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A sequence of factorizable subgroups

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Abstract. Let G be a non-abelian non-simple group. In this article the group G such that $G = MC_G(M)$ will be studied, where M is a proper maximal subgroup of G and $C_G(M)$ is the centralizer of M in G.

1. Introduction

Let G be a group, and let M and N be two subgroups of G. The group G is called *central factorizable* if G can be written as the central product of the subgroups M and N. In this case we say M and N are *CF-subgroups* of G (Central Factorizer subgroup), and we have

$$G/M \cap N \cong G/M \oplus G/N. \tag{1}$$

Since $M \subseteq C_G(N)$ and $N \subseteq C_G(M)$, so $G = MC_G(M) = NC_G(N)$ are the other representations of the central factorizability of G. Therefore M is a CF-subgroup of G whenever $G = MC_G(M)$. One notes that every CF-subgroup is normal, hence simple groups are the first example of groups without any proper CF-subgroups. Clearly every subgroup of an abelian group is a CF-subgroup.

We are interested to the case that M and $C_G(M)$ are proper subgroups. Thus if M is a proper maximal subgroup of G such that $Z(G) \not\subseteq M$, then M is a CF-subgroup (*CF-maximal subgroup*). Indeed, if $Z(G) \not\subseteq \Phi(G)$, the Frattini subgroup of G, then G contains a CF-maximal subgroup.

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Definition 1. Let $\mathcal{S} = \{G_n\}$ be a sequence of subgroups of G, indexed by the non negative integers. We call $\mathcal{S}$ a *CF-sequence* of G if

1. $G_0 = G$,
2. $G_n = G_m Z(G_n)$ for all $m > n$ and
3. G_{n+1} is a proper maximal subgroup of G_n.

According to this definition, G_n is a non-abelian non-simple group for every n, and G_m is a CF-subgroup of G_n for all $m > n$.

Let n be a positive integer and $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ be the prime decomposition of n. Define

$$\Omega(n) = \sum_{i=1}^{t} \alpha_i,$$

$\Omega(1) = 0$ and $\Omega(\infty) = \infty$. Let $D(G) = Z(G) \cap \Phi(G)$ and

$$\Omega_G = \Omega([Z(G) : D(G)]),$$

where $[Z(G) : D(G)]$ denotes the index of $D(G)$ in $Z(G)$. We prove the following theorem.

Theorem 1. If G is a group and $\Phi(H) \subset \Phi(G)$ for every normal subgroup H with finite index, then G has a CF-sequence of length Ω_G.

In the final section we extend this work to abstract classes of groups by defining two closure operations $\boldsymbol{C_0}$ and $\boldsymbol{C}$, finite central product and (infinite) central product, respectively. In particular we prove,

Theorem 2. Let $\mathfrak{A}$ be the class of abelian groups, and $\mathfrak{X}$ and $\mathfrak{Y}$ two $\boldsymbol{C_0}$-closed classes of groups such that $\mathfrak{A} \leq \mathfrak{X}$. Let G be a group and M a CF-maximal subgroup of G.

1. If M is an $\mathfrak{X}$-group then so is G.
2. If $\mathfrak{Y}$ is $\boldsymbol{H}$-closed and M is an $\mathfrak{X}\mathfrak{Y}$-group then G is an $\mathfrak{X}\mathfrak{Y}$-group.

2. Upper central series of a CF-sequence

Let $\mathcal{S} = \{G_n\}$ be a CF-sequence of a group G. In this section we study the upper central series of the terms of $\mathcal{S}$ and we extend it to their lower central and derived series.

Lemma 1. Let M be a CF-subgroup of G and $C = C_G(M)$. Then

1. $G/Z(G) \cong M/Z(M) \oplus C/Z(C)$,

2. $G/Z(M) \cong G/M \oplus M/Z(M)$ and

3. $C = Z(G)$ if M is maximal.

