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Preradicals and characteristic submodules: connections and operations

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Abstract. For an arbitrary module $M \in R$-Mod the relation between the lattice $\boldsymbol{L}^{ch}({}_RM)$ of characteristic (fully invariant) submodules of M and *big lattice* R-pr of preradicals of R-Mod is studied. Some isomorphic images of $\boldsymbol{L}^{ch}({}_RM)$ in R-pr are constructed. Using the product and coproduct in R-pr four operations in the lattice $\boldsymbol{L}^{ch}({}_RM)$ are defined. Some properties of these operations are shown and their relations with the lattice operations in $\boldsymbol{L}^{ch}({}_RM)$ are investigated. As application the case ${}_RM = {}_RR$ is mentioned, when $\boldsymbol{L}^{ch}({}_RR)$ is the lattice of two-sided ideals of ring R.

Introduction

Let R be a ring with unity and R-Mod denote the category of unitary left R-modules. We denote by R-pr the class of all preradicals of the category R-Mod. The ordinary operations of meet and join of preradicals transform R-pr into a *big lattice*, which was studied in a series of works (see, for example, [1]-[4]).

For an arbitrary module ${}_RM \in R$-Mod, in the lattice $\boldsymbol{L}({}_RM)$ of all submodules of ${}_RM$ we distinguish the sublattice $\boldsymbol{L}^{ch}({}_RM)$ of characteristic (fully invariant) submodules with the order relation „ $\subseteq$" (inclusion) and the lattice operations „ $\cap$" (intersection) and „ $+$" (sum).

The aim of this work is to clarify connection between the lattice $\boldsymbol{L}^{ch}({}_RM)$ of characteristic submodules of an arbitrary module ${}_RM$ and the big lattice R-pr of preradicals of R-Mod, as well as the application of

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obtained results to introducing four operations in $\boldsymbol{L}^{ch}({}_RM)$. For that the following mappings are used:

$$\alpha^M: \quad \boldsymbol{L}^{ch}({}_RM) \longrightarrow R\text{-pr}, \qquad N \to \alpha^M_N,$$

$$\omega^M: \quad \boldsymbol{L}^{ch}({}_RM) \longrightarrow R\text{-pr}, \qquad N \to \omega^M_N,$$

where α^M_N and ω^M_N are the preradicals of R-pr defined by the rules:

$$\alpha^M_N({}_RX) = \sum_{f\,:\,M\to X} f(N), \quad \omega^M_N({}_RX) = \bigcap_{f\,:\,X\to M} f^{-1}(N),$$

for every module ${}_RX \in R$-Mod (see: [1, 4, 5]).

The mappings α^M and ω^M define the bijections:

$$\boldsymbol{L}^{ch}({}_RM) \xrightarrow{\alpha^M} \boldsymbol{A}^M = \{\alpha^M_N \,|\, N \in \boldsymbol{L}^{ch}({}_RM)\},$$

$$\boldsymbol{L}^{ch}({}_RM) \xrightarrow{\omega^M} \boldsymbol{\Omega}^M = \{\omega^M_N \,|\, N \in \boldsymbol{L}^{ch}({}_RM)\},$$

which can be transformed in the lattice isomorphisms. Moreover, the equivalence relation $\cong_M$ defined in R-pr by the rule

$$r \cong_M s \Leftrightarrow r(M) = s(M)$$

determines the factor-lattice $R\text{-}pr/\cong_M = \boldsymbol{I}^M$, which is isomorphic to the lattice $\boldsymbol{L}^{ch}({}_RM)$ and consists of the equivalence classes of the form $\mathcal{I}^M_N = [\alpha^M_N, \omega^M_N]$, where $N \in \boldsymbol{L}^{ch}({}_RM)$ and $[\alpha^M_N, \omega^M_N]$ is the interval in R-pr containing all preradicals between α^M_N and ω^M_N. So we have:

$$\boldsymbol{L}^{ch}({}_RM) \cong \boldsymbol{A}^M \cong \boldsymbol{\Omega}^M \cong \boldsymbol{I}^M \quad \big(= R\text{-}pr/\cong_M\big) \ \text{(Proposition 2.3)}.$$

It is proved that the join of preradicals in the lattice $\boldsymbol{A}^M$ coincides with their join in R-pr, and the meet of preradicals in $\boldsymbol{\Omega}^M$ coincides with their meet in R-pr (Propositions 2.4, 2.5).

Using the relations between $\boldsymbol{L}^{ch}({}_RM)$ and R-pr (the mappings α^M and ω^M), as well as the product and coproduct in R-pr, four operations in $\boldsymbol{L}^{ch}({}_RM)$ are defined:

1) α-product: $K \cdot N = \alpha^M_K\, \alpha^M_N(M)$;

2) ω-product: $K \odot N = \omega^M_K\, \omega^M_N(M)$;

3) α-coproduct: $(N : K) = (\alpha^M_N : \alpha^M_K)(M)$;

4) ω-coproduct: $(N \odot K) = (\omega^M_N : \omega^M_K)(M)$,

for every characteristic submodules $K, N \in \boldsymbol{L}^{ch}({}_RM)$.

