Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **25** (2018). Number 1, pp. 27–34

On the Fitting ideals of a multiplication module

Somayeh Hadjirezaei and Somayeh Karimzadeh

Communicated by V. Futorny

Abstract. In this paper, we characterize all finitely generated multiplication R-modules whose the first nonzero Fitting ideal of them is contained in only finitely many maximal ideals. Also, we prove that a finitely generated multiplication R-module M is faithful if and only if M is a projective of constant rank one R-module.

Introduction

Let R be a commutative ring with identity and M be a finitely generated R-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$ where R^n is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$, with λ in some index set Λ. Let $\mathrm{Fitt}_i(M)$ be an ideal of R generated by the minors of size $n - i$ of the matrix

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

For $i > n$, $\mathrm{Fitt}_i(M)$ is defined by R, and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is the invariant ideal determined

2010 MSC: 13C05, 13C99, 11Y50.
Key words and phrases: fitting ideals, multiplication module, projective of constant rank, faithful module.

by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [8]. The ideal $\mathrm{Fitt}_i(M)$ will be called the i-th Fitting ideal of the module M. It follows from the definition of $\mathrm{Fitt}_i(M)$ that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$. Moreover, it is shown that $\mathrm{Fitt}_0(M) \subseteq \mathrm{ann}(M)$ and $(\mathrm{ann}(M))^n \subseteq \mathrm{Fitt}_0(M)$ (M is generated by n elements) and $\mathrm{Fitt}_i(M)_P = \mathrm{Fitt}_i(M_P)$, for every prime ideal P of R [6]. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Note that if $I(M)$ contains a nonzerodivisor then $I(M_P) = I(M)_P$ for every prime ideal P of R. An element of R is called regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Assume that $T(M)$, the torsion submodule of M, be the submodule of M consisting of all elements of M that are annihilated by a regular element of R. An R-module M is a torsion module if $M = T(M)$ and is a torsionfree R-module if $T(M) = 0$. Fitting ideals are strong tools to identify properties of modules and sometimes to characterize modules. For example Buchsbaum and Eisenbud have shown in [2] that for a finitely generated R-module M, $I(M) = R$ if and only if M is a projective of constant rank module. A lemma of Lipman asserts that if R is a local ring and $M = R^m/K$ and $I(M)$ is the $(m-q)$th Fitting ideal of M then $I(M)$ is a regular principal ideal if and only if K is finitely generated free and $M/T(M)$ is free of rank $m-q$ ([11]). Finally it is shown in [9] that if M is a finitely generated module over a Noetherian local UFD (R, P) then $I(M) = P$ if and only if

1. M is isomorphic to $R^n/\langle (a_1, \ldots, a_n)^t \rangle$, where $P = \langle a_1, \ldots, a_n \rangle$ and n is a positive integer if M is torsionfree, and

2. M is isomorphic to $R^n \oplus R/P$, for some positive integer n if M is not torsionfree.

Multiplication modules, first were defined by A. Barnard in [1].

An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$ [15].

1. Fitting ideals of multiplication modules

In this section we study some properties of finitely generated multiplication modules and Fitting ideals of them.

Proposition 1. *Let $M = M_1 \oplus M_2$ be a finitely generated R-module. Then $\mathrm{Fitt}_k(M) = \sum_{i+j=k} \mathrm{Fitt}_i(M_1)\,\mathrm{Fitt}_j(M_2)$. Particularly $I(M) = I(M_1)I(M_2)$.*

Proof. Let $N_i \longrightarrow G_i \longrightarrow M_i \longrightarrow 0$ be exact sequences and G_i be free R-modules of rank r_i for $i = 1, 2$. Let ψ_1 be the matrix presentation of generators of N_1 and ψ_2 be the matrix presentation of generators of N_2. Thus $N_1 \oplus N_2 \longrightarrow G_1 \oplus G_2 \longrightarrow M \longrightarrow 0$ is an exact sequence and $\psi_1 \oplus \psi_2$ is the matrix presentation of generators of $N_1 \oplus N_2$. Let $I_j(\psi_1 \oplus \psi_2)$ be an ideal of R generated by the minors of size j of matrix presentation of $(\psi_1 \oplus \psi_2)$. Since $\psi_1 \oplus \psi_2 = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$, hence $\operatorname{Fitt}_k(M) = I_{r_1+r_2-k}(\psi_1 \oplus \psi_2) = \sum_{i+j=k} I_{r_1-i}(\psi_1) I_{r_2-j}(\psi_2) = \sum_{i+j=k} \operatorname{Fitt}_i(M_1) \operatorname{Fitt}_j(M_2)$. □

