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Modules whose maximal submodules have τ-supplements

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Abstract. Let R be a ring and τ be a preradical for the category of left R-modules. In this paper, we study on modules whose maximal submodules have τ-supplements. We give some characterizations of these modules in terms their certain submodules, so called τ-local submodules. For some certain preradicals τ, i.e. $\tau = \delta$ and idempotent τ, we prove that every maximal submodule of M has a τ-supplement if and only if every cofinite submodule of M has a τ-supplement. For a radical τ on R-Mod, we prove that, for every R-module every submodule is a τ-supplement if and only if $R/\tau(R)$ is semisimple and τ is hereditary.

1. Introduction

Throughout this paper, we assume that all rings are associative with identity and all modules are unital left modules. For a module M, by $N \subseteq M$ we shall mean that N is a submodule of M. A submodule $N \subseteq M$ is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. A module M is called *supplemented* if for any submodule K of M there exists a submodule L of M such that $M = K + L$ and $K \cap L \ll L$. In [2], τ-supplemented modules are defined as a proper generalization of supplemented modules, for an arbitrary preradical τ. Namely, a module M is called *τ-supplemented* if for any submodule K of M there exists a submodule L of M such that $M = K + L$ and $K \cap L \subseteq \tau(L)$. Another generalization of supplemented modules are the modules M whose cofinite submodules (i.e. submodules U of M such that M/U is finitely

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generated) have supplements (see, [1]). These modules are termed as *cofinitely supplemented* modules. A module M is cofinitely supplemented if and only if every maximal submodule of M has a supplement (see, [1, Theorem 2.8]). In [5], a module M is called *cofinitely Rad-supplemented* if every cofinite submodule U of M has a Rad-supplement in M. Cofinitely Rad-supplemented modules are characterized as those modules for which every maximal submodule has a Rad-supplement in M (see, [5, Theorem 3.7]). In light of these characterizations, we study the modules whose maximal submodules have τ-supplements for a preradical τ, and we call these modules *maximally τ-supplemented*. A module M is said to be *cofinitely τ-supplemented* if every cofinite submodule of M has a τ-supplement. From the definitions, it is clear that an R-module M is maximally (Rad-)supplemented if and only if every cofinite submodule of M has a (Rad-)supplement in M.
For the definitions and terminology used in this paper we refer to [6] and [8].

A module N is said to be *hollow* if each proper submodule of N is small in N. A module that has a largest proper submodule is said to be *local*. Clearly each local module is hollow. Hollow modules play an important role in the study of supplemented modules and their generalizations . As a generalization of hollow modules we define τ-local modules. Namely, we call a module N *τ-local* if either $\tau(N) = N$ or $\tau(N)$ is a maximal submodule of N.

The paper is organized as follows. In section 2, we characterize maximally τ-supplemented modules for arbitrary preradicals. First we prove some closure properties of these modules. Namely, we prove that maximally τ-supplemented modules are closed under homomorphic images and arbitrary sums. For any preradical τ, τ-supplements of maximal submodules are τ-local. This fact allows us to give some characterizations of maximally τ-supplemented modules in terms of τ-local submodules. For a module M if τ-local submodules of M are maximally τ-supplemented then, M is maximally τ-supplemented if and only if $M/Loc_\tau(M)$ has no maximal submodules, where $Loc_\tau(M)$ is the sum of all τ-local submodules of M.

In section 3 and section 4, we consider the cases when τ is idempotent and $\tau = \delta$. In these cases, we prove that τ-local modules are cofinitely τ-supplemented. Using this fact, we obtain that M is maximally τ-supplemented if and only if M is cofinitely τ-supplemented. As a consequence we show that, a finitely generated module is τ-supplemented if and only if it is a finite sum of τ-local modules.

