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Diagonal direct limits of finite symmetric and alternating groups

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Abstract. Diagonal direct limits of permutation groups are studied using their representations by homeomorphisms of the boundary of rooted trees.

We describe a new method of classification of such permutation groups, and use this method to find a complete classification of diagonal direct limits of symmetric and alternating groups up to isomorphisms.

Introduction

Classification of simple countable locally finite groups is one of the central problems of the theory of locally finite groups. Big progress in this direction is made in the works of U. Meierfrankenfeld and S. Delcroix [Mei95, DM02]. It is shown there that every simple countable locally finite group belongs to one of four classes. The first class is the class of finitary groups. The other three are defined using the properties of Kegel covers of the groups. One of these three classes is the class of the so-called groups of *1-type*. An important part of the study of this class is classification of locally finite groups which are unions of an ascending chain of finite alternating groups (see [Har95]). Classification of such unions is also closely related to the study of group rings with "small" lattices of ideals and asymptotic theory of characters. A. E. Zalesski has considered in [Zal91] a class of inductive

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limits of finite symmetric and alternating groups that are defined by the so-called *diagonal embeddings*. He proved that the lattice of ideals of the group rings of such groups is a chain, and has formulated several problems for further investigation (see also [Zal98], [HZ97]). In particular, a natural question on classification of such groups was posed.

Note that inductive limits of finite-dimensional C^*-algebras (**so-called AF-algebras**) is already a classical part of the theory of C^*-algebras, and its classification was accomplished by G. Glimm [Gli60] for a partial case of the so-called UHF-algebras, and by J. Dixmier [Dix67] and G. A. Elliott [Ell76] in more general situations.

Inductive limits of finite-dimensional Lie algebras with diagonal embeddings were classified in the work of A. A. Baranov and A. G. Zhilinskii [BZ99].

In the work of N. V. Kroshko and V. I. Sushchansky [KS98] a complete classification of the inductive limits of finite symmetric and alternating groups with strictly diagonal embeddings was given. The classification is formulated in the same terms as the classification of the UHF-algebras by Glimm. The study of such inductive limits of groups was continued in [LS03]. In particular, it was shown that the inductive limits of finite symmetric groups with strictly diagonal embeddings appear naturally in the study of hierarchomorphisms of spherically homogeneous rooted trees.

In [LN07] the first and the second named authors gave a complete classification of the inductive limits of direct products of finite alternating groups that are simple. It was shown that two such groups are isomorphic if and only if the corresponding (i.e., having the same Bratteli diagram) AF-algebras are isomorphic.

Our paper develops classification techniques based on different ideas. We use topological properties of the boundaries of rooted trees and properties of Berhoulli measures on them. Using these techniques, we classify the inductive limits of finite symmetric and alternating groups with respect to arbitrary diagonal embeddings without using theory of C^*-algebras and K-theory. We think that the developed methods are of independent interest. It may be useful in solving classification problems in other categories with inductive limits.

1. Rooted trees

We will study diagonal direct limits of finite symmetric or alternating groups using a representation of these groups by homeomorphisms of the boundaries of rooted trees. We start with introduction of all the notions needed for definition of such representations.

Let T be a locally finite rooted tree with the root vertex v_0. Let us denote by $V(T)$ the set of vertices of T, and by $E(T)$ its set of edges. For every two vertices u, v of the tree T define the *distance* between u and v, denoted $d(u, v)$, to be equal to the length of the shortest path connecting them.

For a rooted tree T with the root v_0 and an integer $n \geq 0$, define the *level number* n (the *sphere of radius* n) to be the set

$$V_n(T) = \{v \in V(T) : d(v_0, v) = n\}.$$

We say that a vertex v of the tree T *lies under a vertex* w, if the path connecting the vertex v and the root contains the vertex w. Let us denote by T_v the subtree consisting of all vertices that lie under the vertex v with the root v.

An *end* of a rooted tree is an infinite path without repetitions which starts in the root. Let us denote by ∂T the *boundary* of T, i.e., the set of all the ends of the tree T. For $V \subseteq V(T)$ put

$$\partial(V) = \bigcup_{v \in V} \partial T_v,$$

where ∂T_v is the boundary of the subtree T_v, i.e., the set of ends passing through v.

For $x \in \partial T$, denote by $x(m)$ the vertex from the level $V_m(T)$ such that x goes through $x(m)$. Let us introduce a natural ultrametric on ∂T putting

$$\rho(\gamma_1, \gamma_2) = 1/(n + 1),$$

where n is the length of the longest common part of the paths γ_1 and γ_2. The topology introduced by the metric ρ is compact, totally disconnected, and has a base of open sets $\{\partial T_v\}_{v \in V(T)}$. Note that ∂T_v is a ball of radius $1/(n + 1)$, where n is such that $v \in V_n(T)$.

If degree of a vertex $v \in V_n(T)$ depends only on n, then the tree T is called *spherically homogeneous*. *Spherical index* of a spherically homogeneous tree T is the sequence

$$\Theta = (a_0, a_1, \ldots),$$

where a_0 is degree of the root and $a_n + 1$ are degrees of any vertex of the nth level (i.e., a_n is the number of "childs" of a vertex of nth level). We assume throughout the paper that $a_n \geq 2$ for all n.

Let T be a spherically homogeneous rooted tree with root v_0 and the spherical index Θ. The tree T is isomorphic to the tree T_Θ whose set of vertices is the set of all finite sequences $(i_0, i_1, \ldots, i_{n-1})$, $n \geq 1$, such that $i_k \in \{1, 2, \ldots, a_k\}$. We include also the empty sequence $\varnothing$ corresponding to $n = 0$. Two vertices are adjacent if and only if they are of the form $(i_0, \ldots, i_{n-1}), (i_0, \ldots, i_{n-1}, i_n)$. We order the vertices of every level lexicographically.

Then every end $x \in \partial T$ is identified with an infinite sequence $(i_0, i_1, \ldots)$, where $1 \leq i_k \leq a_k$ for all $k \geq 0$. Namely, such a sequence is identified with the end

$$\varnothing, (i_0), (i_0, i_1), (i_0, i_1, i_2), \ldots.$$

It is easy to see that this identification is a homeomorphism between ∂T and the direct product $\prod_{k\geq 0}\{1, 2, \ldots, a_k\}$ of discrete sets.

Let us denote by $\operatorname{Homeo}\partial T$ the group of all homeomorphisms of ∂T. For $n \geq 0$, denote by $S(\partial T, n)$ the group of all homeomorphisms of the boundary ∂T which permute the balls T_v, for $v \in V_n(T)$ in a rigid manner, i.e., which change at most the first n coordinates of an end $(i_0, i_1, \ldots)$. The group $S(\partial T, n)$ is naturally isomorphic to the symmetric group $\operatorname{Sym}(V_n)$, where a permutation $\pi \in \operatorname{Sym}(V_n)$ acts on the ends by the rule

$$(i_0, i_1, \ldots)^\pi = ((i_0, i_1, \ldots, i_{n-1})^\pi, i_n, i_{n+1}, \ldots)\,.$$

It is easy to see that $S(\partial T, n) \leq S(\partial T, k)$ for $n \leq k$. Let us define the subgroup $S(\partial T)$ of $\operatorname{Homeo}\partial T$ as the union of the subgroups $S(\partial T, n)$, $n \in \mathbb{N}$.