Proof. Since M and C are CF-subgroups, $Z(M) \subseteq Z(G)$ and $Z(C) \subseteq Z(G)$. Thus $Z(M) = M \cap Z(G)$ and $Z(C) = C \cap Z(G) = Z(G)$. Using equation (1) and $M \cap C = Z(M)$ we have

$$G/Z(G) \cong MZ(G)/Z(G) \oplus C/Z(G) \cong M/Z(M) \oplus C/Z(C),$$

$$G/Z(M) \cong M/Z(M) \oplus C/Z(M)$$

and

$$G/M \cong C/Z(M).$$

Hence

$$G/Z(M) \cong G/M \oplus M/Z(M).$$

Since $M \subseteq C_G(C)$, if M is maximal then $C_G(C) = M$ or $C_G(C) = G$. If $M = C_G(C)$ then $Z(M) = M \cap C = C_G(C) \cap C = Z(C)$ and $G/M \cong C/Z(C)$. This implies the index of $Z(C)$ in C is a prime and C is abelian, which is a contradiction. Hence $C_G(C) = G$ and $C = Z(G)$. $\square$

From the part (3) of Lemma 1 we have

$$G/Z(G) \cong M/Z(M), \tag{2}$$

$$G/M \cong Z(G)/Z(M) \tag{3}$$

and

$$G/Z(M) \cong M/Z(M) \oplus Z(G)/Z(M). \tag{4}$$

Also M is a CF-maximal subgroup of G if and only if $Z(G) \not\subset M$. Therefore, the necessary and sufficient condition for G to contain a CF-maximal subgroup is $Z(G) \not\subset \Phi(G)$. Thus when the centre of G is trivial, G has no CF-maximal subgroups.

The following proposition is a generalization of Lemma 1 for terms of the upper central series of members of a CF-sequence $\{G_n\}$. For simplicity we denote $Z_n = Z(G_n)$ and $Z_{\alpha,n} = Z_\alpha(G_n)$, the α-th term of the upper central series of G_n for each α and n.

Proposition 1. Let $\{G_n\}$ be a CF-sequence of G. Then for every m and n that $m > n$, and each α, we have

1. $G_m \cap Z_{\alpha,n} = Z_{\alpha,m}$,

2. $G_n/G_m \cong Z_{\alpha,n}/Z_{\alpha,m}$,

3. $G_n/Z_{\alpha,n} \cong G_m/Z_{\alpha,m}$ and

4. $G_n/Z_{\alpha,m} \cong G_m/Z_{\alpha,m} \oplus Z_n/Z_m$.

Proof. Let $m = n + k$. We prove it by induction on k and α. Let $\alpha = 1$. If $k = 1$ then G_{n+1} is a CF-maximal subgroup of G_n and equations (2), (3) and (4) result it. Suppose the proposition is correct for $k - 1$, then $Z_{n+k-1} = G_{n+k-1} \cap Z_n$ and $Z_{n+k-1} \subseteq Z_n$. Since $Z_{n+k} \subseteq Z_{n+k-1}$ so $Z_{n+k} \subseteq G_{n+k-1} \cap Z_n$. By assumption $G_n = G_{n+k}Z_n$, thus $G_{n+k} \cap Z_n \subseteq G_{n+k} \cap C_{G_n}(G_{n_k}) = Z_{n+k}$ and we get $G_{n+k} \cap Z_n = Z_{n+k}$, which is

$$G_m \cap Z_n = Z_m. \tag{5}$$

Hence

$$Z_n/Z_m = Z_n/G_m \cap Z_n \cong G_m Z_n/G_m = G_n/G_m,$$

and

$$G_m/Z_m = G_m/G_m \cap Z_n \cong G_m Z_n/Z_n = G_n/Z_n.$$

Using equations (1) and (5) we have

$$G_n/Z_m \cong G_m/Z_m \oplus Z_n/Z_m.$$

This completes the induction on k.

Let the above conclusions be correct for $\alpha - 1$ and $G_n/Z_{\alpha-1,n} \cong G_m/Z_{\alpha-1,m}$. Since the groups of inner automorphisms of two isomorphic groups are isomorphic, so (3) as required.