In the last section, we deal with the modules whose all submodules are τ-supplements, for a radical τ. We prove that if for every module M the submodule $\tau(M)$ is a supplement in M then τ is an idempotent radical. For a ring R we prove that, for each module $M \in$ R-Mod every submodule of M is a τ-supplement in M if and only if $R/\tau(R)$ is semisimple and τ is hereditary.

Let R-Mod be the category of left R-modules. A functor τ : R-Mod $\to$ R-Mod is said to be a *preradical* if $\tau(N) \subseteq N$ for each $N \in$ R-Mod and for each homomorphism $f : M \to M'$ in R-Mod, we have $f(\tau(M)) \subseteq \tau(M')$. A preradical τ is said to be *radical* if $\tau(N/\tau(N)) = 0$ for each $N \in$ R-Mod.

2. Maximally τ-supplemented modules

In this section, unless otherwise stated, we assume that τ is a preradical on R-Mod. In order to give some characterizations of maximally τ-supplemented modules we begin with the following lemma.

Lemma 2.1. *Let M be an R-module and let N be a maximally τ-supplemented submodule of M. If K is a maximal submodule of M such that $K + N = M$ then K has a τ-supplement in M.*

Proof. We have $N/(N\cap K) \simeq (K+N)/K = M/K$ is simple. Then $N\cap K$ is a maximal submodule of N and so $N\cap K$ has a τ-supplement, say L, in N by the hypothesis. That is, $N\cap K+L = N$ and $(N\cap K)\cap L = K\cap L \subseteq \tau(L)$. Also, $M = K+N = K+N\cap K+L = K+L$. Therefore L is a τ-supplement of K in M. □

Lemma 2.2. *Let N be a maximal submodule of a module M and L be a τ-supplement of N in M. Then L is a τ-local submodule of M.*

Proof. Suppose $\tau(L) \neq L$. Since L is a τ-supplement of N in M we have $M = N + L$ and $L \cap N \subseteq \tau(L)$. Now $L/(L \cap N) \simeq M/N$ is simple, and so $L \cap N$ is a maximal submodule of L. Therefore $L \cap N = \tau(L)$ i.e. $\tau(L)$ is a maximal submodule of L. Hence L is a τ-local submodule of M. □

Proposition 2.3. *Let M be an R-module. Suppose $M = \sum_{i\in I} N_i$, where I is an arbitrary index set and N_i is a maximally τ-supplemented submodule of M for each $i \in I$. Then M is a maximally τ-supplemented module.*

Proof. Let K be a maximal submodule of M. Since K is a proper submodule of M, there exists $j \in I$ such that $N_j \nsubseteq K$. Then $K + N_j = M$, and so K has a τ-supplement in M by Lemma 2.1. Therefore M is a maximally τ-supplemented module. □

Lemma 2.4. *Let M be a module and L be a maximally τ-supplemented submodule of M. If M/L has no maximal submodules then M is maximally τ-supplemented.*

Proof. Let K be a maximal submodule of M. Since M/L has no maximal submodules, we have $K + L = M$. Then $L/(L \cap K) \simeq M/K$ is simple, and so $L \cap K$ is a maximal submodule of L. Let L' be a τ-supplement of $L \cap K$ in L. Then $(L \cap K) + L' = L$ and $(L \cap K) \cap L' \subseteq \tau(L')$. Since $M = K + L = K + L'$ and $K \cap L' \subseteq \tau(L')$, the submodule L' is a τ-supplement of K in M. Hence M is maximally τ-supplemented. □

For a module M let $Loc_\tau(M)$ be the sum of all τ-local submodules of M.

Theorem 2.5. *For an R-module M suppose τ-local submodules of M are maximally τ-supplemented. Then the following are equivalent.*

(1) *M is maximally τ-supplemented.*

(2) *$M/Loc_\tau(M)$ has no maximal submodules.*

(3) *$M/\Lambda(M)$ has no maximal submodules, where $\Lambda(M)$ is the sum of maximally τ-supplemented submodules of M.*

Proof. (1) $\Rightarrow$ (2) Let N be a maximal submodule of M such that $Loc_\tau(M) \subseteq N$. By the hypothesis, N has a τ-supplement L in M. By Lemma 2.2, L is τ-local, and so $L \subseteq Loc_\tau(M) \subseteq N$, a contradiction. This implies that $Loc_\tau(M)$ is not contained in any maximal submodule of M. This proves (2).

