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RESEARCH ARTICLE

On the existence of complements in a group to some abelian normal subgroups

Martyn R. Dixon, Leonid A. Kurdachenko, Javier Otal

Abstract. A complement to a proper normal subgroup H of a group G is a subgroup K such that $G = HK$ and $H \cap K = \langle 1 \rangle$. Equivalently it is said that G*splits* over H. In this paper we develop a theory that we call *hierarchy of centralizers* to obtain sufficient conditions for a group to split over a certain abelian subgroup. We apply these results to obtain an entire group-theoretical wide extension of an important result due to D. J. S. Robinson formerly shown by cohomological methods.

Introduction

Let G be a group and let H be a proper subgroup of G. A proper subgroup $K < G$ is called a *supplement* to H in G if $G = HK$ and a *complement* to H in G if $G = HK$ and $H \cap K = \langle 1 \rangle$. If H is a normal subgroup of G and H has a complement K in G, then it is said that G *splits over* H and write this as $G = H \leftthreetimes K$. If all complements to H are conjugate, then it is said that G *conjugately splits over* H.

It is well know that the existence of complements to certain subgroups of a group exerts a big influence on the structure of the group. For example, if every Sylow p–subgroup of a finite group G has a complement, then G is soluble ([5]). Also there are many cases known in which a group splits over several normal subgroups. For example if the nilpotent residual L of a finite group G is abelian, then G splits conjugately over L ([4, 20]); this result has been generalized in many instances for infinite groups (see, for example,

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[7, 17, 18, 19, 21, 22, 23, 28, 29, 30, 31, 32]). Moreover the existence of complements give a possibility to obtain other characterizations of groups. For example in the paper [14], some splitting results have been used for the description of certain groups with no proper abnormal subgroups. In the papers [12, 13], some criteria of existence of supplements to certain normal subgroups have been obtained, and as corollaries of them, the description of certain groups with proper contranormal subgroups have been carried out.

In some sense, this paper is a continuation of the papers [12, 13]. At first, we are able to obtain sufficient conditions for the existence of complements to certain abelian normal subgroups.

Theorem A (Theorem 1.1) *Let G be a group, A be an abelian normal subgroup of G and let $g \in G$ such that $gC_G(A) \in \zeta(G/C_G(A))$ is non-trivial. Suppose that G satisfies the following conditions:*

(i) G/A *is nilpotent;*

(ii) $A = [A, g] \neq \langle 1 \rangle$*; and*

(iii) $C_A(g) = \langle 1 \rangle$.

Then G conjugately splits over A.

To obtain more splitting theorems we need the following construction. Let G be a hypercentral group and let A be a semisimple $\mathbb{Z}G$–module. Suppose that

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda,$$

where A_λ is a simple $\mathbb{Z}G$–module for every $\lambda \in \Lambda$. If H is a normal subgroup of G, then either $[H, A_\lambda] = A_\lambda$ or $[H, A_\lambda] = \{0\}$. Let $C_0 = C_G(A)$, and suppose that that $G \neq C_0$. Being a non-identity hypercentral group, G/C_0 has a non-identity center. Let $C_0 \neq g_0C_0 \in \zeta(G/C_0)$. Thus $[A_\lambda, g_0]$ is a G–invariant subgroup of A_λ so that either $[A_\lambda, g_0] = A_\lambda$ or $[A_\lambda, g_0] = \{0\}$. Put

$$\Sigma = \{\lambda \in \Lambda \mid [A_\lambda, g_0] = \{0\}\},\ Z_1 = C_A(g_0) = \bigoplus_{\lambda \in \Sigma} A_\lambda \text{ and } C_1 = C_G(Z_1).$$

By the election of Z_1, $g_0 \in C_1$ and so $C_1 > C_0$. If $G \neq C_1$, then we continue this process. Pick $C_1 \neq g_1C_1 \in \zeta(G/C1)$ and put $Z_2 = C_A(g_1)$. Suppose that we have constructed the ascending series

$$C_0 < C_1 < \ \cdots \ < C_\beta < \ \cdots$$

for all ordinals $\beta < \alpha$ (α a given ordinal) and respective set of elements $\{g_\beta \mid \beta < \alpha\}$ as above. Let $Z_{\alpha+1} = \bigcap_{\beta<\alpha} C_A(g_\beta)$ and $C_{\alpha+1} = C_G(Z_{\alpha+1})$. If $G = C_{\alpha+1}$, this process finishes. If not we can continue it by picking a non-trivial element $g_{\alpha+1}C_{\alpha+1} \in \zeta(G/C_{\alpha+1})$. At last, there is an ordinal γ such that $G = C_\gamma$. The family $\{C_\alpha; g_\alpha \mid \alpha < \gamma\}$ is called *a hierarchy of centralizers of A*

If G is a group, as usual, we denote by $\Pi(G)$ the set of all primes occurring as divisors of the order of the periodic elements of G. If A is a periodic abelian group, let

$$A = \mathrm{Dr}_{p\in\Pi(A)} A_p$$

be the primary decomposition of A, where A_p is the Sylow p–subgroup of A. If $n \geq 1$, we put

$$A[n] = \mathrm{Dr}_{p\in\Pi(A)} \Omega_n(A_p),$$

where $\Omega_n(A_p) = \{b \in A_p \mid b^{p^n} = 1\}$. In this setting, we have

Theorem B (Theorem 1.8) *Let G be a group and let A be a periodic abelian normal subgroup of G such that G/A is nilpotent. Suppose that $A[1] = S = Dr_{\lambda\in\Lambda} A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(G/A)$–module S has a finite hierarchy of centralizers, then G conjugately splits over A.*

Applying at once Theorem B and [17, Lemma 3], we obtain

Corollary *Let G be a group and let A be a periodic abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is nilpotent. Suppose also that $A[1] = S = Dr_{\lambda\in\Lambda} A_\lambda$, where A_λ is a minimal H–invariant subgroup and $[H, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(H/A)$–module S has a finite hierarchy of centralizers, then G conjugately splits over A.*

We recall that a group G is called *minimax* if G has a finite subnormal series whose factors satisfy either the maximal condition or the minimal condition on subgroups.

Theorem C (Theorem 1.13) *Let G be a group and let A be a periodic abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is nilpotent. Suppose also that $A[1] = Dr_{\lambda\in\Lambda} A_\lambda$, where A_λ is a minimal H–invariant subgroup and $[H, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If H/A is minimax, then G conjugately splits over A.*

As corollaries we are able to obtain the description of some finitely generated soluble groups.

Let G be a group and let L be a normal subgroup of G. We say that L is *G–hyperfinite* if L has an ascending series

$$\langle 1\rangle = L_0 \leq L_1 \leq \cdots \leq L_\alpha \leq L_{\alpha+1} \leq \cdots L_\gamma = L$$

of G–invariant subgroups whose factors are finite.

On the other hand, we recall that the subgroup $Soc_G(L)$ generated by all minimal G–invariant subgroups of G is said to be *the G–socle of L*; we agree $Soc_G(L) = \langle 1\rangle$ provided L has no such subgroups. Starting from the socle we construct *the upper G–socular series of L* as the ascending chain

$$\langle 1\rangle = S_0 \leq S_1 \leq \cdots \leq S_\alpha \leq S_{\alpha+1} \leq \cdots S_\rho,$$

where $S_1 = Soc_G(L)$ and $S_{\alpha+1}/S_\alpha = Soc_G(L/S_\alpha)$ for every ordinal α. We note that $S_\mu = \bigcup_{\beta<\mu} S_\beta$ for any limit ordinal μ. By definition, the least ordinal ρ such that $S_\rho = S_{\rho+1}$ is called *the socular height of L*. The normal subgroup L is called *G–socular* if L is the last term of the upper G–socular series of L and is called *G–socular-finite* if L is G–socular and has finite socular height.