Properties of these operations are studied and some relations between them and lattice operations of $\boldsymbol{L}^{ch}({}_RM)$ are shown. For example, it is proved that α-product is left distributive with respect to sum, ω-product is left distributive with respect to intersection, α-coproduct is right distributive with respect to sum, and ω-coproduct is right distributive with respect to intersection (Propositions 3.3, 3.4, 4.3, 4.4).

The case ${}_RM = {}_RR$ is studied, i.e. when $\boldsymbol{L}^{ch}({}_RR)$ is the lattice of two-sided ideals of the ring R: the mappings α^R and ω^R are specified, as well as the respective operations (two of them coincide with the ordinary product and sum of ideals).

1. Preliminary notions and results

In this auxiliary section we remind some notions and results necessary for the basic material.

Let R be an arbitrary ring with unity and R-Mod is the category of unitary left R-modules. A *preradical* r of the category R-Mod is a subfunctor of identity functor, i.e. r is a function which associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f\,(r(M)) \subseteq r(M')$ for every R-morphism $f: M \to M'$. We denote by R-pr the class of all preradicals of the category R-Mod. The order relation „$\leq$" in R-pr is defined as follows:

$$r \leq s \Leftrightarrow r(M) \subseteq s(M)$$

for every $M \in R$-Mod.

The operations „$\wedge$" (meet) and „$\vee$" (join) in R-pr are defined by the rules:

$$(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha)(M) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M), \qquad (\bigvee_{\alpha\in\mathfrak{A}} r_\alpha)(M) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(M)$$

for every family $\{r_\alpha \,|\, \alpha \in \mathfrak{A}\} \subseteq R$-pr and $M \in R$-Mod.

Then R-pr $(\wedge, \vee)$ has the ordinary properties of lattices with the difference that R-pr is not necessarily a set, and so it is called a *big lattice*. This lattice was studied from different points of view in a series of works, for example in [1]-[4].

Besides the lattice operations in R-pr an important role is played by the following two operations $r \cdot s$ and $(r : s)$ (*product* and *coproduct* of preradicals), which are defined by the rules:

$$(r \cdot s)(M) = r\big(s(M)\big), \qquad [(r:s)(M)]\,/\,r(M) = s\big(M\,/\,r(M)\big),$$

for every $r, s \in R$-pr and $M \in R$-Mod. Some properties and applications of these operations can be found in [1], [4], etc. In particular, is true

Lemma 1.1 ([1], p.36; [4], Theorem 8). *For every preradicals of R-pr the following relations hold:*

(a) $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)\cdot s = \bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \cdot s)$;

(b) $\big(\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)\cdot s = \bigvee\limits_{\alpha\in\mathfrak{A}} (r_\alpha \cdot s)$;

(c) $\big(r : \big(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\big)\big) = \bigwedge\limits_{\alpha\in\mathfrak{A}} (r : s_\alpha)$;

(d) $\big(r : \big(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\big)\big) = \bigvee\limits_{\alpha\in\mathfrak{A}} (r : s_\alpha)$. □

Every preradical $r \in R$-pr defines the following two classes of modules:
$\mathcal{R}(r) = \{M \in R\text{-}Mod \,|\, r(M) = M\}$ is the class of *r-torsion* modules,

$\mathcal{P}(r) = \{M \in R\text{-}Mod \,|\, r(M) = 0\}$ is the class of *r-torsionfree* modules.

For some types of preradicals these classes restore the preradical r ([1]-[3]).

A preradical $r \in R$-pr is called:

- *idempotent* if $r\big(r(M)\big) = r(M)$ for every $M \in R$-Mod;
- *radical* if $r\big(M \,/\, r(M)\big) = 0$ for every $M \in R$-Mod;
- *hereditary* if $r(N) = N \cap r(M)$ for every $N \subseteq M \in R$-Mod;
- *cohereditary* if $r(M \,/\, N) = (r(M) + N) \,/\, N$ for every $N \subseteq M \in R$-Mod.

Now we remind some standard methods of the construction of some preradicals by a module $M \in R$-Mod or by an ideal I of the ring R.