Let $A(\partial T, n) \leq S(\partial T, n)$ be the subgroup coinciding with the alternating group $\operatorname{Alt}(V_n)$. Clearly, $A(\partial T, n) \leq A(\partial T, k)$ for $n \leq k$. Let us define the subgroup $A(\partial T) \leq \operatorname{Homeo}\partial T$ as the union of the subgroups $A(\partial T, n)$ for $n \in \mathbb{N}$.

For the groups $A(\partial T_\Theta)$ and $S(\partial T_\Theta)$, let us define *characteristics* as the supernatural (Steinitz) number $\Omega(\Theta) = \prod_{i=0}^{\infty} a_i$, where $\Theta = (a_0, a_1, \ldots)$ is the spherically index of the spherically homogeneous tree T_Θ.

We will use the following results.

Theorem 1.1 ([LS03] Proposition 10, [LS05] Theorem 2)**.** *Let T_Θ be a spherically homogeneous tree. Every automorphism of the group $S(\partial T_\Theta)$ is locally inner and*

$$\operatorname{Aut} A(\partial T_\Theta) \simeq \operatorname{Aut} S(\partial T_\Theta).$$

Theorem 1.2 ([Rub89] Corollary 3.13c)**.** *Let X_i be locally compact Hausdorff spaces without isolated points, let G_i be subgroups of* $\operatorname{Homeo} X_i$ *and for every open set $D \subseteq X_i$, $x \in D$ and $i = 1, 2$ the set $\{g(x) \mid g \in G_i$ and restriction of g on $X_i \backslash D$ is identity$\}$ be nonempty and somewhere dense. If $\phi : G_1 \to G_2$ is an isomorphism then there is a homeomorphism $h : X_1 \to X_2$ such that for every $g \in G_1$ the equality $\phi(g) = hgh^{-1}$ holds.*

The space ∂T is a compact Hausdorff space, since it is a direct product of compact Hausdorff spaces. Let D be an open subset of ∂T and let $x \in D$. There exists $v \in V(T)$ such that $x \in \partial T_v \subseteq D$. The subgroup $A(\partial T_v) \leq A(\partial T)$ is the maximal subgroup acting trivially outside ∂T_v. Every orbit of the action of $A(\partial T_v)$ on ∂T_v is dense, hence we get the following result.

Lemma 1.3. *Let T be a spherically homogeneous tree. The space ∂T is a compact Hausdorff space without isolated points, and the groups $S(\partial T)$ and $A(\partial T)$ satisfy conditions of Theorem 1.2.*

2. Diagonal embeddings

Definition 1 ([Zal91]). *An embedding d of a transitive permutation group (G, X) into a permutation group (H, Y) is called* diagonal *if the restriction of $d(G)$ onto every G-orbit of length more than 1 is isomorphic to (G, X) as a permutation group.*

A diagonal embedding is called *strictly diagonal* if the length of every orbit of the image $d(G)$ on the set Y is greater than 1.

We say that the group G is a (*strictly*)*diagonal direct limit* of groups if G is the union of an ascending chain of permutation groups G_i $(i \in \mathbb{N})$ where all inclusions $G_i \subset G_{i+1}$ are (strictly) diagonal. It is shown in [LS03] that for a spherically homogeneous tree T the groups $S(\partial T)$ and $A(\partial T)$ are strictly diagonal direct limits of symmetric and alternating groups respectively. Namely, the inclusions $S(\partial T, n) \leq S(\partial T, n+1)$ are strictly diagonal with respect to the natural action of these groups on V_n and V_{n+1}.

Now we construct certain word trees such that (not necessary strictly) diagonal direct limits of finite symmetric groups act on them naturally. Let $\{X_i = \{1, \ldots, n_i\}\}$, $\{Y_{i-1} = \{1, \ldots, k_{i-1}\}\}$ be two infinite sequences of an alphabets $(i \geq 1)$. We take also a symbol "\$" not contained in any of these alphabets.

Consider the tree whose set of vertices is the set

$$\{\underbrace{\$\$\ldots\$}_{l} \mid l \geq 0\} \cup \bigcup_{l \geq 0, m \geq 0} \underbrace{\$\$\ldots\$}_{l} Y_l X_{l+1} X_{l+2} \ldots X_{l+m},$$

where $\underbrace{\$\$\ldots\$}_{l} Y_l X_{l+1} X_{l+2} \ldots X_{l+m} = \{\underbrace{\$\$\ldots\$}_{l} y_l x_{l+1} x_{l+2} \ldots x_{l+m} \mid y_l \in Y_l, x_k \in X_k\}$. We may have Y_l empty, then the corresponding sets

$$\underbrace{\$\$\ldots\$}_{l} Y_l X_{l+1} \ldots X_{l+m}$$

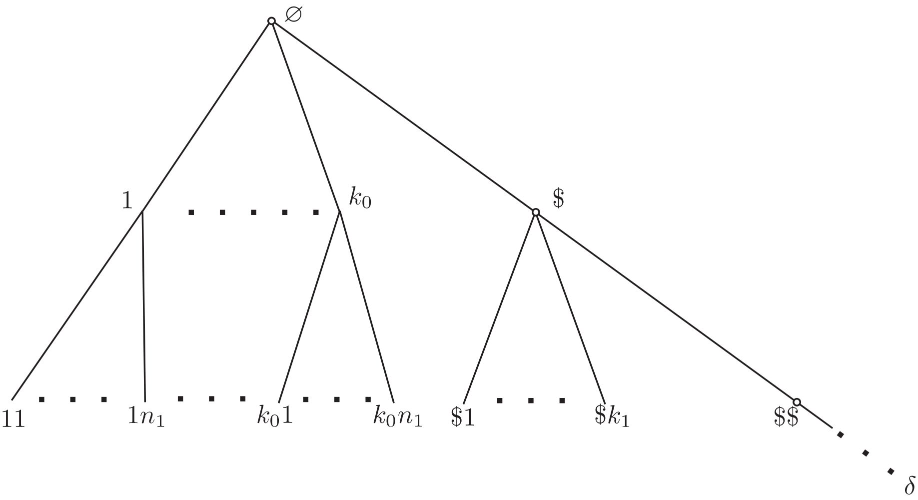

Figure 1:

will be also empty for all m.

The empty word is the root of the tree. Two words are connected by an edge if and only if one is obtained from the other by appending one letter to the right. Let us introduce the lexicographic order on the words. The symbols in each alphabet are ordered in the natural way and

$$x_1 < x_2 < \ldots < y_0 < y_1 < \ldots < \$,$$

for all $x_i \in X_i$, $y_{i-1} \in Y_{i-1}$, $i \in \mathbb{N}$.

Let

$$\chi = \langle (1, k_0), (n_1, k_1), (n_2, k_2), \ldots \rangle.$$

Let us denote the constructed tree T_χ (see Figure 1).