(2) $\Rightarrow$ (3) By hypothesis and by Proposition 2.3, $Loc_\tau(M)$ is maximally τ-supplemented, and so $Loc_\tau(M) \subseteq \Lambda(M)$. Now the proof is obvious.

(3) $\Rightarrow$ (1) By Proposition 2.3, $\Lambda(M)$ is maximally τ-supplemented. Therefore M is maximally τ-supplemented by Lemma 2.4. □

3. Idempotent preradicals

A preradical τ is said to be idempotent if $\tau(\tau(N)) = \tau(N)$ for each R-module N (see, [6, 6.4]). In this section, for an idempotent preradical τ, we shall characterize the modules whose maximal submodules have τ-supplements. We see that these modules coincide with the modules whose cofinite submodules have τ-supplements.

The following lemma is trivial, we include it for completeness.

Lemma 3.1. *Let τ be a preradical and M be an R-module such that $\tau(M) = M$. Then M is τ-supplemented.*

Proof. Let $K \subseteq M$. Then $K + M = M$ and $K \cap M = K \subseteq \tau(M)$. That is M is a τ-supplement of K in M. Hence M is τ-supplemented. □

Proposition 3.2. *[2, 2.3(1)] Let $L_1, U \subseteq L$ be submodules where L_1 is τ-supplemented. If $L_1 + U$ has a τ-supplement in L, then so does U.*

Lemma 3.3. *Arbitrary sum of cofinitely τ-supplemented modules is cofinitely τ-supplemented. That is, for an index set I, if $M = \sum_{i \in I} M_i$, where M_i is cofinitely τ-supplemented for each $i \in I$, then M is cofinitely τ-supplemented.*

Proof. Similar to the proof of [1, Corollary 2.4]. □

Proposition 3.4. *Let τ be an idempotent preradical and M be a τ-local module. Then M is τ-supplemented.*

Proof. If $\tau(M) = M$ then M is τ-supplemented by Lemma 3.1. Suppose $\tau(M)$ is a maximal submodule of M and let U be a submodule of M. Now, we have either $U \subseteq \tau(M)$ or $M = U + \tau(M)$. If $U \subseteq \tau(M)$, then M is a τ-supplement of U in M. Suppose $U + \tau(M) = M$. Since τ is idempotent, $\tau(\tau(M)) = \tau(M)$, and hence $\tau(M)$ is τ-supplemented. So that U has a τ-supplement in M by Proposition 3.2. □

Theorem 3.5. *Let τ be an idempotent preradical and M be an R-module. The following are equivalent.*

(1) *M is cofinitely τ-supplemented.*

(2) *M is maximally τ-supplemented.*

(3) *$M/Loc_\tau(M)$ has no maximal submodules.*

Proof. (1) ⇒ (2) is clear. (2) ⇒ (3) By Proposition 3.4 and Theorem 2.5.

(3) ⇒ (1)Let U be a cofinite submodule of M. Then $U + Loc_\tau(M)$ is also a cofinite submodule of M. If $U + Loc_\tau(M)$ is a proper submodule of M, then we get a maximal submodule containing $U + Loc_\tau(M)$ and hence containing $Loc_\tau(M)$. But this contradicts with the hypothesis. Hence we must have $U + Loc_\tau(M) = M$. Since U is a cofinite submodule of M, we have $M = U + T_1 + T_2 + \cdots + T_n$, where T_i is a τ-local submodule of M for each $i = 1, \ldots, n$. By Proposition 3.4, T_i is τ-supplemented for each $i = 1, \ldots, n$ and hence $T_1 + T_2 + \ldots + T_n$ is τ-supplemented by [2, 2.3(2)]. Then U has a τ-supplement in M by Proposition 3.2. Hence M is cofinitely τ-supplemented. □

Since every submodule of a finitely generated module is cofinite, the notions of being τ-supplemented and being cofinitely τ-supplemented coincide for finitely generated modules. Hence we obtain the following by Theorem 3.5.