Theorem D (Theorem 3.3) *Let G be a finitely generated group and let S a soluble normal subgroup of G. Suppose that the following conditions holds:*

(i) G contains a normal subgroup $L \geq S$ such that L/S is a torsion-free nilpotent minimax group and G/L is abelian - by finite;

(ii) S is G–hyperfinite; and

(iii) every p–factor of S is G–socular-finite for each $p \in \Pi(G)$.

Then the subgroup S is finite.

We mention some application of Theorem D and how we can obtain it. If p is a prime, we recall that a group G is said to have *finite section p–rank* $r_p(G) = r$ if every elementary abelian p–section of G is finite of order at most p^r and there is an elementary abelian p–section A/B of G such that $|A/B| = p^r$. A group G is said to have *finite section rank* if $r_p(G)$ is finite for each prime p. In a celebrate paper [18] of 1982, D. J. S. Robinson was able to prove that a finitely generated soluble group G of finite section rank is minimax. However the proof of this relevant result was obtained by means of cohomological methods. Therefore it is interesting to obtain a group-theoretical proof of that theorem, which can be carried out as an application of the results developed in this paper.

Actually, a sketch of our proof is the following. Let $Tor(G)$ be the largest periodic normal subgroup of such a G. The first step of the proof is to prove that $G/TorG)$ is minimax. The second step is the study of the case in which the Sylow p–subgroups of $Tor(G)$ are finite for all primes p, a fact that appears as a corollary of Theorem D. Finally it remains the consideration of the case when $Tor(G)$ is a divisible abelian subgroup, for which we also obtain a group-theoretical proof.

1. The hierarchy of centralizers

Our first result (Theorem A) describes some sufficient conditions for a group to conjugately split over a normal subgroup. Its proof needs an additional definition. Let G be a group. Given $x, g \in G$, as usually, we put

$$[x,_1 g] = [x, g] \text{ and } [x,_{n+1} g] = [[x,_n g], g] \text{ for all } n \geq 1.$$

The left n–Engelizer $E_{G,n}(g)$ of an element $g \in G$ is the subset of G given by

$$E_{G,n}(g) = \{x \in G \mid [x,_n g] = 1\}.$$

It is worth noting that, in general, $E_{G,n}(g)$ is not a subgroup of G.

Theorem 1.1. *Let G be a group, A be an abelian normal subgroup of G and let $g \in G$ such that $gC_G(A) \in \zeta(G/C_G(A))$ is non-trivial. Suppose that G satisfies the following conditions:*

(i) G/A is nilpotent;

(ii) $A = [A, g] \neq \langle 1\rangle$; and

(iii) $C_A(g) = \langle 1\rangle$.

Then G conjugately splits over A.

Proof. By [13, Proposition 2.4], there is a subgroup L such that $G = AL$ and $L \cap A \subseteq E_{G,m}(g)$ for some positive integer m. Let $E = \langle E_{G,m}(g)\rangle$. By [13, Lemma 2.3], there exists a positive integer k such that $E \subseteq E_{G,k}(g)$. Put $D = A \cap E$. We claim that $D = \langle 1\rangle$. For, otherwise, suppose that $D \neq \langle 1\rangle$. Since $D \subseteq E_{G,k}(g)$, we have that $[D,_k g] = \langle 1\rangle$. If t is the least positive integer such that $C = [D,_t g] \neq \langle 1\rangle$, we have that $[C, g] = [D,_{t+1} g] = \langle 1\rangle$, and therefore $C \leq C_A(g)$ and $C_A(g) \neq \langle 1 >\rangle$, contradicting (iii). Thus $A \cap E = \langle 1\rangle$ as claimed. Let s be the least positive integer such that $L \subseteq E_{G,s}(g)$. By [13, Lemma 2.3], there exists a positive integer r such that $R = \langle E_{G,s}(g)\rangle \subset E_{G,r}(g)$. Since $L \leq R$, $R = L(R \cap A)$. But $R \cap A = \langle 1\rangle$ and so $R = L$. Thus $G = A \rtimes L$. Therefore $L = E_{G,s}(g)$.

Let Y be other complement to A. Since $g \in AY$, we have $g = yb$, where $y \in Y$ and $b \in A$. Equivalently $y = ga$, where $a = b^{-1}$. The subgroups L and Y are isomorphic to G/A, therefore there is an automorphism $\phi : L \leftrightarrow Y$. We note that $\phi(g) = y$. The equality $L = E_{G,s}(g)$ implies that $[Y,_s y] = \langle 1 \rangle$, that is $Y \subseteq E_{G,s}(y)$. Since $A = [A, g] = [g, A]$, we may write $a = [g, c]$, where $c \in A$. Then $y = gg^{-1}c^{-1}gc = c^{-1}gc$. It follows that $E_{G,s}(y) = E_{G,s}(g^c) = E_{G,s}(g)^c$. In particular $E_{G,s}(y)$ is a subgroup. Moreover

$$E_{G,s}(y) \cap A = E_{G,s}(g^c) \cap A = E_{G,s}(g)^c \cap A^c = (E_{G,s}(g) \cap A)^c = \langle 1 \rangle.$$

Now we have $E_{G,s}(y) = Y(E_{G,s}(y) \cap A) = Y$. In particular, $Y = E_{G,s}(y) = E_{G,s}(g)^c = L^c$. □

Corollary 1.2. *Let G be a group and A be a periodic abelian normal subgroup of G. Suppose that G/A is nilpotent and A has a finite series of G–invariant subgroups*

$$\langle 1 \rangle = A_0 \le A_1 \le \cdots \le A_n = A$$

and there are elements $g_1, \cdots, g_n$ satisfying the following conditions:

(i) $g_jC_G(A_j/A_{j-1}) \in \zeta(G/C_G(A_j/A_{j-1})$ *for every* $1 \le j \le n$;

(ii) $A_j/A_{j-1} = [A_j/A_{j-1}, g_jA_j]$ *for every* $1 \le j \le n$; *and*

(iii) the centralizer of g_j in the factor A_j/A_{j-1} is identity for every $1 \le j \le n$.

Then G conjugately splits over A.

Proof. We proceed by induction on n. If $n = 1$, the result follows from Theorem 1.1. Put $B = A_1$ and suppose that we have already proved that G/B conjugately splits over A/B. Let L/B be a complement to A/B so that L/B is nilpotent. If $g \in G$, we have $g = xa$ for some $x \in L$ and $a \in A$. Since $L/B \cong G/A$, $xC_L(B) \in \zeta(L/CL(B))$. Since A is abelian, $b^x = b^g$ for each element b B. It follows that $C_B(x) = \langle 1 \rangle$ and $[B, x] = B$. By Theorem 1.1, $L = B \leftthreetimes K$ and every complement to B in L is conjugate to K. Hence $G = AL = A(BK) = AK$ and

$$A \cap K = A \cap (L \cap K) = (A \cap L) \cap K = B \cap K = \langle 1 \rangle.$$

In other words, K is a complement to A in G.

Let D be another complement to A in G. Then DB/B is a complement to A/B in G/B. By induction hypothesis, there exists an element v such that $D^vB/B = (DB/B)^{vB} = L/B$. In particular, $D^v \le L$, and then $L = D^vB$. Clearly $D^v \cap B = \langle 1 \rangle$, so that D^v is a complement to B in L. Then there is an element $u \in L$ such that $K = (D^v)^u = D^{vu}$, and the proof is now complete. □

We start our specific study on hierarchy of centralizers. Roughly speaking we are going to show that having a certain hierarchy of centralizers implies conjugately splitting.