Now we show $G_m \cap Z_{\alpha,n} = Z_{\alpha,m}$. Since $\{G_n\}$ is a CF-sequence, we have $G_n = G_m Z_n = G_m Z_{\alpha,n}$, thus

$$G_n/Z_{\alpha,n} \cong G_m/G_m \cap Z_{\alpha,n},$$

and using part (3)

$$G_m/Z_{\alpha,m} \cong G_m/G_m \cap Z_{\alpha,n}.$$

Therefore, it is enough to show $G_m \cap Z_{\alpha,n} \subseteq Z_{\alpha,m}$ or

$$(G_m \cap Z_{\alpha,n})/Z_{\alpha-1,m} \subseteq Z(G_m/Z_{\alpha-1,m}) = Z_{\alpha,m}/Z_{\alpha-1,m}.$$

Let $xZ_{\alpha-1,m} \in (G_m \cap Z_{\alpha,n})/Z_{\alpha-1,m}$ and $yZ_{\alpha-1,m} \in G_m/Z_{\alpha-1,m}$, where $x \in G_m \cap Z_{\alpha,n}$, $y \in G_m$ and $x, y \notin Z_{\alpha-1,m}$. Since $x \in G_m$ and

$$G_m \cap Z_{\alpha-1,n} = Z_{\alpha-1,m}, \tag{6}$$

we have $x \notin Z_{\alpha-1,n}$ and $xZ_{\alpha-1,n} \in Z_{\alpha,n}/Z_{\alpha-1,n} = Z(G_n/Z_{\alpha-1,n})$. Also from $y \in G_m \subseteq G_n$ and equation (6) we have $y \notin Z_{\alpha-1,n}$ and $yZ_{\alpha-1,n} \in G_n/Z_{\alpha-1,n}$. This proves

$$xyZ_{\alpha-1,n} = yxZ_{\alpha-1,n}.$$

If $xyZ_{\alpha-1,m} \neq yxZ_{\alpha-1,m}$ then $xyx^{-1}y^{-1} \notin Z_{\alpha-1,m}$. Since $xyx^{-1}y^{-1} \in G_m$ so $xyx^{-1}y^{-1} \notin Z_{\alpha-1,n}$, which is a contradiction. Thus $G_m \cap Z_{\alpha,n} \subseteq Z_{\alpha,m}$ and (1) as required.

Using part (1)

$$Z_{\alpha,n}/Z_{\alpha,m} = Z_{\alpha,n}/G_m \cap Z_{\alpha,n} \cong G_m Z_{\alpha,n}/G_m = G_n/G_m,$$

which results part (2).

Finally by equation (1) and $G_m \cap Z_n \subseteq Z_{\alpha,m} = G_m \cap Z_{\alpha,n}$ we get

$$G_n/Z_{\alpha,m} = G_m/Z_{\alpha,m} \oplus Z_n Z_{\alpha,m}/Z_{\alpha,m} \cong G_m/Z_{\alpha,m} \oplus Z_n/Z_m.$$

This implies part (4). $\square$

If $G = MN$ is a central factorizable group, then it is easy to prove

$$Z_k(G) = Z_k(M)Z_k(N),$$

$$\gamma_k(G) = \gamma_k(M)\gamma_k(N),$$

and

$$G^{(k)} = M^{(k)}N^{(k)},$$

where $Z_k(G), \gamma_k(G)$ and $G^{(k)}$ are k-th term of the upper, lower and derived series of G, respectively. Hence if $\{G_n\}$ is a CF-sequence of G then for each $m > n$,

1. $Z_k(G_n) = Z_k(G_m)Z(G)$ when $k \geq 1$,
2. $G_n^{(k)} = G_m^{(k)}$ when $k \geq 1$ and
3. $\gamma_k(G_n) = \gamma_k(G_m)$ when $k \geq 2$.

In particular, if G is nilpotent then $cl(G) = cl(G_n)$, and if G is soluble then $d(G) = d(G_n)$ for each n, where $cl(G)$ and $d(G)$ are nilpotency class and defect of a given group G, respectively.