Corollary 3.6. *For a finitely generated module M, the following are equivalent.*

(1) *M is τ-supplemented.*

(2) *Every maximal submodule of M has a τ-supplement.*

(3) *$M = T_1 + T_2 + \ldots + T_n$ where T_i is τ-local for each $i = 1, \ldots, n$.*

4. Generalized cofinitely δ-supplemented modules

In this section we shall consider the case $\tau = \delta$. We call an R-module M *generalized (cofinitely) δ-supplemented* if for every (cofinite) submodule U of M, there exists a submodule V of M such that $U + V = M$ and $U \cap V \subseteq \delta(V)$. In this case, the submodule V is called a *generalized δ-supplement* of U in M.

Recall that a module M is said to be *singular* if $M \simeq L/K$ where L, K are R-modules and $K \trianglelefteq L$, that is, $K \cap T \neq 0$ for each nonzero submodule $T \subseteq L$.

For a ring R, let $\mathcal{P}$ be the class of all singular simple left R-modules. Then for an R-module M, as in [7],

$$\delta(M) = \bigcap\{\operatorname{Ker} f \mid f \in \operatorname{Hom}(M, S),\ S \in \mathcal{P}\}.$$

A submodule N of a module M is said to be *δ-small* in M, denoted as $N \ll_\delta M$, if $N + L \neq M$ for any proper submodule L of M with M/L singular.

Lemma 4.1. *[7, Lemma 1.2, Lemma 1.3] Let M be an R-module and $N, L \subseteq M$ then,*

(1) *A submodule $N \subseteq M$ is δ-small if and only if for all submodules $X \subseteq M$:*
$$\textit{if } X + N = M, \textit{ then } M = X \oplus Y$$
for a projective semisimple submodule Y with $Y \subseteq N$.

(2) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*

Lemma 4.2. *Let M be a δ-local module. Then M is cofinitely δ-supplemented.*

Proof. If $\delta(M) = M$ then M is cofinitely δ-supplemented by Lemma 3.1. Suppose $\delta(M)$ is a maximal submodule of M. Let U be a cofinite submodule of M. Since $\delta(M)$ is a maximal submodule of M, we have either $U \subseteq \delta(M)$ or $U + \delta(M) = M$. First suppose $U \subseteq \delta(M)$. In this case, clearly M is a δ-supplement of U in M. Now, suppose $U + \delta(M) = M$. Then there exist δ-small submodules L_1, L_2, $\ldots$, L_n of M such that $U + L_1 + \ldots + L_n = M$. By Lemma 4.1(2), the submodule $N = L_1 + \ldots + L_n$ is δ-small in M. So that by Lemma 4.1(1) there exists a submodule Y of N such that $M = U \oplus Y$. That is, Y is a δ-supplement of U in M. □

From the proof of Lemma 4.2 we have the following.

Corollary 4.3. *Let M be a δ-local module. Then every cofinite submodule of M has a generalized δ-supplement that is a direct summand.*

In [5], for the case $\tau = \mathrm{Rad}$ it is proved that a module M is maximally τ-supplemented if and only if every cofinite submodule of M has a τ-supplement. We have a similar characterization when $\tau = \delta$, as follows.

For a module M let $Loc_\delta(M)$ be the sum of all δ-local submodules of M.