Lemma 1.3. *Let G be a group and let A be an abelian normal subgroup of G such that G/A is hypercentral. Let B a G–invariant subgroup of A and suppose that $B = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If B satisfies the minimal condition on G–invariant centralizers, then the $\mathbb{Z}(G/A)$–module B has a finite hierarchy of centralizers.*

Proof. Put $C_0 = C_G(B)$; by hypothesis, $G \neq C_0$. Since G/C_0 is hypercentral, there exists $C_0 \neq g_0C_0 \in \zeta(G/C0)$. Since $g_0C_0 \in \zeta(G/C_0)$, $[A_\lambda, g_0]$ is a G–invariant subgroup of A_λ, so that either $[A_\lambda, g_0] = A_\lambda$ or $[A_\lambda, g_0] = \langle 1\rangle$, because of the minimality of A_λ. Let

$$\Delta = \{\lambda \in \Lambda \mid [A_\lambda, g_0] = A_\lambda\} \text{ and } \Sigma = \Sigma \setminus \Delta.$$

Then $K_1 = \mathrm{Dr}_{\lambda\in\Delta}A_\lambda = [B, g_0]$ and $Z_1 = \mathrm{Dr}_{\lambda\in\Sigma}A_\lambda = C_B(g_0)$. Clearly K_1 and Z_1 are G–invariant subgroups of B. Let $C_1 = C_G(Z_1)$. If $K_1 = B$, then $Z_1 = \langle 1\rangle$ and hence $C_1 = G$ and the construction of a hierarchy of centralizers of B is finished.

Suppose that $Z_1 \neq \langle 1\rangle$. We claim that $C_1 \neq G$. Otherwise $[G, A_\lambda] = \langle 1\rangle$ for every $\lambda \in \Sigma$, contradicting our conditions. Thus $C_1 \neq G$. Clearly $C_1 \geq C_0$ and, since $g_0 \in C_1$, $C_1 > C_0$. We repeat the above process picking a non-identity element $g_1C_1 \in \zeta(G/C_1)$. As above we construct two G–invariant subgroups of B, namely $K_2 = [Z_1, g_1]$ and $Z_2 = C_{Z_1}(g_1)$ and $Z_1 = K_2 \times Z_2$. Again $Z1 > Z_2$. Otherwise $K_2 = \langle 1\rangle$, and then $g_1 \in C_G(Z_1) = C_1$, contradicting the election of g_1.

Let $C_2 = C_G(Z_2)$. If $K_2 = Z_1$, then $Z_2 = \langle 1\rangle$ and hence $C_2 = G$, and the construction of a hierarchy of centralizers of B is finished. Otherwise we construct a descending series of centralizers

$$Z_1 > Z_2 > \cdots > Z_n > \cdots .$$

Since B satisfies the minimal condition on centralizers, this chain has to break off in finitely many steps and consequently we construct a finite hierarchy of centralizers of B, as required. □

Corollary 1.4. *Let G be a group and let A be an abelian normal subgroup of G such that G/A is hypercentral. Let B a G–invariant subgroup of A such that $B = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If Λ is finite, then the $\mathbb{Z}(G/A)$–module B has a finite hierarchy of centralizers.*

Lemma 1.5. *Let G be a group and let A be an abelian normal subgroup of G such that G/A is nilpotent group and $G/C_G(A)$ is finitely generated. Let B a G–invariant subgroup of A such that $B = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. Then the $\mathbb{Z}(G/A)$–module B has a finite hierarchy of centralizers.*

Proof. Since G/A is finitely generated nilpotent, G/A satisfies the maximal condition on all subgroups. Hence the ascending series

$$C_0 < C_1 < \cdots < C_n < \cdots$$

must breaks off after finitely many steps. □

Proposition 1.6. *Let G be a group and let A be an abelian normal subgroup of G such that G/A is nilpotent. Suppose that $A = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(G/A)$–module A has a finite hierarchy of centralizers, then G conjugately splits over A.*

Proof. Let $\{C_0, \cdots C_n; g_0, \cdots g_n\}$ be a finite hierarchy of centralizers of A. Since we have $C_0 = C_G(A)$, the conditions of this proposition show that $G \neq C_0$. Since $g_0C_0 \in \zeta(G/C_0)$, $[A_\lambda, g_0]$ is a G–invariant subgroup of A_λ, and then either $[A_\lambda, g_0] = A_\lambda$ or $[A_\lambda, g_0] = \langle 1\rangle$, by the minimality of A_λ. Let

$$\Delta = \{\lambda \in \Lambda \mid [A_\lambda, g_0] = A_\lambda\} \text{ and } \Sigma = \Sigma \setminus \Delta.$$

Then $K_1 = \mathrm{Dr}_{\lambda\in\Delta}A_\lambda = [A, g_0]$ and $Z_1 = \mathrm{Dr}_{\lambda\in\Sigma}A_\lambda = C_A(g_0)$. Clearly K_1 and Z_1 are G–invariant subgroups of A. Within the factor-group G/Z_1 we have $[A/Z_1, g_0Z_1] = K_1Z_1/Z_1 = A/Z_1$, and moreover $[A_\lambda Z_1/Z_1, g_0Z_1] = A_\lambda Z_1/Z_1$ for all $\lambda \in \Delta$. It follows that $C_{A/Z_1}(g_0Z_1) = \langle 1\rangle$, so that all conditions of Theorem 1.1 are satisfied. Therefore $G/Z_1 = A/Z_1 \leftthreetimes L/Z_1$, for some subgroup L. Moreover, every complement to A/Z_1 is conjugated with L/Z_1.

Consider now the subgroup L. By construction, we have $g_0 \in L$. Since $G = LA$, every L–invariant subgroup of Z is also G–invariant. Furthermore

$$L/Z_1 = L/(L \cap A) = LA/A = G/A.$$

Therefore we may replace G/A by L/Z_1. In other words, there is no loss if suppose that $C_1 = C_L(Z_1)$. By the election of Z_1, $g_0 \in C_1$. In particular, $C_1 > C_0$. Suppose that $C_1 = L$. Then $[A_\lambda, L] = [A_\lambda, G] = \langle 1\rangle$ for every $\lambda \in \Sigma$, contradicting our conditions. This implies that $L \neq C_1$. As in the proof of the previous results we may express Σ as the disjoint union of two subsets, which define L–invariant subgroups of A (and hence G–invariant), namely $K_2 = [Z_1, g_1]$ and $Z_2 = C_{Z_1}(g_1)$ such that $Z_1 = K_2Z_2$. Once more again $Z_1 > Z_2$.

Within the factor-group L/Z_2 and with respect to the subgroup Z_1/Z_2 and the element g_1Z_2, we see that all conditions of Theorem 1.1 are satisfied, and then $L/Z_2 = Z_1/Z_2 \leftthreetimes L_1/Z_2$, for some subgroup L_1. Moreover, every complement to Z_1/Z_2 in L/Z_2 is conjugated with L_1/Z_2. This and $G = LA$ at once give that $G = LA = (L_1Z_1)A = L_1A$. Moreover, $L_1 \cap A = L_1 \cap L \cap A = L_1 \cap Z_1 = Z_2$, so that $G/Z_2 = A/Z2 \leftthreetimes L_1/Z_2$. Let S be a subgroup of G such that $G/Z_2 = A/Z_2 \leftthreetimes S/Z_2$. Then $G = SA$ and $G/Z_1 = (SZ_1/Z_1)(A/Z_1)$. We have

$$SZ_1 \cap A = SZ_1 \cap (Z_1 \times K_1) = Z_1 \times (SZ_1 \cap K_1) = Z_1.$$

This shows that SZ_1/Z_1 is a complement to A/Z_1 in G/Z_1. Then there exists an element x satisfying $(SZ_1/Z_1)^{xZ_1} = L/Z_1$. It follows that $S^x \leq L$. The equation $G/Z_2 = A/Z_2 \leftthreetimes S/Z_2$ implies $G^x/Z_2 = G/Z_2 = A/Z_2 \leftthreetimes S^x/Z_2$. Therefore $L/Z_2 = (L/Z_2 \cap A/Z_2) \leftthreetimes S^x/Z_2 = Z_1/Z_2 \leftthreetimes Sx/Z2$. In other words, S^x/Z_2 is a complement to Z/Z_2 in L/Z_2. Applying the proved above, there is an element $y \in L$ such that $S^{xy}/Z_2 = L_1/Z_2$, that is $S^{xy} = L_1$. Consequently, every complement to A/Z_2 in G/Z_2 is conjugated with L_1/Z_2.

