Algebra and Discrete Mathematics
Volume 10 **(2010)**. Number 1. pp. 8 – 17

RESEARCH ARTICLE

Associated prime ideals of weak σ-rigid rings and their extensions

V. K. Bhat

Communicated by M. Ya. Komarnytskyj

Abstract. Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ the field of rational numbers). Let σ be an automorphism of R and δ a σ-derivation of R. Let further σ be such that $a\sigma(a) \in N(R)$ implies that $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R. In this paper we study the associated prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a semiprime right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ and δ be as above. Then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R with $\sigma(U) = U$ and $\delta(U) \subseteq U$ and $P = U[x; \sigma, \delta]$.

We also prove that if R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$, σ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$ and $\sigma(U) = U$ for all associated prime ideals U of R (viewed as a right module over itself), then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R such that $(P \cap R)[x; \sigma, \delta] = P$ and $P \cap R = U$.

1. Introduction and preliminaries

Notation: All rings are associative with identity. Throughout this paper R denotes a ring with identity $1 \neq 0$. The prime radical of R is denoted by $P(R)$. The set of nilpotent elements of R is denoted by $N(R)$. The fields

The author would like to express his sincere thanks to the referee for comments and suggestions (regarding inclusion of more examples) to give the paper the present shape.

2000 Mathematics Subject Classification: *16-XX; 16N40, 16P40, 16S36.*

Key words and phrases: *Ore extension, automorphism, derivation, associated prime.*

of rational numbers, real numbers and complex numbers are denoted by $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively. For any subset J of a right R-module M, annihilator of J is denoted by $Ann(J)$. The set of prime ideals of R is denoted by $Spec(R)$, the set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$, and the set of minimal prime ideals of R is denoted by $Min.Spec(R)$. Let R be a right Noetherian ring. For any uniform right R-module J, the assassinator of J is denoted by $Assas(J)$. Let M be a right R-module. Consider the set

$$\{Assas(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}.$$

We denote this set by $\mathbb{A}(M_R)$.

Remark 1.1. If R is viewed as a right module over itself, we note that $Ass(R_R) = \mathbb{A}(R_R)$ (5Y of Goodearl and Warfield [8]).

For any two ideals I, J of R; $I \subset J$ means that I is strictly contained in J.

Let K be an ideal of a ring R such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(K)$ by K^0.

Ore extensions: Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R ($\delta : R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$).

For example let σ be an endomorphism of a ring R and $\delta : R \to R$ any map.
Let $\phi : R \to M_2(R)$ defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R \text{ be a ring homomorphism.}$$

Then δ is a σ-derivation of R.

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable; i.e. $\sigma(I) = I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i, a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients of the polynomials on the right as followed in McConnell and Robson [13].

In case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x; \sigma]$ by $S(I)$.
In case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x; \delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [1, 4, 5, 7, 8, 10, 12, 13].

Prime ideals: This article concerns the study of prime ideals of Ore extensions (skew polynomial rings). Regarding associated prime ideals of Ore extension $R[x;\sigma,\delta]$, we have the following from S. Annin [1]:

Definition (2.1) of Annin [1]: Let R be a ring and M_R be a right R-module. Let σ be an endomorphism of R and δ be a σ-derivation of R. M_R is said to be σ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. Moreover M_R is said to be δ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0$ implies $m\delta(r) = 0$. If M_R is both σ-compatible and δ-compatible, M_R is said to be $(\sigma - \delta)$-compatible.

Theorem (2.3) of Annin [1]: Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R and M_R be a right R-module. If M_R is $(\sigma - \delta)$-compatible, then $Ass(M[x]_S) = \{P[x] \mid P \in Ass(M_R)\}$.

In [12], Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced S-module M_S, where $S = R[x;\sigma,\delta]$ (σ is an automorphism and δ is a σ-derivation of a ring R). They have proved the following:

Theorem (5.7) of [12]: Suppose M_R contains enough prime submodules and let for $Q \in Ass(M_S)$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in Ass(M_R)$.

Motivated by these developments, I investigated the nature of associated prime ideals of $R[x;\sigma,\delta]$ over a right Noetherian ring R and their relation with those of the coefficient ring R. In this way I generalized Theorem (2.4) and Theorem (3.7) of Bhat [4] for associated prime ideals case. The minimal prime ideal case has been generalized in Lemma (2.2) of Bhat [5].

Before we state these known results we require the following notation:

Let R be a right Noetherian ring. We know that $Ass(R_R)$ is finite and $\sigma^j(U) \in Ass(R_R)$ for any $U \in Ass(R_R)$, and for all integers $j \geq 1$, therefore, there exists an integer $m \geq 1$ such that $\sigma^m(U) = U$ for all $U \in Ass(R_R)$. We denote $\cap_{i=1}^{m}\sigma^i(U)$ by U^0 as mentioned in the introduction. Since $Min.Spec(R)$ is also finite, same notation for $Min.Spec(R)$ also.