For every $i \geq 1$ and $v \in V_i(T_\chi)$ the degree of v is equal to $n_i + 1$ if $v \neq \delta(i)$ (i.e. $v \notin \delta$) and is equal to $k_{i+1} + 2$ if $v = \delta(i)$, see Figure 1. For arbitrary $v \in V_i(T_\chi)$ such that $v \neq \delta(i)$ the tree T_v is spherically homogeneous with spherical index $(n_i, n_{i+1}, \ldots)$.

An end $x \in \partial T_\chi$ of the tree T_χ is encoded by a sequence

$$\underbrace{\$\$\ldots\$}_{l}\, y_l x_{l+1} x_{l+2} \ldots,$$

where $l \geq 0$, $y_l \in Y_l$ and $x_i \in X_i$, or x is the end

$$\delta = \$\$\$\ldots.$$

Let $S(\chi, n)$ be the group of homeomorphisms of the ∂T_χ, which rigidly permute the balls ∂T_v, $v \in V_n(T_\chi) \setminus \{\delta(n)\}$. In other words, it is the group of homeomorphisms, which act trivially on the ball $\partial T_{\delta(n)}$ and act outside of it by homeomorphism of the form

$$(\$\$\ldots\$y_l x_{l+1}\ldots x_n x_{n+1}\ldots)^\pi = (\$\$\ldots\$y_l x_{l+1}\ldots x_n)^\pi x_{n+1}\ldots,$$

where π is a permutation of the set $V_n(T_\chi)\setminus\{\delta(n)\}$. The permutation π determines the homeomorphism uniquely and thus $S(\chi,n)$ is isomorphic to the symmetric group $\mathrm{Sym}\,(V_n(T_\chi)\setminus\{\delta(n)\})$. It is also obvious that $S(\chi,n)\le S(\chi,k)$ for $n\le k$. Let us define a subgroup $S_\chi\le \mathrm{Homeo}\,\partial T_\chi$ as the union of the subgroups $S(\chi,n)$ for all $n\in\mathbb{N}$.

The group A_χ is defined in the same way as S_χ, but using the alternating groups $A(\chi,n)$ acting by even permutations of the nth level of the tree.

Let $\mathbf{S}$ be the set of all infinite sequences

$$\langle(1,k_0),(n_1,k_1),(n_2,k_2),\ldots\rangle$$

such that $k_0>0$, $k_i\ge 0$, $n_i\ge 1$ for all $i\ge 1$. Let $\mathbf{S}_1$ be the subset of $\mathbf{S}$ such that $k_0\ge 2$ and $n_i\ge 2$ for all $i\in\mathbb{N}$. We assume also that $n_0=k_0$ for our convenience.

Lemma 2.1. *Let $\chi=\langle(1,k_0),(n_1,k_1),(n_2,k_2),\ldots\rangle\in\mathbf{S}$. If $k_i=0$ for all $i\ge 1$, then $S_\chi\simeq S(\partial T_\Theta)$ and $A_\chi\simeq A(\partial T_\Theta)$, where T_Θ is the homogeneous tree of spherical index $\Theta=(k_0,n_1,n_2,n_3,\ldots)$. Moreover, for all n we have $S(\chi,n)\simeq S(\partial T_\Theta,n)$ and $A(\chi,n)\simeq A(\partial T_\Theta,n)$.*

Proof. The trees T_χ and T_Θ differ only by an additional infinite path δ in T_χ. But the path δ is a fixed point of S_χ, hence we have the necessary isomorphisms. □

Let us define for $\chi=\langle(1,k_0),(n_1,k_1),(n_2,k_2),\ldots\rangle\in\mathbf{S}$

$$r(\chi,i)=|V_{i+1}(T_\chi)|-1=\sum_{j=0}^{i}k_j n_{j+1}\ldots n_i.$$

Let us also define the *characteristics* $\Omega(\chi)$ for S_χ and A_χ as the supernatural number

$$char(S_\chi)=char(A_\chi)=\Omega(\chi)=\prod_{i=1}^{\infty}n_i,$$

and the *characteristic series*

$$M(\chi)=\sum_{i=0}^{\infty}\frac{k_i}{n_1\cdots n_i}. \tag{1}$$

A partial sum $M_j(\chi)$ $(j \geq 0)$ of the series $M(\chi)$ is equal to

$$M_j(\chi) = \sum_{i=0}^{j} \frac{k_i}{n_1 \cdots n_i} = \frac{r(\chi, j)}{n_1 \cdots n_j}. \tag{2}$$

We also consider $\gamma_i(\chi) := M_i(\chi)^{-1}$ and

$$\gamma := \lim_{i \to \infty} M_i(\chi)^{-1}.$$

Since $0 \leq M_i(\chi) \leq M_{i+1}(\chi)$, the number γ is well-defined. The number γ is called the *density index*, following Baranov and Zhilinskii [BZ99].

Now we give a topological interpretation of the density index. In the next two statements the symbol H is either A or S.

Proposition 2.2. *If m is an H_χ-invariant Borel measure on ∂T_χ such that $m(\partial T_v) = 1$, where $v \in V_1 \setminus \{\delta(1)\}$ and $m(\{\delta\}) = 0$, then $m(\partial T_\chi) = M(\chi)$.*

Proof. Let m be an H_χ-invariant Borel measure on ∂T_χ and let $m(\partial T_v) = 1$, where $v \in V_1 \setminus \{\delta(1)\}$. Then

$$m\left(\bigcup_{v \in V_1 \setminus \{\delta(1)\}} \partial T_v\right) = k_0 = M_0(\chi),$$

$$m\left(\bigcup_{v \in V_2 \setminus \{\delta(2)\}} \partial T_v\right) = k_0 + \frac{k_1}{n_1} = M_2(\chi),$$

$$\cdots$$

$$m\left(\bigcup_{v \in V_l \setminus \{\delta(l)\}} \partial T_v\right) = k_0 + \frac{k_1}{n_1} + \cdots + \frac{k_{l-1}}{n_1 \cdots n_{l-1}} = M_l(\chi).$$

Taking into account σ-additivity and

$$\partial T_\chi = \{\delta\} \cup \bigcup_{l=1}^{\infty} \left(\bigcup_{v \in V_l \setminus \{\delta(l)\}} \partial T_v \setminus \bigcup_{v \in V_{l-1} \setminus \{\delta(l-1)\}} \partial T_v\right),$$

we get $m(\partial T_\chi) = M(\chi)$. $\square$

Corollary 2.3. *The space ∂T_χ carries a finite H_χ-invariant measure m if and only if the density index is strictly positive, i.e.,*

$$\gamma = M(\chi)^{-1} > 0.$$

In this case, if $m(\partial T_\chi) = 1$ and $m(\{\delta\}) = 0$, then

1. *$\gamma = m(\partial T_v)$, where $v \in V_1 \setminus \{\delta(1)\}$.*

2. *$m(\partial T_{\delta(l)}) = 1 - \gamma M_l(\chi) \to 0$ for $l \to \infty$.*

Proof. If $\gamma > 0$, then existance of an H_χ-invariant probability measure is an immediate corollary of Proposition 2.2. Let $\gamma = 0$, and suppose there exists an H_χ-invariant probability measure m. Then Proposition 2.2 implies that $m(\partial T_v) = 0$. Since m is H_χ-invariant, $m(\partial T_\chi) = 0$. We get a contradiction finishing our proof. □