Proceeding as above and using the fact that A has a finite hierarchy of centralizers, after finitely many steps we prove the required result. □

Corollary 1.7. *Let G be a group and let A be an abelian normal subgroup of G such that G/A is nilpotent. Suppose that $A = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(G/A)$–module A satisfies the minimal condition on G–invariant centralizers, then G conjugately splits over A.*

Proof. Apply Lemma 1.3 and Proposition 1.6 at once. □

If C/B is a normal factor of a group G, we recall that C/B is said to be G*–central* if $C_G(C/B) = G$ and G*–eccentric* otherwise.

Theorem 1.8. *Let G be a group and let A be a periodic abelian normal subgroup of G such that G/A is nilpotent. Suppose that $A[1] = S = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(G/A)$–module S has a finite hierarchy of centralizers, then G conjugately splits over A.*

Proof. Let $\{C_0, \cdots, C_n; g_0, \cdots, g_n\}$ be a finite hierarchy of centralizers of S. We have $C_0 = C_G(S)$ and then the conditions of this proposition show that $G \neq C_0$. Since $g_0C_0 \in \zeta(G/C_0)$, we have that $[A_\lambda, g_0]$ is a

G–invariant subgroup of A_λ for all $\lambda \in \Lambda$, so that either $[A_\lambda, g_0] = A_\lambda$ or $[A_\lambda, g_0] = \langle 1\rangle$, by the minimality of A_λ. Let Let

$$\Delta = \{\lambda \in \Lambda \mid [A_\lambda, g_0] = A_\lambda\} \text{ and } \Sigma = \Sigma \setminus \Delta.$$

Then $K_1 = \mathrm{Dr}_{\lambda\in\Delta} A_\lambda = [S, g_0]$ and $Z_1 = \mathrm{Dr}_{\lambda\in\Sigma} A_\lambda = C_S(g_0)$. As the next step we consider the subgroup $C_1 = C_G(Z_1)$. Since $g_1C_1 \in \zeta(L/C_1)$, $[A_\lambda, g_1]$ is an L–invariant (and hence G–invariant) subgroup of A_λ for all $A_\lambda \in \Sigma$. The minimality de A_λ again ensures us that either $[A_\lambda, g_1] = A_\lambda$ or $[A_\lambda, g_1] = \langle 1\rangle$. Once more again, we decompose $\Sigma = \Xi \cup \Omega$ as the disjoint union of two subsets, which give rise two G–invariant subgroups of A, namely $K_2 = [Z_1, g_1]$ and $Z_2 = C_{Z_1}(g_1)$, which allow us to continue the process. Proceeding in this way, after finitely many steps we has to obtain that $[A_\lambda, \langle g_0, \ \cdots \ , g_n\rangle] = A_\lambda$ for all $\lambda \in \Lambda$. Labeling $D_1 = K_1$, $D_2 = D_1 \times K_2$, ... we construct a finite series of G–invariant subgroups of S,

$$\langle 1\rangle = D_0 \le D_1 \le \cdots \le D_n = S$$

satisfying the following conditions:

- $g_jC_G(D_j/D_{j-1}) \in \zeta(G/C_G(D_j/D_{j-1})$ for every $1 \le j \le n$;
- $D_j/D_{j-1} = [D_j/D_{j-1}, g_jD_j]$ for every $1 \le j \le n$; and
- the centralizer of g_j in the factor D_j/D_{j-1} is identity for every $1 \le j \le n$.

Put $H = \langle g_0, \ \cdots \ , g_n\rangle$ and $L = SH$. We have $D_1 = \mathrm{Dr}_{\lambda\in\Delta} A_\lambda$. Since every subgroup A_λ is minimal G–invariant, $\zeta(G/C_G(A_\lambda))$ is periodic (see [10, Theorem 3.1]). It follows that $g_0^s \in C_G(A_\lambda)$ for some positive integer s. Therefore A_λ includes a finite minimal $\langle g_0\rangle$–invariant subgroup W. Then there exists a subset $X \subseteq G$ such that $A_\lambda = \mathrm{Dr}_{x\in X} W^x$ (see [11, Lemma 5.4]). If we suppose that $[W, g_0] = \langle 1\rangle$, the latter yields $[A_\lambda, g_0] = \langle 1\rangle$, and we get a contradiction. Hence $[W, g_0] = W$. Thus D_1 is a direct product of finite minimal $\langle g_0\rangle$–invariant subgroups, every of which is $\langle g_0\rangle$–eccentric. In the same way we establish similar assertions for the elements g_j and the factors D_{j+1}/D_j.

Since H is finitely generated and $H/(H\cap A)$ is nilpotent, by [6, Theorem 3] $H \cap A$ satisfies the maximal condition on H–invariant subgroups. In particular, there exists a positive integer r such that $(H \cap A)[r] = H \cap A$. Put $T = L \cap A$ so that $T[r] = T$. We have $T = \mathrm{Dr}_{p\in\Pi(T)} T_p$, where T_p is the Sylow p–subgroup of T. Clearly the mapping $\phi_p : a \to a^p,\ a \in \Omega_2(T_p)$, is a $\mathbb{Z}H$–homomorphism of $\Omega_2(T_p)$ in $\Omega_1(T_p)$. Since $\mathrm{Ker}\ \phi_p = \Omega_1(T_p)$, we have that $\Omega_2(T_p)/\Omega_1(T_p)$ is isomorphic to some H–invariant subgroup of $\Omega_1(T_p)$. Let $D_{1,p} = D_1 \cap \Omega_1(T_p)$ and let $D_{2,p}$ be the preimage of $D_{1,p}$ by

ϕ_p. Thus $D_{2,p}/\Omega_1(T_p)$ is isomorphic to some H–invariant subgroup of $D_{1,p}$. Being H–invariant, this subgroup is also $\langle g_0\rangle$–invariant. Since D_1 can be decomposed into a direct product of minimal $\langle g_0\rangle$–invariant subgroups, every $\langle g_0\rangle$–invariant subgroup of D_1 is a direct product of minimal $\langle g_0\rangle$–invariant subgroups. It follows that $D_{2,p}/\Omega_1(T_p)$ is a direct product of minimal $\langle g_0\rangle$–invariant subgroups. Then $[D_{2,p}/\Omega_1(T_p), g_0] = D_{2,p}/\Omega_1(T_p)$ and the centralizer of g_0 in $D_{2,p}/\Omega_1(T_p)$ is identity. Proceeding in this way, we obtain that $T[2]$ has a finite series of H–invariant subgroups

$$\langle 1\rangle = D_0 \leq D_1 \leq \cdots \leq D_n \leq D_{n+1} \leq \cdots \leq D_{2n} = T[2]$$

satisfying the following conditions:

- $g_jC_G(D_j/D_{j-1}) \in \zeta(G/C_G(D_j/D_{j-1})$ for every $1 \leq j \leq 2n$;
- $D_j/D_{j-1} = [D_j/D_{j-1}, g_jD_j]$ for every $1 \leq j \leq 2n$; and
- the centralizer of g_j in the factor D_j/D_{j-1} is identity for every $1 \leq j \leq 2n$.

Proceeding in this way we construct a series of H–invariant subgroups of T satisfying the conditions of Corollary 1.2. Therefore L conjugately splits over T, that is $L = T \leftthreetimes K$ for some subgroup K and every complement to T in L is conjugated to K. Since $K \cap A = \langle 1\rangle$, $HA = A \leftthreetimes K$.

Put $S_1 = A[2]$ and $L_1 = S_1H$. We have

$$A = \mathrm{Dr}_{p\in\Pi(A)}A_p,$$

where A_p is the Sylow p–subgroup of A. Clearly the mapping $\psi_p : a \mapsto a^p,\ a \in \Omega_2(A_p)$ is a $\mathbb{Z}G$–homomorphism of $\Omega_2(A_p)$ in $\Omega_1(A_p)$. Since Ker $\psi_p = \Omega_1(A_p)$, it follows that $\Omega_2(A_p)/\Omega_1(A_p)$ can be decomposed as a direct product of minimal G–invariant subgroups, and every direct factor is H–eccentric. Then the same holds for S_1/S. As we showed above, every complement to S_1/S in L_1/S is conjugated with KS/S.