Theorem 1. *Let R be a ring and M be a finitely generated R-module. If $\operatorname{Fitt}_0(M) = Q$ be a maximal ideal of R then $M \cong (R/Q)^n$, for some positive integer n.*

Proof. By [6, Proposition 20-7], $Q = \operatorname{Fitt}_0(M) \subseteq \operatorname{ann}(M)$. So M is a vector space over the field R/Q. Hence there exists a positive integer n such that $M \cong (R/Q)^n$. □

The next Proposition asserts the relation between the 0-th Fitting ideal of a module and the 0-th Fitting ideal of it's submodules.

Proposition 2. *Let M be a finitely generated module and N be a submodule of M generated by k elements. Then $\operatorname{Fitt}_0(M)^k \subseteq \operatorname{Fitt}_0(N)$.*

Proof. By [6, Proposition 20-7] we have $\operatorname{Fitt}_0(M) \subseteq \operatorname{ann}(M) \subseteq \operatorname{ann}(N)$ and $\operatorname{ann}(N)^k \subseteq \operatorname{Fitt}_0(N)$. Thus $\operatorname{Fitt}_0(M)^k \subseteq \operatorname{Fitt}_0(N)$. □

A Theorem of Barnard [1] asserts that every multiplication module is locally cyclic [1]. Here we give another proof for this result using Fitting ideals.

Lemma 1. *Let M be a finitely generated multiplication R-module. Then $\operatorname{Fitt}_1(M) = R$.*

Proof. Let M be generated by $\{x_1, \ldots, x_n\}$. Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where $\varphi(e_j) = x_j$ and $N = Ker(\varphi)$. Put $B_i = (Rx_i : M)$, for $i, 1 \leqslant i \leqslant n$. For the moment fix $i, 1 \leqslant i \leqslant n$. Let $a_{ji} \in B_i$, $1 \leqslant j \leqslant n, j \neq i$. Then there exist some $b_{ij} \in R$ such that

$a_{ji}x_j = b_{ij}x_i, 1 \leqslant j \leqslant n, j \neq i$. Consider the matrix

$$\begin{pmatrix} a_{1i} & 0 & \dots & \dots & \dots & 0 \\ 0 & a_{2i} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -b_{i1} & -b_{i2} & \dots & \ddots & \dots & -b_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & a_{ni} \end{pmatrix}.$$

Since each columns of this matrix belongs to N, so we have

$$\begin{vmatrix} a_{1i} & 0 & \dots & 0 \\ 0 & a_{2i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{ni} \end{vmatrix} \in \mathrm{Fitt}_1(M).$$

This implies that $B_i^{n-1} \subseteq \mathrm{Fitt}_1(M)$, for all $i, 1 \leqslant i \leqslant n$. Hence $B_1^{n-1} + \cdots + B_n^{n-1} \subseteq \mathrm{Fitt}_1(M)$. So by [13, 2.25],

$$\sqrt{B_1^{n-1} + \cdots + B_n^{n-1}} = \sqrt{\sqrt{B_1^{n-1}} + \cdots + \sqrt{B_n^{n-1}}}$$
$$= \sqrt{\sqrt{B_1} + \cdots + \sqrt{B_n}} \subseteq \sqrt{\mathrm{Fitt}_1(M)}.$$

Since $(B_1 + \cdots + B_n)M = M$, by [14, Corollary], $R = B_1 + \cdots + B_n + \mathrm{ann}(M)$. on the other hand we have $\mathrm{ann}(M) \subseteq B_i = (Rx_i : M)$. Thus $R = B_1 + \cdots + B_n$. Therefore $\sqrt{\mathrm{Fitt}_1(M)} = R$. Thus $\mathrm{Fitt}_1(M) = R$. □