Definition 2. *Let χ_1, χ_2 be sequences from $\mathbf{S}$. Let u, v be some positive integers. We call the sequences χ_1, χ_2 (u, v)-commensurable if*

1. *$u\Omega(\chi_1) = v\Omega(\chi_2)$;*

2. *the characteristic series $M(\chi_1)$, $M(\chi_2)$ are convergent or divergent simultaneously;*

3. *if $M(\chi_1)$ and $M(\chi_2)$ are convergent, then $vM(\chi_1) = uM(\chi_2)$;*

4. *sequences χ_1 and χ_2 have finitely or infinitely many nonzero members k_i simultaneously.*

We call the sequences χ_1, χ_2 commensurable *if there exist positive integers u, v such that χ_1, χ_2 are (u, v)-commensurable.*

We can change the first three conditions of the definition to equivalent two conditions without mentioning u and v, namely:

1. $\Omega(\chi_1)/\Omega(\chi_2)$ is a rational number;

2. if one of the series $M(\chi_1)$ and $M(\chi_2)$ is convergent, then the other is convergent too, and
$$\Omega(\chi_1)M(\chi_1) = \Omega(\chi_2)M(\chi_2).$$

Lemma 2.4. *Let $\chi \in \mathbf{S}_1$. Then $\partial T_\chi \setminus \{\delta\}$ is a locally compact Hausdorff space without isolated points, and the actions of the groups S_χ and A_χ on it satisfy the conditions of Theorem 1.2.*

Proof. Consider the set
$$U_{\delta(i)} = \partial T_{\delta(i)} \setminus \partial T_{\delta(i+1)}.$$

It is a compact Hausdorff space, since it is a union of boundaries of a finite number of spherically homogeneous trees. But $\partial T_\chi \setminus \{\delta\} = \bigcup_{i \geq 0} U_{\delta(i)}$, hence $\partial T_\chi \setminus \{\delta\}$ is a locally compact Hausdorff space. The group A_χ contains the subgroup $A(\partial T_v)$ for every $v \in V(T_\chi) \setminus \{\delta\}$, hence A_χ satisfies the conditions of Theorem 1.2. □

Let V be a subset of $V_i(T_\chi)$ for $i > 0$, and let H be one of the symbols S or A. Consider the subgroup of all homeomorphisms of ∂T_χ which act trivially outside $\partial(V) = \bigcup_{v \in V} \partial T_v$. Let us denote this subgroup $H(\partial(V))$. We also consider the rooted tree T_V obtained by taking all the subtrees T_v of T_χ for $v \in V$, and connecting them together by a root, so that the first level of the tree T_V is $V_1(T_V) = V$. The group $H(\partial T_V)$ acting naturally on ∂T_χ coincides with $H(\partial(V))$.

If $\delta(i) \notin V$ then $H(\partial(V))$ is a strictly diagonal direct limit of symmetric (resp. alternating) groups. In this case, if the tree T_χ is constructed using the sequence

$$\chi = \langle (1, k_0), (n_1, k_1), (n_2, k_2), \ldots \rangle \in \mathbf{S},$$

then the characteristics of $H(\partial(V))$ is $|V| \cdot \prod_{j=i}^{\infty} n_j$.

Let us define the *standard* diagonal embedding $u(r,s) : \mathrm{Sym}(A) \hookrightarrow \mathrm{Sym}(B)$, where $A = \{1, 2, \ldots, n\}$, $B = \{1, 2, 3, \ldots, nr + s\}$, for $n, r, s \in \mathbb{N}$, as follows.

For $\alpha \in \mathrm{Sym}(A)$, we set

$$(ri - k)^{u(r,s)(\alpha)} = ri^{\alpha} - k \text{ if } 0 \le k \le r - 1,\ i \ge 1,$$

and

$$i^{u(r,s)(\alpha)} = i \text{ if } nr + 1 \le i \le nr + s.$$

It is easy to verify that the map $u(r,s) : \alpha \mapsto u(r,s)(\alpha)$ is a diagonal embedding of $\mathrm{Sym}(A)$ into $\mathrm{Sym}(B)$.

Note that the natural embedding of the subgroup $S(\chi, i)$ into the subgroup $S(\chi, j)$ of S_χ, for $j > i$, is an example of a standard diagonal embedding, if we number the sets $V_i(T_\chi)$ and $V_j(T_\chi)$ lexicographically.

Lemma 2.5. *Denote $M_1 = \{1, \ldots, m_1\}$, $M_2 = \{1, \ldots, m_2\}$, and $M_3 = \{1, \ldots, m_3\}$, where $m_2 = m_1 n_1 + r_1$, $m_3 = m_2 n_2 + r_2$ for some integers $n_i > 0, r_i \ge 0$, $i = 1, 2$.*

Let

$$\begin{aligned} u(n_1, r_1) &: \mathrm{Sym}(M_1) \to \mathrm{Sym}(M_2), \\ u(n_2, r_2) &: \mathrm{Sym}(M_2) \to \mathrm{Sym}(M_3), \\ u(n_1 n_2, n_2 r_1 + r_2) &: \mathrm{Sym}(M_1) \to \mathrm{Sym}(M_3) \end{aligned}$$

be the standard diagonal embeddings.

Then

$$u(n_1, r_1) u(n_2, r_2) = u(n_1 n_2, n_2 r_1 + r_2).$$

Proof. Let us consider the sequence

$$\chi = \langle (1, m_1), (n_1, r_1), (n_2, r_2), (2, 0), (2, 0), \ldots \rangle \in \mathbf{S}_1 .$$

The bijections $M_1 \to V_1(T_\chi) \setminus \{\delta(1)\}$, $M_2 \to V_2(T_\chi) \setminus \{\delta(2)\}$, and $M_3 \to V_3(T_\chi) \setminus \{\delta(3)\}$ induce isomorphisms $\mathrm{Sym}(M_1) \to S(\chi, 1)$, $\mathrm{Sym}(M_2) \to S(\chi, 2)$, and $\mathrm{Sym}(M_3) \to S(\chi, 3)$ respectively.

The embedding of $S(\chi, i)$ into $S(\chi, i+1)$ coincides with the standard diagonal embedding $u(n_i, r_i)$ ($i = 1, 2$), and the embedding of $S(\chi, 1)$ in $S(\chi, 3)$ coincides with $u(n_1 n_2, n_2 r_1 + r_2)$. Hence $u(n_1, r_1)u(n_2, r_2) = u(n_1 n_2, n_2 r_1 + r_2)$. □

Lemma 2.6 ([KS98]). *Let T_1, T_2 be spherically homogeneous rooted trees such that*

$$char(S(\partial T_1)) = char(S(\partial T_2)).$$

Then the groups $S(\partial T_1)$ and $S(\partial T_2)$ (resp., $A(\partial T_1)$ and $A(\partial T_2)$) are isomorphic.

Proof. Let us construct an isomorphism $\phi : S(\partial T_1) \to S(\partial T_2)$. Since characteristics of $S(\partial T_1)$ and $S(\partial T_2)$ are equal, for every $i \geq 1$ there is l_i such that $|V_i(T_1)|$ is a factor of $|V_{l_i}(T_2)|$. We can assume that sequence $\{l_i, i \mid \ \in \mathbb{N}\}$ is increasing.