Let R be another complement to S_1 in L_1. Then RS/S is a complement to S_1/S in L_1/S and hence conjugated to KS/S in L_1/S. Thus, there is some $x \in L_1$ such that $R^x \leq KS = L$. We have $L_1 = L_1^x = S_1 \leftthreetimes R^x$, and so

$$L = L \cap L_1 = (S_1 \cap L) \leftthreetimes R^x = S \leftthreetimes R^x.$$

We have already proved that R^x and K are conjugated in L, that is $(R^x)^y = K$ for some element $y \in L$. This shows that K is a complement to S_1 in L_1 and therefore conjugated to K.

Proceeding in this way and applying induction, we see that K is a complement to $A[n]$ in $A[n]H$ and moreover every complement to $A[n]$

in $A[n]H$ is conjugated with K. Since $A = \bigcup_{n\in\mathbb{N}} A[n]$, this implies that $AH = A \leftthreetimes K$. Let Y be another complement to A in AH. Then

$$Y \cong Y/(Y \cap A) \cong YA/A = HA/A \cong H/(H \cap A),$$

which gives that Y is finitely generated. Let $Y = \langle y_1, \cdots, y_m\rangle$. Since $AH = \bigcup_{n\in\mathbb{N}} A[n]H$, there exist a positive integer k such that y_1, ..., $y_m \in A[k]H$. In other words, $Y \leq A[k]H$. Furthermore H is a finitely generated abelian–by–nilpotent group, and so we may apply a result due to P. Hall [6, Theorem 3] to obtain that H satisfies the maximal condition on normal subgroups. In particular, $H \cap A$ satisfies the maximal condition for H–invariant subgroups. It follows that there is a number r such that $H \cap A \leq A[r]$. There is no loss if we assume that $r = k$. Then

$$A[k]H = A[k]H \cap AH = (A[k]H \cap (A \leftthreetimes Y) =$$

$$= (A[k]H \cap A) \leftthreetimes Y = (A[k](H \cap A)) \leftthreetimes Y = A[k] \leftthreetimes Y.$$

But in this case there is an element $z \in A[k]H$ such that $Y^z = K$. Hence every complement to A in AH is conjugated with K. Since G/A is hypercentral, HA/A is ascendant in G. By applying [17, Lemma 3], we deduce that G conjugately splits over A. □

Corollary 1.9. *Let G be a group and let A be a periodic abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is nilpotent. Suppose also that $A[1] = S = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal H–invariant subgroup and $[H, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(H/A)$–module S has a finite hierarchy of centralizers, then G conjugately splits over A.*

Proof. It suffices to apply at once Theorem 1.8 and [17, Lemma 3]. □

Corollary 1.10. *Let G be a group and let A be a periodic abelian normal subgroup of G such that G/A is nilpotent. Suppose that $A[1] = S = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal G–invariant subgroup and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(G/A)$–module S satisfies the minimal condition on H–invariant centralizers, then G conjugately splits over A.*

Proof. G has a finite hierarchy of centralizers by Lemma 1.3, so it suffices to apply Theorem 1.8. □

Corollary 1.11. *Let G be a group and let A be a periodic abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is nilpotent. Suppose also that $A[1] = S = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal H–invariant subgroup and $[H, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. If the $\mathbb{Z}(H/A)$–module S satisfies the minimal condition on H–invariant centralizers, then G conjugately splits over A.*

Proof. It suffices to apply at once Corollary 1.10 and [17, Lemma 3]. □

A group G is said to have *finite Hirsch-Zaitsev rank* $r_{hz}(G) = r$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r. Otherwise it is said that *G has infinite Hirsch-Zaitsev rank* ([2]). It is not hard to show that $r_{hz}(G)$ is an invariant of the group G. We mention that this definition is a slightly generalization of a former definition by D. I. Zaitsev, who considered groups with a finite subnormal series whose factors are either infinite cyclic or periodic. The latter is known as *the torsion-free rank* or *0–rank* $r_0(G)$ of G. This numerical invariant was an important tool in the study of polycyclic–by–finite groups started in the celebrated paper of K. A. Hirsch [9], and later on was called *the Hirsch number*. In the study of polyrational groups this numerical invariant appeared as *the rational rank* of the group, a definition also due to D. I. Zaitsev [24]. In another paper by D. I. Zaitsev [26] this concept was applied to locally polycyclic–b! y–finite groups, and in more general form this concept was extended to arbitrary groups by D. I. Zaitsev in the paper [27].

Proposition 1.12. *Let G be a group and let A be a periodic abelian normal subgroup of G such that G/A is nilpotent. Suppose that $G/C_G(A[1])$ is minimax and $A[1] = S = Dr_{\lambda\in\Lambda} A_\lambda$, where A_λ is a minimal G–invariant subgroups and $[G, A_\lambda] \neq \langle 1\rangle$, for every $\lambda \in \Lambda$. Then S has a finite hierarchy of centralizers.*

Proof. Let $C_0 = C_G(S)$. Since a periodic soluble minimax group is Chernikov, the torsion subgroup $Tor(G/C_0) = P/C_0$ is Chernikov. Let D/C_0 be the divisible part of P/C_0. Since A has finite section rank, every A_λ is finite and so the index $|G : C_G(A_\lambda)|$ is finite too. Since $C_0 \leq C_G(A_\lambda)$ and D/C_0 has no proper subgroups of finite index, $D \leq C_G(A_\lambda)$. This holds for all $\lambda \in \Lambda$, therefore

$$D \leq \bigcap_{\lambda\in\Lambda} C_G(A_\lambda) = C.$$

In other words, the periodic part of G/C_0 has to be finite. In particular, G/C_0 is periodic provided it is finite. The finiteness of $Tor(G/C_0)$ implies that $\zeta(G/C_0)$ cannot be periodic. Therefore we may pick $g_0C_0 \in \zeta(G/C_0)$ of infinite order. Let $K_1 = [A, g_0]$ and $Z_1 = C_A(g_0)$. Clearly K_1 and Z_1 are G–invariant subgroups of A. Put $C_1 = C_G(Z_1)$. If $K_1 = A$, then $Z_1 = \langle 1\rangle$ and hence $C_1 = G$. Therefore we have just constructed a finite hierarchy of centralizers.

Suppose that $Z_1 \neq \langle 1\rangle$. We may assume that $C_1 \neq G$. Otherwise, $[G, A_\lambda] = \langle 1\rangle$ for some $\lambda \in \Lambda$, contradicting our hypothesis. Clearly

$C_1 \geq C_0$ and, since $g_0 \in C_1$, we have that $C_1 > C_0$. It follows that $r_{hz}(G/C_1) < r_{hz}(G/C_0)$. As above, we have that $Tor(G/C_1)$ is finite, which implies that $\zeta(G/C_1)$ cannot be periodic. Therefore we may pick $g_1C_1 \in \zeta(G/C_1)$ of infinite order and construct a second step: $K_2 = [Z_1, g_1]$, $Z_2 = C_{Z_1}(g_1)$ and $C_2 = C_G(Z_2)$. Clearly $Z_1 > Z_2$ because $g_1 \notin C_1$. Again, we construct a finite hierarchy of centralizers if $Z_2 = \langle 1 \rangle$.