3. Groups with a CF-sequence

Proof of Theorem 1. **Case I)** $\Omega_G = 0$. In this case $Z(G) \subseteq \Phi(G)$ and G has no CF-maximal subgroup. Thus G has no CF-sequences.

Case II) $\Omega_G = \infty$. Let $G_0 = G$. Then $Z(G_0)$ has infinite order. Since $\Omega_{G_0} = \infty, Z(G_0) \nsubseteq \Phi(G_0)$. In this case there exist a CF-maximal subgroup G_1 of G_0 such that $G_0 = G_1 Z(G_0)$ and $G_0/G_1 \cong Z(G_0)/Z(G_1)$. Hence $[Z(G_0) : Z(G_1)]$ is prime and $Z(G_1)$ has infinite order. Since G_1 is normal in G_0 and has a finite index, $\Phi(G_1) \subseteq \Phi(G_0)$. Thus $\Omega_{G_1} = \infty$ and $Z(G_1) \nsubseteq \Phi(G_1)$. So there exists a CF-maximal subgroup G_2 of G_1 such that $G_1 = G_2 Z(G_1)$, $[G_1 : G_2] = [Z(G_1) : Z(G_2)]$ is prime, and $Z(G_2)$ has infinite order. If $G_0 = G_1 Z(G) = G_2 Z(G_1) Z(G)$ then $G_0 = G_2 Z(G)$, G_2 is normal in G_0 and $[G_0 : G_2] = [Z(G_0) : Z(G_1)][Z(G_1) : Z(G_2)] < \infty$. Hence $\Phi(G_2) \subseteq \Phi(G_0)$ and $\Omega_{G_2} = \infty$. This procedure gives an infinite sequence $\{G_n\}$ of subgroups of G such that $G_0 = G$ and

$$G_n = G_{n-1} Z(G_n). \tag{7}$$

We show $G_n = G_m Z(G_n)$ for $m > n$. Let $m = n + k$. We prove it by induction on k. For $k = 1$ it is (7). Let $G_n = G_{n+k-1} Z(G_n)$. Since $G_{n+k-1} = G_{n+k} Z(G_{n+k-1})$, $G_n = G_{n+k} Z(G_{n+k-1}) Z(G_n)$ and $Z(G_{n+k-1}) \subseteq Z(G_n)$. Thus $G_n = G_{n+k} Z(G_n)$ and the induction is completed. This shows $\{G_n\}$ is an infinite CF-sequence.

Case III) $0 < \Omega_G < \infty$. Let $G_0 = G$. Since $\Omega_G > 0$, thus $Z(G_0) \not\subset \Phi(G_0)$ and there exists a CF-maximal subgroup G_1 of G_0 such that $G_0 = G_1 Z(G_0)$, and $[G_0 : G_1] = [Z(G_0) : Z(G_1)]$ is a prime. So $\Omega(|Z(G_1)|) = \Omega(|Z(G_0)|) - 1$. On the other hand G_1 is normal in G_0. Therefore $\Phi(G_1) \subseteq \Phi(G_0)$. If $Z(G_1) \subset \Phi(G_0)$ then $\Omega_{G_0} = [Z(G_0) : D(G_0)] = 1$ and

$$G_1 \leq G_0$$

is a CF-sequence of G. Otherwise, $Z(G_1) \nsubseteq \Phi(G_1)$ and there is a CF-maximal subgroup G_2 of G_1 of prime index such that $G_1 = G_2 Z(G_1)$. Hence $G_0 = G_2 Z(G_0)$, $G_2 \triangleleft G_0$, $\Phi(G_2) \subset \Phi(G_0)$ and $\Omega(|Z(G_2)|) = \Omega(|Z(G_0)|) - 2$. Since Ω_{G_0} is finite, after a finite steps we obtain a CF-maximal subgroup G_l of G_{l-1} of prime index such that $G_{l-1} = G_l Z(G_{l-1})$, $\Omega(|Z(G_l)|) = \Omega(|Z(G_0)|) - l$ and $Z(G_l) \subseteq \Phi(G_0)$. Hence $\Omega_G = l$ is the length of the sequence