Let $\phi_i = u\left(\frac{|V_{l_i}(T_2)|}{|V_i(T_1)|}, 0\right)$ be the standard strictly diagonal embedding of $S(\partial T_1, i)$ into $S(\partial T_2, l_i)$, $i \geq 1$.

The next diagram is commutative for all $1 \leq i < j$ by Lemma 2.5

$$\begin{array}{ccc} S(\partial T_1, i) & \xrightarrow{\psi_1(i,j)} & S(\partial T_1, j) \\ \phi_i \Big\downarrow & & \Big\downarrow \phi_j \\ S(\partial T_2, l_i) & \xrightarrow{\psi_2(l_i,l_j)} & S(\partial T_2, l_j) \end{array}$$

where $\psi_1(i,j)$ and $\psi_2(l_i,l_j)$ are the diagonal embeddings induced by inclusions of corresponding groups into $S(\partial T_1)$ and $S(\partial T_2)$, respectively. Consequently, there is an isomorphism $\phi : \bigcup_i S(\partial T_1, i) \to \bigcup_i \phi_i(S(\partial T_1, i))$, which is equal to the inductive limit $\lim_{\to} \phi_i$.

Since $\bigcup_i S(\partial T_1, i) = S(\partial T_1)$, we need to prove that $\bigcup_j \phi_i(S(\partial T_1, i)) = S(\partial T_2)$. The characteristics of $S(\partial T_1)$ and $S(\partial T_2)$ are equal, therefore for every $k \geq 0$ there is $i \geq 0$ such that $|V_k(T_2)|$ is a divisor of $|V_i(T_1)|$. Then $\phi(S(\partial T_1, i)) \geq S(\partial T_2, k)$ by Lemma 2.5. Hence, ϕ is an isomorphism of $S(\partial T_1)$ and $S(\partial T_2)$. □

Let us call the above constructed isomorphism $\phi : S(\partial T_1) \to S(\partial T_2)$ *canonical.*

Lemma 2.7. *Let T_1, T_2 be spherically homogeneous rooted trees such that $S(\partial T_1) \simeq S(\partial T_2)$. We assume that vertices of every level of T_1 and T_2 are numbered in the lexicographic order. Let V_1 be the subset of the first i_n vertices of $V_n(T_1)$, let V_2 be the subset of the first j_k vertices of $V_k(T_2)$, and suppose that $S(\partial(V_1)) \simeq S(\partial(V_2))$. Let $\phi : S(\partial T_1) \to S(\partial T_2)$ be the canonical isomorphism. Then restriction of ϕ onto $S(\partial(V_1))$ is an isomorphism of $S(\partial(V_1))$ with $S(\partial(V_2))$.*

Proof. It follows from construction of canonical isomorphism. □

Since the canonical isomorphism has the properties required for Lemma 2.7, we can prove the following lemma by restricting the isomorphism onto the subgroups $A(\partial T_1) < S(\partial T_1)$ and $A(\partial T_2) < S(\partial T_2)$.

Lemma 2.8. *Let T_1, T_2 be spherically homogeneous rooted trees such that $A(\partial T_1) \simeq A(\partial T_2)$. We assume that the vertices of every level of T_1 and T_2 are numbered in the lexicographic order. Let V_1 be the set of the first i_n vertices of $V_n(T_1)$, let V_2 be the set of the first j_k vertices of $V_k(T_2)$, and suppose that $A(\partial(V_1)) \simeq A(\partial(V_2))$. Then there is an isomorphism $\psi : A(\partial T_1) \to A(\partial T_2)$ such that its restriction onto $A(\partial(V_1))$ is an isomorphism of $A(\partial(V_1))$ with $A(\partial(V_2))$.*

The following is straightforward.

Lemma 2.9.

1. *Let $f(r,s)$ be a diagonal embedding of* $\mathrm{Sym}(n)$ *into* $\mathrm{Sym}(nr+s)$ *with s fixed points. Then the subgroups $f(r,s)(\mathrm{Sym}(n))$ and $u(r,s)(\mathrm{Sym}(n))$ are conjugate in* $\mathrm{Sym}(nr+s)$.
2. *Let $h(r,s)$ be a diagonal embedding of* $\mathrm{Alt}(n)$ *into* $\mathrm{Alt}(nr+s)$ *with s fixed points. Then the subgroups $h(r,s)(\mathrm{Alt}(n))$ and $u(r,s)(\mathrm{Alt}(n))$ are conjugate in* $\mathrm{Sym}(nr+s)$.

Every diagonal direct limit $H = \lim((G_i, X_i), \phi_i)$ of symmetric (alternating) groups has a sequence from $\mathbf{S}$ naturally corresponding to it. Let us set n_i to be equal to the number of the natural orbits of (G_i, X_i) on X_{i+1}, and k_i to be the number of the trivial orbits of the action of (G_i, X_i) on X_{i+1}. We put $k_0 = n_0 = |X_1|$.

Proposition 2.10. *Every diagonal direct limit H of symmetric (alternating) groups is isomorphic to the standard diagonal limit S_χ (resp. A_χ), where χ is corresponding sequence.*

Proof. It follows from Lemma 2.3 of [Bur68] and Lemma 2.9 of our paper. □

3. Classification

Theorem 3.1. *Let $\chi \in \mathbf{S}$. Then*

1. *$S_\chi = A_\chi$ if and only if $\Omega(\chi)$ is divisible by 2^∞;*

2. *if $\Omega(\chi)$ is not divisible by 2^∞, then $[S_\chi : A_\chi] = 2$;*

3. *A_χ is the commutator subgroup S_χ;*

4. *A_χ is a simple group.*

Proof. 1. If $\Omega(\chi)$ is divisible by 2^∞, then for every $n \geq 1$ there is $k > n$ such that $A(\chi, k) > S(\chi, n)$, thus $S_\chi = A_\chi$.

If $\Omega(\chi)$ is not divisible by 2^∞, then there is $n \geq 1$ such that for every $k > n$ the group $A(\chi, k)$ does not contain any odd permutations from $S(\chi, n)$. So, $S_\chi \neq A_\chi$.

2. For every n the group $A(\chi, n)$ is an index 2 subgroup of $S(\chi, n)$. Since χ is odd, we have $S_\chi \neq A_\chi$, and $[S_\chi : A_\chi] = 2$.

Statement (3) is a corollary of a standard statement on verbal subgroups of locally finite groups. Statement (4) is a corollary of Theorem 4.1 in [KW73], p. 112. □

Theorem 3.2. *Let $\chi_1, \chi_2 \in \mathbf{S}$. The direct limits of finite symmetric (alternating) groups corresponding to the sequences χ_1 and χ_2 are isomorphic if and only if the sequences χ_1 and χ_2 are commensurable.*

Note that this theorem is completely analogous to the J. Dixmier's classifications [Dix67] of diagonal direct limits of C^*-algebras.

We need some auxiliary statements.