If $Z_2 \neq \langle 1 \rangle$, and we proceed in this way we eventually construct a strict descending series

$$Z_1 > Z_2 > \cdots > Z_n > \cdots,$$

such that if $C_\alpha = C_G(Z_\alpha)$, then $r_{hz}(G/C_{\alpha+1}) > r_{hz}(G/C_\alpha)$. Thus the above chain gives rise to the following sequence of nonnegative numbers

$$r_{hz}(G/C_0) > r_{hz}(G/C_1) > \cdots > r_{hz}(G/C_n) > ... > rhz(G/Cn) > \cdots,$$

and thus there is some number t such that G/C_t is periodic. In this case G/C_t is necessarily finite, which shows that we can construct a finite hierarchy of centralizers. □

Theorem 1.13. *Let G be a group and let A be a periodic abelian normal subgroup of G. Suppose that G has an ascendant subgroup $H \geq A$ such that H/A is nilpotent. Suppose also that $A[1] = Dr_{\lambda\in\Lambda}A_\lambda$, where A_λ is a minimal H–invariant subgroup and $[H, A_\lambda] \neq \langle 1 \rangle$, for every $\lambda \in \Lambda$. If H/A is minimax, then G conjugately splits over A.*

Proof. It suffices to apply at once Proposition 1.11, Theorem 1.8 and [17, Lemma 3]. □

Corollary 1.14. *Let G be a group and let A be an abelian normal subgroup of G. Suppose that the following conditions holds:*

(i) G has a normal subgroup $L \geq A$ such that L/A is torsion-free nilpotent minimax; and

(ii) $A = Dr_{\lambda\in\Lambda}A_\lambda$ and the A_λ are finite minimal G–invariant subgroups such that $[G, A_\lambda] \neq \langle 1 \rangle$.

If $Z = C_A(L)$, then G/Z conjugately splits over A/Z.

Proof. Since L is a normal subgroup of G, $[L, A_\lambda]$ is G–invariant, and then either $[L, A_\lambda] = A_\lambda$ or $[L, A_\lambda] = \langle 1 \rangle$. Let

$$\Delta = \{\lambda \in \Lambda \mid [L, A_\lambda] = A_\lambda\},$$

so that

$$D = \mathrm{Dr}_{\lambda\in\Delta}A_\lambda = [L, A] \text{ and } Z = \mathrm{Dr}_{\lambda\in\Lambda\setminus\Delta}A_\lambda = C_A(L).$$

We remark that the subgroups D and Z are G–invariant. Within the factor-group G/Z we have that $A/Z = DZ/Z$ and $[L/Z, A_\lambda Z/Z] = A_\lambda Z/Z$ for each $\lambda \in \Delta$. By Theorem 1.13, G/Z conjugately splits over A/Z, that is $G/Z = A/Z \leftthreetimes K/Z$ for some subgroup K/Z. □

2. Some locally nilpotent groups

This Section is entirely auxiliary. We develop here some results that are consequence of the splitting criteria obtained in the above Section in order to obtain some applications in the next Section.

Let G be an abelian group of finite 0–rank. Let M be a maximal $\mathbb{Z}$–independent subset of G and put $A = \langle M \rangle$ so that the factor-group G/A is periodic. Let

$$\mathrm{Sp}(G) = \{p \text{ prime} \mid \text{the Sylow } p\text{–subgroup of } G/A \text{ is infinite}\}.$$

If $B \le G$ is free (abelian) and G/B is periodic, then the factors $A/(A \cap B)$ and $B/(A \cap B)$ are finite, which shows that the set $\mathrm{Sp}(G)$ is independent of A, i.e. $\mathrm{Sp}(G)$ is an invariant of G. This set $\mathrm{Sp}(G)$ is called *the spectrum of* G. If $H \trianglelefteq G$, it is not hard to see that

$$\mathrm{Sp}(G) = \mathrm{Sp}(H) \cup \mathrm{Sp}(G/H).$$

Let G be a soluble group of finite Hirsch-Zaitsev rank. We define $\mathrm{Sp}(G)$ to be the union of the spectrums of the factors of the derived series of G. It is rather easy to see that $\mathrm{Sp}(G)$ can be defined as the union of the spectrums of the factors of an arbitrary normal series of G with abelian factors. Clearly, a soluble–by–finite minimax group has a finite spectrum.

For our purposes we need the following auxiliary result. Even though the result is known, a reference is difficult to find in the literature. For this reason and for the reader's convenience we give a proof of the result. We recall that a group G is said to have *finite special rank* $r(G) = r$ if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

Lemma 2.1. *If A and B are abelian minimax groups then so is their tensor product $A \otimes B$.*

Proof. Let $r = \max\{r_0(A), r_0(B)\}$ and $s = \max\{r(A), r(B)\}$. We recall that $\mathrm{Sp}(A)$ and $\mathrm{Sp}(B)$ are finite sets. Being minimax, there exist finitely generated subgroups $H \le A$ and $K \le B$ such that

$$A/H = \mathrm{Dr}_{p \in \mathrm{Sp}(A)} A_p \text{ and } B/K = \mathrm{Dr}_{p \in \mathrm{Sp}(A)} B_p,$$

where A_p and B_p are direct products of finitely many Prüfer p–groups. Suppose that we have

$$H = (Z_1 \times \cdots \times Z_k) \times (\mathrm{Dr}_{p \in \mathrm{Sp}(A)} C_p),$$

where $k \leq r$, each Z_j is an infinite cyclic group, and each C_p is a direct product of at most s cyclic p–groups. Now the sequence

$$\langle 1 \rangle \longrightarrow H \longrightarrow A \longrightarrow A/H \longrightarrow \langle 1 \rangle$$

is exact. Applying a theorem due to Dieudonné (see, for example, [3, Theorem 60.6]), we have that the sequence

$$\langle 1 \rangle \to H \otimes (B/Tor(B)) \to A \otimes (B/Tor(B)) \to A/H \otimes (B/Tor(B)) \to \langle 1 \rangle$$

is also exact. Standard properties of the tensor product of abelian groups (see [3, p. 255]) give that

$$H \otimes (B/Tor(B)) \cong (Y_1 \times \cdots \times Y_k) \times (\mathrm{Dr}_{p \in \mathrm{Sp}(A)} D_p),$$

where each $Y_j = B/Tor(B)$, $D_p = B/B^{t_p}Tor(B)$, where $t_p = p^{m_p}$ is a power of p and $p \in \mathrm{Sp}(A)$. From this, we conclude that $H \otimes (B/Tor(B))$ is also minimax. Furthermore,

$$A/H \otimes (B/Tor(B)) \cong (\mathrm{Dr}_{p \in \mathrm{Sp}(A)} A_p) \otimes (B/Tor(B)),$$

where each A_p is a direct product of finitely many Prüfer p–groups. Let $E = X_1 \times \cdots \times X_u$ be the p–basic subgroup of $B/Tor(B)$, where each X_j is an infinite cyclic subgroup (see [3]). By [3, Theorem 61.1],

$$C_{p^\infty} \otimes (B/Tor(B)) \cong C_{p^\infty} \otimes E \cong W_1 \times \cdots \times W_u,$$

where each $W_j \cong C_{p^\infty}$. From this it follows that $A/H \otimes (B/Tor(B))$ is also minimax and has spectrum $\mathrm{Sp}(A)$. Consequently $A \otimes (B/Tor(B))$ is minimax. We have that

$$\langle 1 \rangle \to Tor(A) \to A \to A/Tor(A) \to \langle 1 \rangle$$

is also exact. By [3, Theorem 60.6], so is the sequence

$$\langle 1 \rangle \to Tor(A) \otimes (B/Tor(B)) \to \cdots$$

$$\cdots \to A \otimes (B/Tor(B)) \to (A/Tor(A)) \otimes (B/Tor(B)) \to \langle 1 \rangle.$$