Theorem (2.4) of [4]: Let R be a right Noetherian ring and σ be an automorphism of R. Then:

1. $P \in Ass(S(R)_{S(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $S(P \cap R) = P$ and $(P \cap R) = U^0$.

2. $P \in Min.Spec(S(R))$ if and only if there exists $U \in Min.Spec(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.

Theorem (3.7) of [4]: Let R be a right Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then:

1. $P \in Ass(D(R)_{D(R)})$ if and only if $P = D(P \cap R)$ and $P \cap R \in Ass(R_R)$.

2. $P \in Min.Spec(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in Min.Spec(R)$.

Before we state the main result, we require the following:

Weak σ-rigid rings:
Let R be a ring and σ be an endomorphism of R. Recall that in [11], σ is called a rigid endomorphism if $a\sigma(a) = 0$ implies a $= 0$ for $a \in R$, and R is called a σ-rigid ring.

Example 1.2. Let $R = \mathbb{C}$, and $\sigma : R \to R$ be the map defined by $\sigma(a + ib) = a - ib$, a, $b \in \mathbb{R}$. Then it can be seen that R is a σ-rigid ring.

Definition 1.3. (Ouyang [14]): *Let R be a ring and σ be an endomorphism of R. Then R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.*

Example 1.4. (Example (2.1) of Ouyang [14]: Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid \text{a, b, c, d} \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. The it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

2. Main results

We now state the main result in the form of the following Theorem:

Theorem A: Let R be a semiprime right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R. Then $P \in Ass(O(R)_{O(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $O(P \cap R) = P$ and $(P \cap R) = U$. This result has been proved in Theorem (2.6).

Towards the proof of the above Theorem, we require the following:

Recall that an ideal of a ring R is said to be completely semiprime if $a^2 \in R$ implies that $a \in R$.

Let R be a Noetherian ring and σ an automorphism of R. We now give a necessary and sufficient condition for R to be a weak σ-rigid ring in the following Theorem:

Theorem 2.1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $N(R)$ is completely semiprime.*

Proof. First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R$ such that $\sigma(a) = n \in N(R)\}$ is an ideal of R. Now I is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then
$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R)$.
Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Conversely let $N(R)$ be completely semiprime. We will show that R is a weak σ-rigid ring. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Hence R is a weak σ-rigid ring. □

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i.e. if the prime radical is a completely semiprime ideal. We note that a reduced is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [3, 9].

Proposition 2.2. *Let R be a 2-primal right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in Min.Spec(R)$.*

Proof. Let R be 2-primal weak σ-rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$.

We will now show that $P(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

We next show that $\sigma(U) = U$ for all $U \in Min.Spec(R)$. Let $U = U_1$ be a minimal prime ideal of R. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of

R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Now P(R) is completely semiprime implies that $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Let now $V = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$.

First of all, we will show that V is an ideal of R. Let $a, b \in V$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1\}$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in V$. Also it is easy to see that for any $a \in V$ and for any $r \in R$, $ar \in V$ and $ra \in V$. Therefore V is a δ-invariant ideal of R.

We will now show that $V \in Spec(R)$. Suppose $V \notin Spec(R)$. Let $a \notin V$, $b \notin V$ be such that $aRb \subseteq V$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq V$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $V \in Spec(R)$. Now $V \subseteq U$, so $V = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$. □

Corollary 2.3. *Let R be a 2-primal right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that $\sigma(U) = U$ for all $U \in Min.Spec(R)$. Let δ be a σ-derivation of R. Then $\delta(U) \subseteq U$.*

Lemma 2.4. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then*

1. *If U is a minimal prime ideal of R, then O(U) is a minimal prime ideal of of O(R) and $O(U) \cap R = U$.*

2. *If P is a minimal prime ideal of O(R), then $P \cap R$ is a minimal prime ideal of R.*

Proof. **(1)** Let U be a minimal prime ideal of R. Then by Proposition (2.2) $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now on the same lines as in Theorem (2.22) of Goodearl and Warfield [8] we have $O(U) \in Spec(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in Min.Spec(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now the proof follows on the same lines as in Theorem (2.22) of Goodearl and Warfield [8] using Lemma (2.1) and Lemma (2.2) of Bhat [2] and Proposition (2.2). □

Theorem 2.5 (Hilbert Basis Theorem). *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

Proof. See Theorem (2.6) of Goodearl and Warfield [8]. □

With this we now state and prove Theorem A:

Theorem 2.6. *Let R be a semiprime right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R. Then $P \in Ass(O(R)_{O(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. $O(R)$ is right Noetherian by Theorem (2.5). Let $P \in Ass(O(R)_{O(R)})$. Now by Remark (1.1) $Ass(O(R)_{O(R)}) = \mathbb{A}(O(R)_{(R)})$. Let $P = Ann(I) = Assas(I)$ for some ideal I of $O(R)$ such that I is uniform as a right $O(R)$-module. Choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Let $U = Ann(a_nR) = Assas(a_nR)$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$, and since R is semiprime, $U \in Min.Spec(R)$ by Proposition (2.2.14) of McConnell and Robson [13]. Now R is a weak σ-rigid ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$, i.e. $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = Ann(cR) = Assas(cR)$, $c \in R$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$, and since R is semiprime, $U \in Min.Spec(R)$ by Proposition (2.2.14) of McConnell and Robson [13]. Now R is a weak σ-rigid ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now it can be easily seen that $O(U) = Ann(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = Ann(cO(R)) = Assas(cO(R))$. □

Example 2.7.

1. R as in Example 1.2 is a semiprime weak σ-rigid ring, but R being a field has no ideals and is therefore a trivial example.

2. Let τ be the conjugacy map on $\mathbb{C}$. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

Define $\sigma : R \to R$ by $\sigma((a_{ij})) = (\tau(a_{ij}))$. Then it can be seen that σ is an endomorphism of R and R is a weak σ-rigid ring.

Now for any $s \in R$, define $\delta_s : R \to R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ-derivation of R.

Let

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\} \in Ass(R_R).$$

In fact $U = Ann(I) = Assas(I)$, where $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \,\middle|\, c \in \mathbb{C} \right\}$ is a right ideal of R. Now we note that $\sigma(I) = I$, $\delta_s(I) \subseteq I$, Then it can be seen that σ is an endomorphism of R and $\sigma(U) \subseteq U$. and $\delta_s(U) \subseteq U$. Also $O(U) \in Ass(O(R)_{O(R)})$. In fact $O(U) = Ann(O(I)) = Assas(O(I))$.

Example 2.8. Now let $R = F \times F$, F a field and $\sigma : R \to R$ defined by $\sigma((u, v)) = (v, u)$ for $u, v \in F$. Then σ is an automorphism of R. But R is not a weak σ-rigid ring as for any $0 \neq a \in F$, we have $(a, 0)\sigma((a, 0)) = (0, 0) \in N(R)$, but $(a, 0) \notin N(R)$.

Proposition 2.9. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $U \in Min.Spec(R)$ with $\sigma(U) = U$ implies that $\delta(U) \subseteq U$.*

Proof. See Lemma (2.6) of Bhat [6]. □

We now prove the following Theorem:

Theorem 2.10. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$, σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$ and $\sigma(U) = U$ for all $U \in \mathbb{A}(R_R)$. Then $P \in Ass(O(R)_{O(R)})$ if and only if there exists $U \in Ass(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. $O(R)$ is right Noetherian by Theorem (2.5). Let $J \in Ass(O(R)_{O(R)})$. Now by Remark (1.1) $Ass(O(R)_{O(R)}) = \mathbb{A}(O(R)_{(R)})$. Let $P = Ann(I) = Assas(I)$ for some ideal I of $O(R)$ such that I is uniform as a right $O(R)$-module. Choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Let $U = Ann(a_nR) = Assas(a_nR)$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$. Now by hypothesis $\sigma(U) = U$, and therefore, Proposition (2.9) implies that $\delta(U) \subseteq U$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$. So $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = Ann(cR) = Assas(cR)$, $c \in R$. Now R is right Noetherian implies that $Ass(R_R) = \mathbb{A}(R_R)$. Now by hypothesis $\sigma(U) = U$, and therefore, Proposition (2.9) implies that $\delta(U) \subseteq U$. Now it can be easily seen that $O(U) = Ann(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = Ann(cO(R)) = Assas(cO(R))$. □

Example 2.11. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$. Then

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} \in Ass(R_R).$$

In fact $U = Ann(I) = Assas(I)$, where $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$ is a right ideal of R.

Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and $\sigma(U) \subseteq U$.
For any $s \in R$, define $\delta_s : R \to R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ-derivation of R. Also we see that $\sigma(\delta_s(u)) = \delta_s(\sigma(u))$ for all $u \in R$. For let $u = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $s = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix}$. Then

$$\sigma(\delta_s(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \delta_s(\sigma(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now we note that $\sigma(I) = I$, $\delta_s(I) \subseteq I$ and $\delta_s(U) \subseteq U$. Also $O(U) \in Ass(O(R)_{O(R)})$. In fact $O(U) = Ann(O(I)) = Assas(O(I))$.