Theorem 2. *Let M be a finitely generated multiplication module over a ring R. Then M is locally cyclic.*

Proof. Let M be a finitely generated multiplication module over a local ring (R, P). Let $\{x_1, \dots, x_n\}$ be a minimal generator set for M. Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_i = a_{1i}e_1 + \dots + a_{ni}e_n$, with i in some index set I. Since $\{x_1, \dots, x_n\}$ is a minimal generator set for M, hence $a_{ij} \in P$, for all i, j. Thus $\mathrm{Fitt}_{n-1}(M) \subseteq P$. On the other hand by Lemma 1, we have $\mathrm{Fitt}_{n-1}(M) = R$, for $n \geqslant 2$. Hence $n = 1$. This means that M is cyclic. □

Proposition 3. *Let M be a finitely generated multiplication R-module. Then* $\mathrm{Fitt}_0(M) = \mathrm{ann}(M)$.

Proof. By 1, M_Q is cyclic. Thus $\mathrm{Fitt}_0(M_Q) = \mathrm{ann}(M_Q)$, for every prime ideal Q of R. Since M is finitely generated, hence $\mathrm{ann}(M_Q) = \mathrm{ann}(M)_Q$. Thus by [6, Corollary 20.5], $\mathrm{Fitt}_0(M)_Q = \mathrm{ann}(M)_Q$, for every prime ideal Q of R. Therefore $\mathrm{Fitt}_0(M) = \mathrm{ann}(M)$. □

Lemma 2. *Let R be an integral domain and M be an R-module. Then $T(M_P) = T(M)_P$, for every prime ideal P of R.*

Proof. Let P be a prime ideal of R. It is easily seen that $T(M)_P \subseteq T(M_P)$. Now let R be an integral domain. Assume that $\frac{x}{1} \in T(M_P)$. Thus there exists a regular element $\frac{r}{s} \in R_P$ such that $\frac{r}{s}\frac{x}{1} = \frac{0}{1}$. So there exists an element $t \in R \setminus P$ such that $trx = 0$. Thus $r(tx) = 0$. It is sufficient to prove that r is a regular element of R. Then $tx \in T(M)$ and consequently $\frac{x}{1} = \frac{tx}{t} \in T(M)_P$. To prove the regularity of r, assume that $ar = 0$, for some element a of R. Thus $\frac{r}{s}\frac{a}{1} = \frac{0}{1}$, in R_P. Since $\frac{r}{s}$ is a regular element of R_P, So $\frac{a}{1} = \frac{0}{1}$. Thus there exists an element $b \in R \setminus P$ such that $ab = 0$. Since R is an integral domain and $b \in R \setminus P$, hence $a = 0$ and we are done. □

Theorem 3. *Let M be a finitely generated multiplication module over an integral domain R. Then M is a torsionfree R-module or M is a torsion R-module.*

Proof. Let $M = \langle x_1, \ldots, x_n \rangle$ be a finitely generated nontorsionfree multiplication module over an integral domain R. So $T(M) \neq 0$. Since M is a multiplication module, hence there exists an ideal I of R such that $T(M) = IM$. Let $0 \neq a \in I$ be arbitrary. For $1 \leqslant i \leqslant n$, there exist some $0 \neq r_i$ such that $r_i a x_i = 0$. So $r_1 \ldots r_n a \in \mathrm{ann}(M)$. Since $r_1 \ldots r_n$ is a regular element so $M = T(M)$. □

Theorem 4. *Let M be a finitely generated multiplication module over an integral domain R. Then the following conditions are equivalent.*

1) M is a torsionfree R-module.

2) M is a projective of constant rank one R-module.

Proof. ($1 \Longrightarrow 2$) Let M be a finitely generated torsionfree multiplication module over an integral domain R. So $\mathrm{ann}(M) = 0$. Thus $M_P \cong R_P/\mathrm{ann}(M)_P = R_P/\mathrm{ann}(M_P) = R_P$, for every prime ideal P of R. Hence by [3, 5&3, Theorem 2] M is a projective of constant rank one R-module.