$$G_l \leq G_{l-1} \leq \cdots \leq G_1 \leq G_0 = G,$$

where

$$G_n = G_{n+1} Z(G_n) \quad \text{for} \;\; 1 \leq n \leq \Omega_G.$$

Now it is easy to see that

$$G_n = G_m Z(G_n) \text{ for } 1 \leq n < m \leq \Omega_G.$$

This completes the proof. □

Recall a group G is called with max if each non-empty set of subgroups of G has a maximal element.

Corollary 1. If G is with max then G has a CF-sequence of length Ω_G.

Proof. If G is with max then every subgroup of G is finitely generated and $\Phi(H) \subseteq \Phi(G)$ for every normal subgroup H of G. □

The most well known classes of groups with max are finitely generated nilpotent groups and polycyclic groups. Therefore, using Theorem 1 and Corollary 1, each finitely generated nilpotent group and polycyclic group has a CF-sequence of length Ω_G. As we showed in section 2, when G is a finitely generated group then each term of its CF-sequence is nilpotent of class $cl(G)$. Similarly, when G is a polycyclic group then each term of its CF-sequence is soluble with defect $d(G)$.

4. On abstract classes of groups

In this section we generalize some senses of pervious sections and discuss about the invariant properties by the central product. Suppose A and B are two subgroups of a group G such that $[A, B] = 1$. Then the central product AB is a subgroup of G. It is also correct for $\prod_{i \in I} C_i$ where $\{G_i\}_{i \in I}$ is a collection of distinct subgroups of G, such that $[G_i, G_j] = 1$ for $i \neq j$. Our interest is the cases that AB and $\prod_{i \in I} C_i$ are $\mathfrak{X}$-groups, whenever A, B and every C_i are $\mathfrak{X}$-groups for an abstract class of groups $\mathfrak{X}$.

Let $\mathfrak{X}$ be a class of groups. We say the class $\mathfrak{X}$ is closed under taking finite central product, or $\boldsymbol{C_0}$-*closed*, if C_1, C_2 are the $\mathfrak{X}$-subgroups of G such that $[G_1, G_2] = 1$ and $C_1 C_2$ is a $\mathfrak{X}$-subgroup of G. Similarly $\mathfrak{X}$ is $\boldsymbol{C}$-*closed* or central product closed, if $\prod_{i \in I} C_i$, the (central) product of the subgroups C_i, is an $\mathfrak{X}$-group, for a collection $\{G_i\}_{i \in I}$ of distinct $\mathfrak{X}$-subgroups of G. It is easy to see that $\boldsymbol{C_0}$ and $\boldsymbol{C}$ are two closure operations.

By definition, the class of abelian groups is $\boldsymbol{C}$-closed (and so $\boldsymbol{C_0}$-closed). It is proved that if $\mathfrak{X}$ is a $\{\boldsymbol{S}, \boldsymbol{H}, \boldsymbol{P}\}$-closed class of groups and $G = AB$ is a group, which is the product of $\mathfrak{X}$-groups A and B, then G is an $\mathfrak{X}$-group, whenever one of them, A or B, are subnormal in G [1]. Since in the central product AB both of A and B are normal in G, thus every abstract class of groups $\mathfrak{X}$ which is $\{\boldsymbol{S}, \boldsymbol{H}, \boldsymbol{P}\}$-closed, is $\boldsymbol{C_0}$-closed.

Proposition 2.

1. $\boldsymbol{D_0} \leq \boldsymbol{C_0} \leq \boldsymbol{N_0}$.

2. $\boldsymbol{D} \leq \boldsymbol{C} \leq \boldsymbol{N}$.