Proposition 3.3. *Let $\chi = \langle (1, k_0), (n_1, k_1), (n_2, k_2), \ldots \rangle \in \mathbf{S}$. The direct limit of finite symmetric (alternating) groups with corresponding sequence χ is*

1. *isomorphic to $S_{\chi'}$ (resp., $A_{\chi'}$) for some $\chi' \in \mathbf{S}_1$ commensurable with χ, if and only if $n_i \geq 2$ for infinitely many $i \in \mathbb{N}$;*

2. *finite symmetric (alternating) group if and only if $n_i \leq 2$ and $k_i = 0$ for all but a finite number of indices $i \in \mathbb{N}$;*

3. *finitary symmetric (alternating) group if and only if $n_i \leq 2$ for all but a finite number of indices i, and $k_i \geq 1$ for infinitely many $i \in \mathbb{N}$.*

Proof. If $n_i \geq 2$ for infinitely many i, then we can consider a subsequence $S(\chi, l_i)$ (or of $A(\chi, l_i)$) such that the correspondent subsequence χ' belongs to $\mathbf{S}_1$.

Obviously, then $S_\chi \simeq S_{\chi'}$ and $A_\chi \simeq A_{\chi'}$. It is sufficient to use the formula (2) for partial sums of $M(\chi_1)$ and $M(\chi_2)$ in order to prove commensurability of χ and χ'.

It is straightforward that if we have $n_i \geq 2$ only for finitely many indices i, then conditions (2) or (3) hold, accordingly to the number of indices for which $k_i \leq 0$. □

Lemma 3.4. *If $\chi \in \mathbf{S}_1$, then the groups S_χ and A_χ are isomorphic neither to the finitary symmetric, nor to the finitary alternating groups.*

Proof. One can find elements $g \in A_\chi$ whose centralizers $C_{A_\chi}(g)$ contain direct products of index at most 2 of two infinite subgroups. Centralizers of elements of the finitary alternating group does not have this property. Thus, A_χ and the finitary alternating group are not isomorphic. The symmetric groups are treated similarly. □

The following notation is used in statements 3.5–3.10. The symbol H denotes either A or S (the alternating and symmetric group, respectively). The sequences $\chi_1, \chi_2 \in \mathbf{S}_1$ are such that ∂T_{χ_1} and ∂T_{χ_2} are homeomorphic, and the homeomorphism $h : \partial T_{\chi_1} \to \partial T_{\chi_2}$ induces an isomorphism $\phi : H_{\chi_1} \to H_{\chi_2}$, i.e., $\phi(g) = hgh^{-1}$ for every $g \in H_{\chi_1}$. Let m_i be an H_{χ_i}-invariant Borel measure on ∂T_{χ_i} and $m_i(\{\delta_i\}) = 0$. To avoid ambiguity, let us denote the subtree of T_{χ_i} with the root at v by T^i_v.

Lemma 3.5. *For every ball ∂T^1_v ($v \in V(T_{\chi_1}) \setminus \{\delta_1\}$) there exist l and k such that the set $h(\partial T^1_v)$ is a disjoint union of the balls $\partial T^2_{v_i}$ for $\{v_1, \ldots, v_k\} \subset V_l(T_{\chi_2}) \setminus \{\delta_1(l)\}$.*

Proof. Since H_{χ_i} fixes δ_i and does not fix any other end, the set $h(\partial T^1_v)$ does not contain any ball ∂T^2_w such that $w \in \delta_2$. Since ∂T^1_v is compact set, the number k is finite. □

Lemma 3.6. *If the measures m_i are H_{χ_i}-invariant and such that $m_i(\partial T^i_v) = 1$ for $v \in V_1$, and $m_i(\delta) = 0$, for $i = 1, 2$, then the homeomorphism h preserves the measures m_i, i.e., the push forward measure $h_*(m_1)$ is equal to m_2.*

Proof. It is easy to prove that m_i are uniquely defined by the conditions of the lemma (see the proof of Proposition 2.2). But $h_*(m_1)$ is also such a measure on ∂T_{χ_2}, hence $h_*(m_1) = m_2$, and h is measure-preserving. □

Corollary 3.7. *If $m_1(\partial T_{\chi_1})$ and $m_2(\partial T^2_v)$ (for any $v \in V(T_{\chi_2})$) are finite, then $m_2(\partial T_{\chi_2})$ is finite too.*

Theorem 3.8 ([KS98])**.** *Let $\chi_i \in \mathbf{S}_1$ be such that $T_{\chi_i} \setminus \{\delta_i\}$ are spherically homogeneous trees ($i = 1, 2$). Then H_{χ_1} and H_{χ_2} are isomorphic if and only if*

$$char(H_{\chi_1}) = char(H_{\chi_2}).$$

Proof. Let $\phi : H_{\chi_1} \to H_{\chi_2}$ be an isomorphism. The spaces ∂T_{χ_1} and ∂T_{χ_2} are homeomorphic. Then by Theorem 1.2, Lemmata 1.3 and 2.1 there exists a homeomorphism $h : \partial T_{\chi_1} \to \partial T_{\chi_2}$ such that $\phi(g) = hgh^{-1}$ for every $g \in H_{\chi_1}$. Let m_i be the probabilistic H_{χ_i}-invariant measure on ∂T_{χ_i} for $i = 1, 2$. Such measures exist by Lemma 2.3.

Let

$$h(\partial T^1_v) = \bigcup_{i=1}^{k} \partial T^2_{v_i},$$

where $\{v_1, \ldots, v_k\} \in V_l(T_{\chi_2})\setminus\{\delta_2(l)\}$, $v \in V_s(T_{\chi_1})\setminus\{\delta_1(s)\}$. By Lemma 3.6, we have $m_1(\partial T^1_v) = m_2\left(\bigcup_{i=1}^{k} \partial T^2_{v_i}\right)$. Taking into account transitivity of the action of H_{χ_1} on $V_s(T_{\chi_1})\setminus\{\delta_1(s)\}$, we get that $k_{0,1}n_{1,1}\cdots n_{s-1,1}$ divides $k_{0,2}n_{1,2}\cdots n_{l-1,2}$. Since s is arbitrary, we have proved that $char(H_{\chi_1})$ divides $char(H_{\chi_2})$. It is easy to see that $char(H_{\chi_2})$ also divides $char(H_{\chi_1})$. Hence,

$$char(H_{\chi_1}) = char(H_{\chi_2}).$$

Implication in the other direction was proved in Lemma 2.6. □

Lemma 3.9. *Let*

$$h(\partial T^1_v) = \bigcup_{i=1}^{k} \partial T^2_{v_i},$$

where $\{v_1, \ldots, v_k\} \in V_l(T_{\chi_2}) \setminus \{\delta_1(l)\}$, $v \in V_1(T_{\chi_1}) \setminus \{\delta_1(1)\}$. Then

$$\Omega(\chi_1) = \frac{k}{n_{1,2}\cdots n_{l-1,2}}\Omega(\chi_2).$$

Proof. The subgroup $H_1 = H(\partial T_v) < H_{\chi_1}$ is the largest subgroup acting trivially outside of ∂T^1_v. Therefore, the largest subgroup acting trivially outside of $\bigcup_{i=1}^{k} \partial T^2_{v_i}$, i.e., $H_2 = H(\partial\{v_i \mid i = 1, \ldots, k\})$, coincides with $\phi(H_{\chi_1})$. Since the groups H_{χ_1} and H_{χ_2} are isomorphic, we have, by Theorem 3.8,