Theorem 4.4. *For an R-module M, the following are equivalent.*

(1) *M is generalized cofinitely δ-supplemented.*

(2) *M is maximally δ-supplemented.*

(3) *$M/Loc_\delta(M)$ has no maximal submodules.*

(4) *$M/\Lambda(M)$ has no maximal submodules, where $\Lambda(M)$ is the sum of maximally δ-supplemented submodules of M.*

Proof. (1) $\Rightarrow$ (2) is clear. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) By Theorem 2.5. (3) $\Rightarrow$ (1) Similar to the proof of Theorem 3.5. □

Corollary 4.5. *For a finitely generated module M, the following are equivalent.*

(1) *M is generalized δ-supplemented.*

(2) *Every maximal submodule of M has a generalized δ-supplement.*

(3) *$M = D_1 + D_2 + \ldots + D_n$, where D_i is δ-local for each $i = 1, \ldots, n$.*

5. When all submodules of a module are τ-supplements

Let τ be a radical on R-Mod and M be an R-module. Recall that a preradical τ is said to be *hereditary (or left exact)* if for any module N and $K \subseteq N$ we have $\tau(K) = K \cap \tau(N)$. Hereditary preradicals are idempotent (see, [6, 6.9 (1)]).

Proposition 5.1. *[3, proposition 4.1]Let τ be radical and V be a τ-supplement submodule of M. Then $\tau(V) = V \cap \tau(M)$.*

Theorem 5.2. *Let τ be a radical on R-Mod. If $\tau(M)$ is a τ-supplement in M for every left R-module M, then τ is an idempotent radical.*

Proof. Let N be an R-module. By hypothesis $\tau(N)$ is a τ-supplement in N. So that $\tau(\tau(N)) = N \cap \tau(N) = \tau(N)$ by Proposition 5.1. This implies that τ is idempotent. □

Lemma 5.3. *Let M be a module such that each submodule of M is a τ-supplement in M. Then $M/\tau(M)$ is semisimple.*

Proof. Let $K/\tau(M)$ be a submodule of $M/\tau(M)$. By hypothesis K is a τ-supplement in M, that is, $K + L = M$ and $K \cap L \subseteq \tau(K)$ for some submodule L of M. Then we have

$$M/\tau(M) = K/\tau(M) + (L + \tau(M))/\tau(M)$$

and

$$K/\tau(M) \cap (L + \tau(M))/\tau(M) = (K \cap L + \tau(M))/\tau(M) = 0.$$

That is, $K/\tau(M)$ is a direct summand of $M/\tau(M)$. Hence $M/\tau(M)$ is semisimple. □

Theorem 5.4. *For a ring R and a radical τ on* R-Mod, *the following are equivalent.*

(1) *For each $M \in$* R-Mod, *every submodule of M is a τ-supplement in M.*

(2) *$R/\tau(R)$ is semisimple and τ is hereditary.*

Proof. (1) ⇒ (2) By hypothesis every submodule of ${}_RR$ is a τ-supplement, so $R/\tau(R)$ is semisimple by Lemma 5.3. Let N be an R-module and $K \subseteq N$. Since K is a τ-supplement in N, we have $\tau(K) = K \cap \tau(N)$ by Proposition 5.1. Hence τ is hereditary by [6, 6.9.(1)].

(2) ⇒ (1) Since $\tau(R)M \subseteq \tau(M)$, the module $M/\tau(M)$ is an $R/\tau(R)$-module. So that $M/\tau(M)$ is a semisimple $R/\tau(R)$-module. Hence $M/\tau(M)$ is a semisimple R-module. Let K be a submodule of M. Since $M/\tau(M)$ is semisimple,

$$M/\tau(M) = [(K + \tau(M))/\tau(M)] \oplus L/\tau(M)$$

for some submodule $L \subseteq M$. That is, $K + L = M$ and $K \cap L \subseteq \tau(M)$. Then $K \cap L \subseteq K \cap \tau(M) = \tau(K)$, by [6, 6.9.(1)(b)]. So that K is a τ-supplement of L in M. Hence every submodule of M is a τ-supplement in M. □

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