Proof. It is enough to prove (2) then (1) is clear. Let $\mathfrak{X}$ be an $\boldsymbol{N}$-closed class of groups and $\{G_i\}_{i\in I}$ be a collection of distinct $\mathfrak{X}$-subgroups C_i of G such that $[C_i, C_j] = 1$ for $i \neq j$. Since C_i is normal in $C = \prod_{i\in I} C_i$, thus $C \in \mathfrak{X}$ and this requires $\mathfrak{X}$ is $\boldsymbol{C}$-closed. If $C_i \cap C_j = 1$ for every $i, j \in I$, which $i \neq j$, then $C = Dr_{i\in I} C_i$. This implies $\mathfrak{X}$ is $\boldsymbol{D}$-closed. □

As a corollary, the class of nilpotent groups is $\boldsymbol{N_0}$-closed, so is $\boldsymbol{C_0}$-closed. Also by [2, page 34], the class of periodic groups is $\boldsymbol{N}$-closed, which is $\boldsymbol{C}$-closed. In the following theorem we obtain some new $\boldsymbol{C}$-closed classes of groups.

Theorem 3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two $\boldsymbol{C}$-closed abstract classes of groups.

1. If $\mathfrak{X}$ is $\boldsymbol{S}$-closed then $\boldsymbol{L_\lambda}\mathfrak{X}$ is $\boldsymbol{C}$-closed for every cardinal number λ.

2. If $\mathfrak{Y}$ is $\boldsymbol{H}$-closed then $\mathfrak{X}\mathfrak{Y}$ is $\boldsymbol{C}$-closed.

The class $\boldsymbol{L_\lambda}\mathfrak{X}$ is defined to consist of all groups G in which every subset of cardinality at most λ is contained in a $\mathfrak{X}$-subgroup of G, for a cardinal number λ. Before the proof of Theorem 3, we refer to this fact that, if H and K are two subgroups of G and $N \triangleleft G$ then

$$[HN/N, KN/N] = [H, K]N/N. \tag{8}$$

Proof of Theorem 3. Let $\{G_i\}_{i\in I}$ be a collection of distinct subgroups of G such that $[C_i, C_j] = 1$ for $i, j \in I$ and $i \neq j$. Let C_i be an $\boldsymbol{L_\lambda}\mathfrak{X}$-group for each $i \in I$. We show the central product $C = \prod_{i\in I} C_i$ is also an $\boldsymbol{L_\lambda}\mathfrak{X}$-group.

Suppose $X = \langle x_k | k \in K\rangle$ is a subgroup of C, generated by elements x_k for $k \in K$, where K is an index set with cardinal number at most λ. As $x_k \in C$ and $[C_i, C_j] = 1$ for $i \neq j$, we can write down $x_k = c_{k_1} \cdots c_{k_t}$, where $c_{k_i} \in C_{k_i}$ and $k_i \neq k_j$ for $1 \leq i \neq j \leq t$. Let $C^*_{k_i}$ be the subgroup of C_{k_i} generated by all elements c_{k_i} of C_{k_i} which appear in the representation of x_k. Then it is clear that for every $k_i \in I$, $C^*_{k_i}$ has the cardinal number at most λ and so it is an $\mathfrak{X}$-group. But $\mathfrak{X}$ is $\boldsymbol{C}$-closed class of groups and $[C^*_{k_i}, C^*_{k_j}] = 1$ for $k_i, k_j \in I$ and $k_i \neq k_j$. Hence the central product $C^* = \prod_{k\in I} C^*_{k_i}$ is an $\mathfrak{X}$-group. Since $\mathfrak{X}$ is $\boldsymbol{S}$-closed, it is enough to show $X \leq C^*$.

Let x be an element of X. Then x is a product of some generated elements x_k. Since for every pair elements c_{k_i} and c_{k_j} that $k_i \neq k_j$,

$[c_{k_i}, c_{k_j}] = 1$, thus we can write x as a product of elements c_{k_i} such that no pair of them belongs to the same C_i. This implies $x \in C^*$.