($2 \Longrightarrow 1$) Since M is a projective of constant rank one R-module, hence for every prime ideal p of R, we have $M_p = R_p$. So $T(M)_p = T(M_p) = 0$, for every prime ideal p of R. Therefore $T(M) = 0$. □

Corollary 1. *Let M be a finitely generated multiplication module over an integral domain R. Then M is a projective of constant rank R-module or M is a torsion R-module.*

Proof. By Theorem 3 and Theorem 4. □

Theorem 5. *Let M be a finitely generated multiplication R-module. If $I(M)$ is contained in only finitely many maximal ideals of R, then M is cyclic.*

Proof. If $\mathrm{ann}(M) = 0$, then by Proposition 3, $\mathrm{Fitt}_0(M) = 0$. So by Lemma 1, $I(M) = R$ that is not contained in any maximal ideal of R. Thus $\mathrm{ann}(M) \neq 0$. Then by Proposition 3, $I(M) = \mathrm{Fitt}_0(M) = \mathrm{ann}(M)$. Since there exist only finitely many maximal ideals of R containing $I(M)$, hence by [1, Lemma 10], M is a cyclic R-module. □

Corollary 2. *Let M be a finitely generated multiplication module. Let $I(M) = P_1^{n_1} \dots P_k^{n_k}$, for some maximal ideals P_i of R and for some positive integers n_i, $1 \leqslant i \leqslant k$. Then $M \cong R/P_1^{n_1} \oplus \dots \oplus R/P_k^{n_k}$.*

Proof. Since $I(M)$ is contained in only finitely many maximal ideals P_1, $\dots$, P_n, hence by Theorem 5, $M \cong R/P_1^{n_1} \dots P_k^{n_k} \cong R/P_1^{n_1} \oplus \dots \oplus R/P_k^{n_k}$. □

Corollary 3. *Let M be a finitely generated multiplication module. Then M is a faithful R-module if and only if M is a projective of constant rank one R-module.*

Proof. By Lemma 1, $\mathrm{Fitt}_1(M) = R$ and by Proposition 3, $\mathrm{Fitt}_0(M) = \mathrm{ann}(M) = 0$. So $I(M) = R$. Thus by [2, Lemma 1], M is a projective of constant rank R-module. On the other hand by Theorem 2, M is locally cyclic. Hence M is a projective of constant rank one R-module. Now Let M be a projective of constant rank R-module. So $\mathrm{ann}(M_P) = \mathrm{ann}(M)_P = 0$. Thus $\mathrm{ann}(M) = 0$. □

Note that if M is a projective R-module then the converse of the previous lemma is not true always. See the following Lemma.

Lemma 3. *Let M be a finitely generated multiplication module. If $I(M) = \langle e \rangle$, where e is an idempotent element of R, then M is a projective R-module.*

Proof. If $I(M) = \mathrm{Fitt}_0(M)$, then by Proposition 3, $\mathrm{Fitt}_0(M) = \mathrm{ann}(M) = \langle e \rangle$. So by [4, Theorem 2.1], M is a projective R-module. If $\mathrm{Fitt}_0(M) = 0$, then by Lemma 1, $\mathrm{Fitt}_1(M) = R$. So by [2, Lemma 1], M is a projective R-module. □

Theorem 6. *Let M be a finitely generated multiplication module. If e is an idempotent element of R such that $\mathrm{ann}(M) \subsetneq \langle e \rangle \subsetneq R$, then $M \cong eM \oplus \dfrac{M}{eM}$.*

Proof. Since $\langle e \rangle \neq R$, hence $eM \neq M$. It is easily seen that $\dfrac{M}{eM}$ is a multiplication module and we have $\mathrm{ann}(\dfrac{M}{eM}) = (eM : M) \supseteq \langle e \rangle$. Let $r \in (eM : M)$ and $m \in M$. Thus $rm \in eM$. So there exists an element $m' \in M$, such that $rm = em'$. Hence $rem = e^2m' = em' = rm$. Therefore $(re-r)m = 0$. So $re-r \in \mathrm{ann}(M) \subseteq \langle e \rangle$. Thus $r \in \langle e \rangle$. So $\mathrm{ann}(\dfrac{M}{eM}) = \langle e \rangle$. By Lemma 3, $\dfrac{M}{eM}$ is a projective R-module. So $M \cong eM \oplus \dfrac{M}{eM}$. □

