences of the Ukraine, Mathematics.* – 2009. – No. 2. – P. 14–19 (in Russian).

[7] Fuchs L. Infinite Abelian Groups. *Mir, Moscow.* – 1974. – V.1. – 336 p.(in Russian).

[8] Wehrfritz B.A.F. Infinite linear Groups. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York, Heidelberg, Berlin.*– 1973. – 229 p.

Contact information

O. Yu. Dashkova 49010, Ukraine, Dniepropetrovsk, prospekt Gagarina, 72, Dniepropetrovsk National University, Department of Mathematics and Mechanics
E-Mail: odashkova@yandex.ru

Received by the editors: 06.02.2011
and in final form 03.03.2011.