Let $C_i \in \mathfrak{X}\mathfrak{Y}$ for every $i \in I$. Then there exists a normal subgroup D_i in C_i such that $D_i \in \mathfrak{X}$ and $C_i/D_i \in \mathfrak{Y}$. As $\mathfrak{X}$ is $\boldsymbol{C}$-closed and $[D_i, D_j] = 1$ for $i \neq j$, the central product $D = \prod_{i\in I} D_i$ is a normal subgroup of C and $D \in \mathfrak{X}$. Now it is enough to show $C/D \in \mathfrak{Y}$. We have

$$C/D = (\prod_{i\in I} C_i)/D = \prod_{i\in I}(C_iD/D) \cong \prod_{i\in I}(C_i/(C_i \cap D)), \tag{9}$$

$$C_i/(C_i \cap D) \cong (C_i/D_i)/((C_i \cap D)/D_i)$$

and $\mathfrak{Y}$ is $\boldsymbol{H}$-closed, thus $C_i/(C_i \cap D)$, which is the homomorphic image of C_i/D_i, is a $\mathfrak{Y}$-group. On the other hand, by equation (8)

$$[C_iD/D, C_jD/D] = [C_i, C_j]D/D = 1 \quad \text{for every} \quad i, j \in I \quad \text{and} \quad i \neq j$$

and $\mathfrak{Y}$ is $\boldsymbol{C}$-closed. Therefore using (9) we get $C/D \in \mathfrak{Y}$. This completes the proof. $\square$

If we substitute $\boldsymbol{C_0}$-closed instead of $\boldsymbol{C}$-closed in the theorem above then the proof will be correct in a similar way.

Corollary 2. If $\mathfrak{X}$ is a $\{\boldsymbol{S}, \boldsymbol{C}\}$-closed ($\{\boldsymbol{S_0}, \boldsymbol{C_0}\}$-closed) class of groups then the class $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{C}$-closed ($\boldsymbol{C_0}$-closed).

Proof. As for every finite cardinal number λ, $\boldsymbol{L_\lambda} \leq \boldsymbol{L}$ so as required. $\square$

The class of locally soluble groups is $\boldsymbol{C_0}$-closed, while it is not $\boldsymbol{N_0}$-closed [2, page 90]. It is easy to see that the classes of FC-groups and CC-groups are $\boldsymbol{C_0}$-closed but they are not $\boldsymbol{N_0}$-closed. Also the class of abelian groups which is $\boldsymbol{C}$-closed, is not $\boldsymbol{N}$-closed. The class $\mathfrak{X}$ of groups of even powers of a prime number p is an example of $\boldsymbol{D_0}$-closed class of groups which is not $\boldsymbol{C_0}$-closed: Let G be an abelian group and $A = \langle r\rangle \oplus \langle s\rangle$, $B = \langle t\rangle \oplus \langle s\rangle$ be two $\mathfrak{X}$-subgroups of G such that $|r| = |s| = p^3$ and $|s| = p$. Then $[A, B] = 1$ and AB is not an $\mathfrak{X}$-group. Hence $\mathfrak{X}$ is not $\boldsymbol{C_0}$-closed, while it is obviously $\boldsymbol{D_0}$-closed.

Proof of Theorem 2. Since M is a CF-maximal $\mathfrak{X}$-subgroup of G, so $G = MZ(G)$. By assumption, $Z(G)$ is an $\mathfrak{X}$-group and $\mathfrak{X}$ is $\boldsymbol{C_0}$-closed, hence G is an $\mathfrak{X}$-group. This proves (1).

Since any given group G is an extension of itself by the trivial group, thus if $G \in \mathfrak{X}$ then $G \in \mathfrak{X}\mathfrak{Y}$ for the class of groups $\mathfrak{Y}$. Hence $\mathfrak{A} \leq \mathfrak{X}\mathfrak{Y}$. On the other hand, $\mathfrak{Y}$ is $\boldsymbol{H}$-closed and using Theorem 3, the class $\mathfrak{X}\mathfrak{Y}$ is $\boldsymbol{C_0}$-closed. Using part (1) if M is a CF-maximal $\mathfrak{X}\mathfrak{Y}$-subgroup of G then G is an $\mathfrak{X}\mathfrak{Y}$-group. $\square$

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