Since a homomorphic image of an abelian minimax group with spectrum π is minimax and has spectrum $\rho \subseteq \pi$, $(A/Tor(A)) \otimes (B/Tor(B))$ is minimax with spectrum contained in Sp(A). Applying [3, Theorem 61.5], we deduce that $Tor(A \otimes B)$ is isomorphic to

$$(Tor(A) \otimes Tor(B)) \times (Tor(A) \otimes B/Tor(B)) \times ((A/Tor(A)) \otimes Tor(B))$$

and

$$(A \otimes B)/(Tor(A \otimes B)) \cong (A/Tor(A)) \otimes (B/Tor(B)).$$

We have already seen that $Tor(A) \otimes B/Tor(B)$ is minimax since it is a subgroup of $A \otimes B/Tor(B)$. Similarly $A/Tor(A) \otimes Tor(B)$ is minimax. From the above paragraph, we deduce that $(A/Tor(A)) \otimes (B/Tor(B))$ is also minimax. Thus it remains to show that $Tor(A) \otimes Tor(B)$ is minimax. We note that $Tor(A)$ and $Tor(B)$ satisfy the minimal condition on subgroups, and so they are direct products of finitely many Prüfer p–groups and finitely many finite cyclic groups. Therefore to compute its tensor product, it will suffice to compute the tensor product of two arbitrary factors of these mentioned above. Standard properties of tensor products (see, for example, [3, p. 255]), give that

$$C_{p^\infty} \otimes C_{p^\infty} = \langle 1 \rangle, \; C_{p^\infty} \otimes C_{q^\infty} = C_{p^\infty} \otimes C_{q^k} = C_{p^m} \otimes C_{q^k} = \langle 1 \rangle \; (p \neq q)$$

and

$$C_{p^m} \otimes C_{p^k} = C_{p^t} \; (t = \min\{k, m\}).$$

Thus $Tor(A) \otimes Tor(B)$ has to be finite, which finishes the proof. □

Let G be a group. It is well-known that the universal property of the tensor product of two abelian groups allows to define an epimorphism

$$\vartheta : G/[G,G] \bigotimes \gamma_j(G)/\gamma_{j+1}(G) \to \gamma_{j+1}(G)/\gamma_{j+2}(G)$$

given by $\vartheta(a[G,G] \bigotimes b\gamma_{j+1}(G)) = [a,b]\gamma_{j+2}(G), \; a \in G, b \in \gamma_j(G)$, which is extremely useful for studying nilpotent groups.

Proposition 2.2. *Let G be a nilpotent group. If $G/[G,G]$ is minimax, then G is minimax.*

Proof. By Lemma 2.1, the tensor product of two minimax abelian groups is minimax. It suffices to apply this to all lower central factors of G. □

Lemma 2.3. *Let G be a finitely generated group that has a bounded abelian normal subgroup A such that G/A is polycyclic–by–finite. If A is G–hyperfinite, then A is finite.*

Proof. Since G is finitely generated, the normal subgroup A satisfies the maximal condition on G–invariant subgroups ([6, Theorem 3]). Since A is G–hyperfinite, A is finite. □

Corollary 2.4. *Let G be a finitely generated group that has a bounded nilpotent normal subgroup H such that G/H is polycyclic–by–finite. If H is G–hyperfinite, then H is finite.*

Proof. Put $D = [H, H]$. If H is infinite, since H is nilpotent, H/D has to be also infinite. But H/D is clearly G/D–hyperfinite so that H/D is finite by Lemma 2.3. This contradiction shows that H is finite. □

Corollary 2.5. *Let G be a finitely generated group. Suppose that G has a nilpotent normal subgroup H satisfying the following conditions:*

(i) $H/Tor(H)$ is minimax;

(ii) $Tor(H)$ is bounded G–hyperfinite; and

(iii) G/H is polycyclic–by–finite.

Then $Tor(H)$ is finite and hence H is minimax.

Proof. Since $T = Tor(H)$ is bounded, by [8, Proposition 2], H contains a torsion-free normal subgroup B such that H/B is bounded. Let k be a positive integer such that $(H/B)^k = \langle 1 \rangle$. We put $C = H^k$ so that $C \leq B$. In particular, C is a torsion-free minimax Ginvariant subgroup. Moreover the subgroup TC/C of the factor-group G/C is clarly G/C–hyperfinite and the factor H/TC is a bounded nilpotent minimax group, therefore it is finite. It follows that H/C is bounded G/C–hyperfinite. By Corollary 2.4, H/C is finite. Since C is torsion-free, $T \cap C = \langle 1 \rangle$ and then T is finite, as required. □

Corollary 2.6. *Let G be a finitely generated group. Suppose that G has a nilpotent normal subgroup H satisfying the following conditions:*

(i) $H/Tor(H)$ is minimax;

(ii) $T = Tor(H)$ is G–hyperfinite;

(iii) for every $p \in \Pi(T)$, the Sylow p–subgroup T_p of T is bounded; and

(iv) G/H is polycyclic–by–finite.

Then T_p is finite for every $p \in \Pi(T)$.

Proof. We have $T = \mathrm{Dr}_{p \in \Pi(T)} T_p$. Given $p \in \Pi(T)$, we put $Q_p = \mathrm{Dr}_{q \neq p} T_q$ so that Q_p is G–invariant and $T/Q_p \cong T_p$. By Corollary 2.5, T/Q_p is finite and hence T_p is finite, as required. □

Lemma 2.7. *Let G be a finitely generated group. Suppose that G has an abelian normal subgroup A satisfying the following conditions:*

(i) A includes a periodic subgroup T such that A/T is minimax;

(ii) for every $p \in \Pi(T)$, the Sylow p–subgroup T_p of T is finite; and

(iii) G/A is polycyclic–by–finite.

Then T is finite and A is minimax.

Proof. Since G is finitely generated and G/A is polycyclic–by–finite, A has a free abelian subgroup C such that A/C is periodic and $\Pi(A/C)$ is finite (see [10, Corollary 1.8]). The factor-group A/C is an extension of $TC/C \cong T$ by the minimax group A/TC. By condition (ii), the Sylow p–subgroup of A/C is Chernikov for each prime p. Then, the finiteness of $\Pi(A/C)$ implies that A/C is Chernikov group. It follows that A is a minimax group. In particular, $\Pi(T)$ is finite, and then T is finite. □

Corollary 2.8. *Let G be a finitely generated group. Suppose that G has a nilpotent normal subgroup H satisfying the following conditions:*

(i) If $T = Tor(H)$, H/T is minimax;

(ii) for every $p \in \Pi(T)$, the Sylow p–subgroup T_p of T is finite; and

(iii) G/H is polycyclic–by–finite.

Then T is finite and hence H is minimax.

Proof. Put $D = [G, G]$ so that the factor-group H/D has a subgroup TD/D, whose Sylow p–subgroups are all finite and H/TD is minimax. By Lemma 2.7, H/D is minimax. Proposition 2.2 implies that H is minimax too. Therefore $\Pi(T)$ is finite and then T is finite. □

Corollary 2.9. *Let G be a finitely generated group. Suppose that G has a nilpotent normal subgroup H satisfying the following conditions:*

(i) If $T = Tor(H)$, H/T is minimax;

(ii) T is G–hyperfinite;

(iii) for every $p \in \Pi(T)$, the Sylow p–subgroup T_p of T is bounded; and

(iv) G/H is polycyclic–by–finite.

Then T is finite and hence H is minimax.

Proof. By Corollary 2.6, the Sylow p–subgroup T_p of T is finite for each prime $p \in \Pi(T)$ and it suffices to apply Corollary 2.8. □

Proposition 2.10. *Let G be a finitely generated group. If G contains a torsion-free nilpotent normal subgroup H of finite Hirsch-Zaitsev rank such that G/H is polycyclic–by–finite, then H is minimax and hence G is minimax.*

Proof. We put $D = [H, H]$ so that H/D has a free abelian subgroup C/D such that H/C is periodic and $\Pi(H/C)$ is finite (see, for example, [10, Corollary 1.8]). Since $r_{hz}(H)$ is finite, C/D is finitely generated. Similar reasons ensure us that the Sylow p–subgroups of H/C are Chernikov for every prime p. Since $\Pi(H/C)$ is finite, H/C is a Chernikov group and it follows that H/D is minimax. By Proposition 2.2, H is also minimax. □

3. Some applications

Before to show the main result of this Section, we need some auxiliary results in which we will need to apply results of the above Sections.