$$char(H_{\chi_1}) = char(H_{\chi_2}).$$

Taking into account

$$\Omega(\chi_1) = char(H_{\chi_1})$$

and

$$\Omega(\chi_2) = \frac{n_{1,2}\cdots n_{l-1,2}}{k}\, char(H_{\chi_2})$$

we get

$$\Omega(\chi_1) = \frac{k}{n_{1,2}\cdots n_{l-1,2}}\Omega(\chi_2).$$

□

Lemma 3.10. *Suppose that $\gamma_i < M_l(\chi_i)^{-1}$ for all natural numbers l and $i = 1, 2$ and for all $\gamma_1 > 0$, $\gamma_2 > 0$. Suppose also that $h(\partial T_v^1) = \bigcup_{i=1}^k \partial T_{v_i}^2$, where $\{v_1, \ldots, v_k\} \in V_l(T_{\chi_2})$, $v \in V_1(T_{\chi_1}) \setminus \{\delta_1(1)\}$. Then*

$$\gamma_1 = \frac{n_{1,2}\cdots n_{l-1,2}}{k}\gamma_2.$$

Proof. We may assume that $m_1(\partial T_{\chi_1}) = m_2(\partial T_{\chi_2}) = 1$. By Lemma 3.6,

$$\gamma_1 = m_1(\partial T_v^1) = m_2\left(\bigcup_{i=1}^k \partial T_{v_i}^2\right) = k\ m_2(\partial T_{v_i}^2).$$

We also have $\gamma_2 = m_2(\partial T_{v_i}^2) n_{1,2}\cdots n_{l-1,2}$, hence

$$\gamma_1 = \frac{n_{1,2}\cdots n_{l-1,2}}{k}\gamma_2.$$

□

Lemma 3.11. *Let u be a positive integer. Let $\chi_1, \chi_2 \in \mathbf{S}_1$ be $(u, 1)$-commensurable sequences. Then $S_{\chi_1} \simeq S_{\chi_2}$ and $A_{\chi_1} \simeq A_{\chi_2}$.*

Proof. Let us show that there is an increasing sequence $\{l_i\}$ such that for every $j \geq 0$ the number

$$t_{l_j} = r(\chi_1, j) n_{1,2}\cdots n_{l_j,2}(u n_{1,1}\cdots n_{j,1})^{-1}$$

is an integer and less than or equal to $|V_{l_j}(T_{\chi_2}) \setminus \{\delta_2(l_j)\}| = r(\chi_2, l_j)$.

Since $u\Omega(\chi_1) = \Omega(\chi_2)$ there is a positive integer l_0' such that for arbitrary $l_0 \geq l_0'$ the number u is a factor of $n_{1,2}\cdots n_{l_0,2}$. We can choose $l_0 \geq l_0'$ such that $ur(\chi_2, l_0) \geq n_{1,2}\cdots n_{l_0,2} k_{0,1}$, because for any partial sum of $M(\chi_1)$ there is a greater or equal partial sum of $uM(\chi_2)$. Hence

$$|V_{l_0}(T_{\chi_2}) \setminus \{\delta_2(l_0)\}| = r(\chi_2, l_0) \geq n_{1,2}\cdots n_{l_0,2} u^{-1} k_{0,1}.$$

Let $j \geq 1$. Since $u\Omega(\chi_1) = \Omega(\chi_2)$ there is an integer $l_j' > l_{j-1}$ such that for arbitrary $l_j \geq l_j'$ the product $n_{1,1}\cdots n_{j,1}u$ is a factor of $n_{1,2}\cdots n_{l_j,2}$. We can choose $l_j \geq l_j'$ such that

$$\frac{ur(\chi_2, l_j)}{n_{1,2}\cdots n_{l_j,2}} \geq \frac{r(\chi_1, j)}{n_{1,1}\cdots n_{j,1}},$$

because for an arbitrary partial sum of $M(\chi_1)$ there is a greater or equal partial sum of $uM(\chi_2)$. Then

$$|V_{l_j}(T_{\chi_2})\backslash\{\delta_2(l_j)\}| = r(\chi_2, l_j) \geq t_{l_j} = r(\chi_1, j)n_{1,2}\cdots n_{l_j,2}(un_{1,1}\cdots n_{j,1})^{-1}.$$

Let $V^j_{l_j}(T_{\chi_2}) = \{v^{l_j}_1, \ldots, v^{l_j}_{t_{l_j}}\}$ be the set of the first t_{l_j} vertices of the level number l_j of T_{χ_2}. Let us denote the sets $\partial(V_j(T_{\chi_1}) \setminus \{\delta_1(j)\})$ and $\partial(V^j_{l_j}(T_{\chi_2}))$ by U^1_j and $U^2_{l_j}$, respectively. Since

$$char(H(U^1_j)) = \frac{r(\chi_1, j)\Omega(\chi_1)}{n_{1,1}\cdots n_{j,1}} = \frac{t_{l_j}\Omega(\chi_2)}{n_{1,2}\cdots n_{l_j,2}} = char(H(U^2_{l_j})),$$

by definition of t_{l_j}, the groups $H(U^1_j)$ and $H(U^2_{l_j})$ are isomorphic for all $j \geq 0$. Let ϕ_j be the canonical isomorphism of $H(U^1_j)$ with $H(U^2_{l_j})$.

We have

$$t_{l_j} \geq t_{l_{j-1}} n_{l_{j-1}+1,2}\cdots n_{l_j,2},$$

since by the definition of t_{l_j}, this inequality is equivalent to $r(\chi_1, j) \geq r(\chi_1, j-1)n_j$, which is always true. Hence, the inclusion

$$U^2_{l_{j-1}} \subseteq U^2_{l_j}$$

holds, and therefore

$$H(U^2_{l_{j-1}}) \leq H(U^2_{l_j}).$$

It follows from Lemma 2.7 that the restriction of ϕ_j onto $H(U^1_{j-1})$ coincides with ϕ_{j-1}.

Consequently, we have the following commutative diagram

$$\begin{tikzcd}
H(U^1_1) \arrow[r] \arrow[d, "\phi_1"] & H(U^1_2) \arrow[r] \arrow[d, "\phi_2"] & H(U^1_3) \arrow[r] \arrow[d, "\phi_3"] & \ldots \\
H(U^2_{l_1}) \arrow[r] & H(U^2_{l_2}) \arrow[r] & H(U^2_{l_3}) \arrow[r] & \ldots
\end{tikzcd}$$

and we get in the limit an isomorphism

$$\phi : \bigcup_{j\geq 0} H(U^1_j) \to \bigcup_{j\geq 0} H(U^2_{l_j}).$$

We have

$$\bigcup_{j\geq 0} H(U^1_j) = S_{\chi_1},$$

therefore, it remains to prove that

$$\bigcup_{j\geq 0} H(U^2_{l_j}) = S_{\chi_2}.$$

It is sufficient to show that for every $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that

$$t_{l_j} \geq r(\chi_2, i) n_{i+1,2} \cdots n_{l_j,2}, \tag{3}$$

since it will imply

$$\phi(H(\partial(V_j(T_{\chi_1})))) \geq H(\partial(V_i(T_{\chi_2}))).$$

For an arbitrary partial sum of $uM(\chi_2)$ there is a greater or equal partial sum of $M(\chi_1)$. That is, for all $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that the inequality

$$\frac{r(\chi_1, j)}{n_{1,1} \cdots n_{j,1}} \geq \frac{ur(\chi_2, l_j)}{n_{1,2} \cdots n_{l_j,2}}$$

holds, which implies, by the definition of t_{l_j}, the inequality (3). $\square$

Proof of Theorem 3.2. We will prove the theorem for the case of symmetric groups. The proof for the alternating groups is similar.