Example 2.12. Let $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. Then $P = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} \in Ass(R_R)$. In fact $P = Ann(I)$ where $I = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{R} \end{pmatrix}$ is a right ideal of R. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and $\sigma(P) \subseteq P$.
For any $s \in R$, define $\delta_s : R \to R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ-derivation of R. But we see that $\sigma(\delta_s(u)) \neq \delta_s(\sigma(u))$ for all $u \in R$. Let $u = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$. Then

$$\sigma(\delta_s(u)) = \begin{pmatrix} 0 & pb + qc - aq \\ 0 & 0 \end{pmatrix} \text{ and } \delta_s(\sigma(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 2.13. Let $R = \mathbb{R} \times \mathbb{R}$, $\sigma : R \to R$ defined by $\sigma((a,b)) = (b,a)$ for $a, b \in \mathbb{R}$. Then σ is an automorphism of R. Let now $r \in \mathbb{R}$. Define $\delta_r : R \to R$ by $\delta_r((a,b)) = (a,b)r - r\sigma((a,b))$ for $a, b \in R$. Then δ is a σ-derivation. Now for any $(a,b) \in R$,

$$\sigma(\delta_r((a,b))) = \sigma((u,v)r - r\sigma((u,v))) =$$
$$= \sigma((u,v)r - r(v,u)) = \sigma((ur,vr) - \sigma(vr,ur)) = (vr,ur) - (ur,vr)).$$

Also

$$\delta_r(\sigma((u,v))) = \delta_r(v,u) = (v,u)r - r\sigma((v,u)) =$$
$$= (v,u)r - r(u,v) = (vr,ur) - (ur,vr)).$$

Therefore $\sigma(\delta((u,v))) = \delta(\sigma((u,v)))$ for all $(u,v) \in R$. We see that $U = 0 \times \mathbb{R} \in Ass(R_R)$. In fact $U = Ann(\mathbb{R} \times \{0\}) = Assas(\mathbb{R} \times \{0\})$. But we note that $\sigma(U) \neq U$.

References

[1] S. Annin, *Associated primes over Ore extension rings*, J. Algebra Appl. 3 (2004), no. 2, 193-205.

[2] V. K. Bhat, *Polynomial rings over pseudovaluation rings*, Int. J. Math. and Math. Sc. 2007 (2007), Art. ID 20138.

[3] V. K. Bhat, *On 2-primal Ore extensions*, Ukr. Math. Bull., Vol. 4(2) (2007), 173-179.

[4] V. K. Bhat, *Associated prime ideals of skew polynomial rings*, Beitrдge Algebra Geom., Vol. 49(1) (2008), 277-283.

[5] V. K. Bhat, *On near-pseudo-valuation rings and their extensions*, Int. Electron. J. Algebra, Vol. 5 (2009), 70-77.

[6] V. K. Bhat, *Transparent rings and their extensions*, New York J. Math., Vol. 15 (2009), 291-299.

[7] W. D. Blair and L. W. Small, *Embedding differential and skew-polynomial rings into artinain rings*, Proc. Amer. Math. Soc. 109(4) 1990, 881-886.

[8] K. R. Goodearl and R. B. Warfield Jr., *An introduction to non-commutative Noetherian rings*, Second Edition, Cambridge Uni. Press, 2004.

[9] N. K. Kim and T. K. Kwak, *Minimal prime ideals in 2-primal rings*, Math. Japon., Vol. 50(3) (1999), 415-420.

[10] T. K. Kwak, *Prime radicals of skew-polynomial rings*, Int. J. Math. Sci., Vol. 2(2) (2003), 219-227.

[11] J. Krempa, *Some examples of reduced rings*, Algebra Colloq., Vol. 3(4) (1996), 289-300.

[12] A. Leroy and J. Matczuk, *On induced modules over Ore extensions*, Comm. Algebra 32 (2004), no. 7, 2743-2766.

[13] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley 1987; revised edition: Amer. Math. Soc. 2001.

[14] L. Ouyang, *Extensions of generalized α-rigid rings*, Int. Electron. J. Algebra, Vol. 3 (2008), 103-116.

[15] A. Seidenberg, *Differential ideals in rings of finitely generated Type*, Amer. J. Math. 89 (1967), 22-42.

Contact information

V. K. Bhat

School of Mathematics, SMVD University, P/o SMVD University, Katra, J and K, India-182320

E-Mail: vijaykumarbhat2000@yahoo.com

Received by the editors: 16.10.2009
and in final form 16.10.2009.