Lemma 3.1. *Let G be a finitely generated group and let A be an abelian normal subgroup of G. Suppose that the following conditions holds:*

(i) G has a normal subgroup $L \geq A$ such that L/A is a torsion-free nilpotent minimax group and G/L is abelian–by–finite; and

(ii) $A = Dr_{p\in\Pi(A)}A_p$, where A_p is the Sylow p–subgroup of A, and $A_p = Dr_{\lambda\in\Lambda(p)}A_\lambda$, where A_λ is a finite minimal G–invariant subgroup for all $\lambda \in \Lambda(p)$ and $p \in \Pi(A)$.

Then A is finite.

Proof. We note that A is G–hyperfinite. Since L is a normal subgroup of G, as other times we have that either $[L, A_\lambda] = A_\lambda$ or $[L, A_\lambda] = \langle 1\rangle$ for every $\lambda \in \Lambda(p)$ and $p \in \Pi(A)$. Let

$$\Delta = \{\lambda \in \cup_{p\in\Pi(A)}\Lambda(p) \mid [L, A_\lambda] = A_\lambda\} \text{ and } \Sigma = (\cup_{p\in\Pi(A)}\Lambda(p)) \setminus \Delta,$$

so that

$$D = \mathrm{Dr}_{\lambda\in\Delta}A_\lambda = [L, A] \text{ and } Z = \mathrm{Dr}_{\lambda\in\Sigma}A_\lambda = C_A(L)$$

are G–invariant. Since $A/D \leq \zeta(L/D)$, L/D is nilpotent. By Corollary 2.9, $Z \cong A/D$ is finite. By Corollary 1.13, G/Z conjugately splits over A/Z, that is $G/Z = A/Z \leftthreetimes K/Z$, for some subgroup K/Z. Thus $G = AK$ and $A \cap K = Z$ is finite. In particular, K is minimax. If $G = \langle g_1, \cdots, g_m\rangle$, for each j, we express $g_j = x_j a_j$ where $x_j \in K$ and $a_j \in A$. Then $G = K\langle a_1, \cdots, a_m\rangle^K$. Since A is periodic abelian, $\langle a_1, \cdots, a_m\rangle^K$ is finite, so that G is minimax. In particular, A is finite. □

Proposition 3.2. *Let G be a finitely generated group and A be an abelian normal subgroup of G. Suppose that the following conditions holds:*

(i) G has a normal subgroup $L \geq A$ such that L/A is a torsion-free nilpotent minimax group and G/L is abelian–by–finite; and

(ii) A is G–hyperfinite; and

(iii) every p–factor of A is G–socular-finite for each $p \in \Pi(A)$.

Then A is finite.

Proof. We have $A = \mathrm{Dr}_{p\in\Pi(A)} A_p$, where A_p is the Sylow p–subgroup of A. Since A_p is G–invariant, condition (ii) shows that A_p has a finite G–socular series

$$\langle 1\rangle = A_{p,0} \leq A_{p,1} \leq \cdots \leq A_{p,k(p)} = Ap.$$

For every $p \in \Pi(A)$, we put $B_p = A_{p,k(p)}$ and $B = \mathrm{Dr}_{p\in\Pi(A)} B_p$. By the election of B_p we have $A/B = Soc_G(A/B)$. By Lemma 3.1, A/B is finite. Since $\Pi(A/B) = \Pi(A)$, $\Pi(A)$ is finite.

Suppose that there exists some prime p such that A_p is infinite. Then there exists a non-negative integer d such that $A_p/A_{p,d}$ is finite but $A_{p,d}/A_{p,(d+1)}$ is infinite. Put $Q = \mathrm{Dr}_{q\neq p} A_q$, $D = Q \times A_{p,d}$, $T = Q \times A_{p,(d+1)}$, and consider the factor-group G/T. We have A/D finite, D/T infinite and $D/T = Soc_G(D/T)$. There is a normal subgroup H/D such that G/H is finite and $H \cap A = D$ ([24, Lemma 3]). Since G/H is finite, H is finitely generated too. Let U/V be a G–chief factor of D/T, so that U/V is finite by condition (i). It follows that U/V includes a minimal H–invariant subgroup, say W/V. We put $Y/V = \langle (W/V)^{gV} \mid g \in G\rangle$ so that $Y/V = Soc_{L/V}(Y/V)$. Since Y/V is G–invariant, $Y/V = U/V$ and then $U/V = Soc_{L/V}(U/V)$. Applying Lemma 3.1 to L/T, we deduce that D/T has to be finite, a contradiction. Therefore A_p is finite for each prime p and hence A is finite too. □

Theorem 3.3. *Let G be a finitely generated group and let S a soluble normal subgroup of G. Suppose that the following conditions holds:*

(i) G contains a normal subgroup $L \geq S$ such that L/S is a torsion-free nilpotent minimax group and G/L is abelian - by finite;

(ii) S is G–hyperfinite; and

(iii) every p–factor of S is G–socular-finite for each $p \in \Pi(G)$.

Then the subgroup S is finite.

Proof. Let
$$S \geq S_1 \geq \cdots \geq S_d = \langle 1 \rangle$$
be the derived series of S. Suppose that the result is false, that is S is infinite. Then there is a non-negative integer k such that S/S_k is finite, but S_k/S_{k+1} is infinite. The factor-group G/S_k has a normal subgroup L/S_k such that G/L is finite and $L \cap S = S_k$ ([24, Lemma 3]). Since G/L is finite, L is finitely generated. The abelian factor-group $(L \cap S)/S_{k+1} = S_k/S_{k+1}$ is L–hyperfinite. Proceeding as in the proof of Proposition 3.2, we see that every p–factor of S_k/S_{k+1} is L–socular-finite for each prime $p \in \Pi(G)$. Application of Proposition 3.2 to L/S_{k+1} gives rise to a contradiction, which shows the required result. □

As a consequence, we obtain the first part of Robinson's result [18].

Corollary 3.4. *Let G be a finitely generated soluble group of finite section rank. If the Sylow p–subgroups of G are finite, then G is residually finite minimax.*

Proof. Let U/V be an abelian section of G. Clearly $r_0(U/V)$ is finite. Applying [16, Theorems 3 and 4], we see that G has a finite series of normal subgroups
$$T \leq L \leq G,$$
where T is periodic, L/T is a nilpotent group of finite Hirsch-Zaitsev rank and G/L is abelian–by–finite. By Proposition 2.10, L/T is minimax and hence G/T is minimax too. Since the Sylow p–subgroups of T are finite, G satisfies conditions (ii) and (iii) of Theorem 3.3. Then T is finite and therefore G is minimax. □

With some extra work we can widely generalize the last result.

Theorem 3.5. *Let G be a finitely generated radical group of finite section rank. If the Sylow p–subgroups of G are finite, then G is residually finite minimax.*

Proof. Put $T = Tor(G)$. Then G/T contains a torsion-free nilpotent normal subgroup L/T such that G/L is a finitely generated abelian–by–finite group (see, for example, [2, Theorem A]). Since the Sylow p–subgroups of T are finite, T is hyperabelian. Suppose that T is infinite. By [2, Theorem A], T has a G–invariant subgroup R such that T/R is infinite soluble. Since G/T is soluble, G/R is soluble too. Being finitely generated, G/R is minimax by Corollary 3.4 and thus T/R is finite, a contradiction. Hence T is finite, G is soluble and it suffices to apply Corollary 3.4 again. □

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Contact information

M. R. Dixon
Department of Mathematics, University of Alabama at Tuscaloosa, AL 35487-0350, U.S.A.
E-Mail: `mdixon@gp.as.ua.edu`

L. A. Kurdachenko
Department of Algebra, National University of Dnepropetrovsk, Dnepropetrovsk 10, 49010, Ukraine
E-Mail: `lkurdachenko@i.ua`

J. Otal
Departamento de Matemáticas – IUMA, Universidad de Zaragoza, 50009 Zaragoza, SPAIN
E-Mail: `otal@unizar.es`

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