By Proposition 2.10 it is sufficient to prove the theorem for the groups of the form S_{χ_1} and S_{χ_2}. The group S_{χ_i} is finite if and only if $\Omega(\chi_i) \in \mathbb{N}$, and $M(\chi_i)$ is convergent.

In this case we have

$$S_{\chi_i} \simeq \mathrm{Sym}(\Omega(\chi_i) M(\chi_i)), \quad (i = 1, 2).$$

Putting $u = \Omega(\chi_2)$ and $v = \Omega(\chi_1)$, we obtain the necessity condition.

Sufficiency follows from the equality

$$\Omega(\chi_1) M(\chi_1) = \Omega(\chi_2) M(\chi_2),$$

which we get by multiplying $u\Omega(\chi_1) = v\Omega(\chi_2)$ and $vM(\chi_1) = uM(\chi_2)$.

The group $S(\chi_i)$ is infinite for a sequence $\chi_i \in \mathbf{S} \setminus \mathbf{S}_1$ if and only if exactly one of the following conditions holds

1. $\Omega(\chi_i) \in \mathbb{N}$ and $M(\chi_i)$ is divergent $(i = 1, 2)$;
2. $\Omega(\chi_i) \notin \mathbb{N}$.

In the first case the group S_{χ_i} is isomorphic to the finitary symmetric group on the set $\mathbb{N}$. According to Proposition 3.3, if $\chi_1 \in \mathbf{S}$ is such that $\Omega(\chi_1) \in \mathbb{N}$, $M(\chi_1)$ is divergent, and the groups S_{χ_1} and S_{χ_2} are isomorphic, then $\chi_2 \in \mathbf{S}$ has the same properties.

In the second case, according to Proposition 3.3, without loss of generality, we can assume that $\chi_1, \chi_2 \in \mathbf{S}_1$.

Let us prove the "if" direction of the theorem, i.e., that $S_{\chi_1} \simeq S_{\chi_2}$ implies commensurability of the sequences χ_1 and χ_2.

If the series $M(\chi_1)$ and $M(\chi_2)$ are divergent, then the members $k_{i,1}$ of χ_1 are positive for infinitely many i, and the same is true for χ_2. The converse statements are also true. Hence, if the series $M(\chi_1)$ and $M(\chi_2)$ are not convergent, then commensurability of χ_1 and χ_2 follows from Lemma 3.9.

Suppose that one of the series, for instance $M(\chi_1)$, is convergent. Let $h(\partial T^1_v) = \bigcup_{i=1}^k \partial T^2_{v_i}$, where $\{v_1, \ldots, v_k\} \in V_l(T_{\chi_1})$, $v \in V_1(T_{\chi_1}) \setminus \{\delta_1(1)\}$.

By Lemma 3.9, we have

$$n_{1,2} \cdots n_{l-1,2}\Omega(\chi_1) = k\Omega(\chi_2).$$

Then the following cases are possible

(1) $\gamma_i < M_l(\chi_i)^{-1}$ for all natural l and $i = 1, 2$, and

  a) $\gamma_1 > 0$ and $\gamma_2 > 0$, or

  b) $\gamma_1 > 0$ and $\gamma_2 = 0$;

(2) $\gamma_i = M_l(\chi_i)^{-1}$ for both $i = 1, 2$, and for some l;

(3) γ_1 is such as in the first case, and γ_2 is such as in the second case.

Let us consider at first case (1). If $\gamma_1 > 0$ and $\gamma_2 > 0$, then we may assume that $m_1(X_1) = m_2(X_2) = 1$. By Lemma 3.10, we have then

$$k\gamma_1 = n_{1,2} \cdots n_{l-1,2}\gamma_2.$$

If $\gamma_1 > 0$ and $\gamma_2 = 0$, then by Corollary 3.7, the groups S_{χ_1} and S_{χ_2} are not isomorphic.

In the case (2) both groups are inductive limits with strictly diagonal embeddings, and by Theorem 3.8, we have

$$\frac{r(\chi_1, l)}{n_{1,1} \cdots n_{l,1}}\Omega(\chi_1) = \frac{r(\chi_2, l)}{n_{1,2} \cdots n_{l,2}}\Omega(\chi_2).$$

Since $M(\chi_1) = \frac{r(\chi_1,l)}{n_{1,1}\cdots n_{l,1}}$ and $M(\chi_2) = \frac{r(\chi_2,l)}{n_{1,2}\cdots n_{l,2}}$, we have that χ_1 and χ_2 are (u, v)-commensurable for

$$u = r(\chi_1, l)n_{1,2} \cdots n_{l,2}, \quad v = r(\chi_2, l)n_{1,1} \cdots n_{l,1}.$$

In case (3) the spaces are not homeomorphic, since one is compact and the other is not. Then, by Theorem 1.2, the groups are not isomorphic.

We have shown that in all three cases the isomorphism of the groups S_{χ_1} and S_{χ_2} implies commensurability of χ_1 and χ_2.

Let us prove the "only if" implication of the theorem. Suppose that $\chi_1, \chi_2 \in \mathbf{S}_1$ are (u, v)-commensurable.

Let us define $\chi_3 \in \mathbf{S}_1$ such that χ_1 and χ_3 are $(1, v)$-commensurable. Let m be such that v is a divisor of $\prod_{i=1}^{m} n_{i,1}$. Let $n_{1,3} = v^{-1} \prod_{i=1}^{m} n_{i,1}$ and $n_{i,3} = n_{m+i-1,1}$ for $i > 1$. Then $v\Omega(\chi_3) = \Omega(\chi_1)$. Let $k_{0,3} = vk_{0,1}$,

$$k_{1,3} = \prod_{i=1}^{m} n_{i,1} \sum_{i=1}^{m} \frac{k_{i,1}}{n_{1,1} \cdots n_{i,1}},$$

$k_{i,3} = k_{m+i-1,1}$ for $i > 1$. Then $M_i(\chi_3) = vM_{m+i-1}(\chi_1)$ for $i > 0$. Therefore χ_1 and χ_3 are $(1, v)$-commensurable. Then χ_2 and χ_3 are $(1, u)$-commensurable.

By the Lemma 3.11 we have that the pairs of groups S_{χ_1}, S_{χ_3}, and S_{χ_2}, S_{χ_3} are isomorphic. So, the groups S_{χ_1} and S_{χ_2} are isomorphic. □

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