Theorem 7. *Let M be a finitely generated multiplication R-module. If $I(M)$ is a primary ideal of R then M is an indecomposable R-module.*

Proof. Let $M = N \oplus K$, for some R-submodules N and K of M. Assume that $\pi_1, \pi_2 : M = N \oplus K \longrightarrow M$ be defined by $\pi_1(n + k) = n$ and $\pi_2(n + k) = k$, for $n \in N$ and $k \in K$. Since M is a finitely generated multiplication module, so by [5, Theorem 3], there exist $0 \neq r_1$ and $0 \neq r_2$ in R such that $\pi_1(m) = r_1 m$ and $\pi_2(m) = r_2 m$, for every $m \in M$. Since $\pi_1 o \pi_2 = \pi_2 o \pi_1 = 0$, hence $r_1 r_2 M = 0$. So $r_1 r_2 \in \mathrm{ann}(M) = \mathrm{Fitt}_0(M)$. Since $I(M) = \mathrm{Fitt}_0(M) = \mathrm{ann}(M)$ is a primary ideal of R, hence $r_1^{n_1} \in \mathrm{ann}(M)$ or $r_2^{n_2} \in \mathrm{ann}(M)$, for some positive integers n_1 and n_2. If $r_1^{n_1} \in \mathrm{ann}(M)$, then $\pi_1(m) = \pi_1^{n_1}(m) = r_1^{n_1} m = 0$, for every element $m \in M$. Therefore $N = 0$. Similarly if $r_2^{n_2} \in \mathrm{ann}(M)$, then $\pi_2(m) = \pi_2^{n_2}(m) = r_2^{n_2} m = 0$, for every element $m \in M$. Thus $K = 0$. □

References

[1] A. Barnard, *Multiplication modules*, J. Algebra, **71**, 1981, pp.174-178.

[2] D. A. Buchsbaum and D. Eisenbud, *What makes a complex exact*, J. Algebra, **25**, 1973, pp.259-268.

[3] N. Bourbaki, *Commutative Algebra*, Springer-Verlag, 1989.

[4] A. Campillo, T. S. Giralda, *Finitely generated projective modules and Fitting ideals*, Collect. Math., **30**, 1979, pp.97-102.

[5] C. W. Choi, P. F. Smith, *On endomorphism of multiplication modules*, J. Korean Math. Soc., **31(1)**, 1994, pp.89-95.

[6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-verlag, New York, 1995.

[7] Z. A. El-Bast and P.F. Smith, *Multiplication modules* , Comm. Algebra, **16**, 1988, pp.755-779.

[8] H. Fitting, *Die Determinantenideale eines Moduls*, Jahresbericht d. Deutschen Math.-Vereinigung, **46**, 1936, pp.195-228.

[9] S. Hadjirezaei, S. Hedayat, *On the first nonzero Fitting ideal of a module over a UFD*, Comm. Algebra, **41**, 2013, pp.361-366.

[10] V. Kodiyalam, *Integrally closed modules over two-dimensional regular local ring*, Trans. Amer. math. soc., **347(9)**, 1995, pp.3551-3573.

[11] J.Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc., **21**, 1969, pp.423-426.

[12] J.Ohm, *On the first nonzero Fitting ideal of a module*, J. Algebra, **320**, 2008, pp.417-425.

[13] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge university press, 2000.

[14] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50**, 1988, pp.223-235.

[15] Y. Tiras, M. Alkan, *Prime modules and submodules*, Comm. Algebra, **31**, 2003, pp.5253-5261.

Contact information

S. Hadjirezaei, S. Karimzadeh
Department of Mathematics,
Vali-e-Asr University of Rafsanjan,
P.O.Box 7718897111, Rafsanjan, Iran
E-Mail(s): s.hajirezaei@vru.ac.ir, karimzadeh@vru.ac.ir

Received by the editors: 28.09.2015.