For a fixed module $M \in R$-Mod we can define an *idempotent preradical* r^M by the rule:

$$r^M(X) = \sum_{f\,:\,M\,\to\,X} Im\, f,$$

for every module $X \in R$-Mod (i.e. $r^M(X)$ is the *trace* of M in X). This idempotent preradical is defined by the class of modules generated by module M:

$$\mathcal{R}(r^M) = Gen({}_R M) =$$
$$= \{X \in R\text{-}Mod \,\big|\, \exists\, epi \dot{\sum_{\alpha\in\mathfrak{A}}} M_\alpha \to X \to 0, M_\alpha \cong M\} \qquad ([1]\text{-}[3]).$$

Dually, the module $M \in R$-Mod defines a *radical* r_M by the rule:

$$r_M(X) = \bigcap_{f\,:\,X\,\to\,M} Ker\, f,$$

for every module $X \in R$-Mod (i.e. $r_M(X)$ is the *reject* of M in X). It is determined by the class of modules cogenerated by M:

$$\mathcal{P}(r_M) = Cog({}_R M) =$$
$$= \{X \in R\text{-}Mod \,\big|\, \exists\, mono\ 0 \to X \to \prod_{\alpha \in \mathfrak{A}} M_\alpha, M_\alpha \cong M\}.$$

Further, if an ideal I of ring R is fixed, then some associated preradicals are known, in particular:

- *idempotent radical* r^I, determined by the class $\mathcal{R}(r^I) = \{{}_R X \,\big|\, IX = X\}$;
- *torsion* (hereditary radical) r_I such that $\mathcal{P}(r_I) = \{{}_R X \,\big|\, Ix = 0 \Rightarrow x = 0\}$;
- *pretorsion* (hereditary preradical) $r_{(I)}$ such that $\mathcal{R}\,(r_{(I)}) = \{{}_R X \,\big|\, IX = 0\}$, i.e. $r_{(I)}(X) = \{x \in X \,\big|\, Ix = 0\}$;
- *cohereditary radical* $r^{(I)}$ with $\mathcal{P}\,(r^{(I)}) = \{{}_R X \,\big|\, IX = 0\}$, i.e. $r^{(I)}(X) = IX$ (see: [1, 3, 7]).

Let M be an arbitrary R-module and $\boldsymbol{L}({}_R M)$ be the lattice of its submodules. A submodule $N \in \boldsymbol{L}({}_R M)$ is called *characteristic* (or *fully invariant*) in M if $f(N) \subseteq N$ for every R-endomorphism $f : {}_R M \to {}_R M$. This means that N is an R-S-subbimodule of bimodule ${}_R M_S$, where $S = End\,({}_R M)$. We denote by $\boldsymbol{L}^{ch}({}_R M)$ the set of all characteristic submodules of ${}_R M (0,\, M \in \boldsymbol{L}^{ch}({}_R M))$. It is clear that the intersection and the sum of characteristic submodules are submodules of the same type, so $\boldsymbol{L}^{ch}({}_R M)\,(\subseteq, \cap, +)$ is a complete sublatice of $\boldsymbol{L}({}_R M)$.

The following well known fact shows the relation between the characteristic submodules of ${}_R M$ and preradicals of R-pr (see: [1, 4], etc.).

Lemma 1.2. *A submodule $N \in \boldsymbol{L}({}_R M)$ is characteristic in ${}_R M$ if and only if $N = r(M)$ for some preradical $r \in R$-pr.* □

For a characteristic submodule $N \in \boldsymbol{L}^{ch}({}_R M)$ many preradicals $r \in R$-pr with $N = r(M)$ can exist. To describe all preradicals with this property we use the preradicals α_N^M and ω_N^M, defined by the rules:

$$\alpha_N^M(X) = \sum_{f\,:\,M \to X} f(N), \qquad \omega_N^M(X) = \bigcap_{f\,:\,X \to M} f^{-1}(N)$$

for every $X \in R$-Mod (see: [4]-[6]; in [1] these preradicals are defined for every $N \in \boldsymbol{L}({}_R M)$ and are denoted by $t_{(N \subseteq M)}$ and $t^{(N \subseteq M)}$, respectively).

For every $N \in \boldsymbol{L}^{ch}({}_R M)$ the relation $\alpha_N^M \leq \omega_N^M$ is true. Moreover, the following fact is proved (see [1, 4], etc.).

Lemma 1.3. *Let* ${}_RM$ *be a fixed module and* $N \in \boldsymbol{L}^{ch}({}_RM)$. *A preradical* $r \in R$*-pr has the property* $r(M) = N$ *if and only if* r *belongs to the interval* $\mathfrak{I}_N^M = [\alpha_N^M, \omega_N^M]$ *of* R*-pr.* $\square$

So α_N^M is the least among the preradicals r of R-pr with $r(M) = N$ and ω_N^M is the greatest among such preradicals.

We remark that the similar results as in Lemmas 1.2 and 1.3 for special types of preradicals (pretorsions, torsions, etc.) were obtained in [1, 8].

2. The relation between the lattices $\boldsymbol{L}^{ch}({}_RM)$ and R-pr

We fix an arbitrary module $M \in R$-Mod and consider the lattice $\boldsymbol{L}^{ch}({}_RM)$ of characteristic submodules of ${}_RM$. Using the indicated above constructions, we obtain the mappings:

$$\alpha^M : \quad \boldsymbol{L}^{ch}({}_RM) \longrightarrow R\text{-pr}, \qquad N \to \alpha_N^M,$$

$$\omega^M : \quad \boldsymbol{L}^{ch}({}_RM) \longrightarrow R\text{-pr}, \qquad N \to \omega_N^M.$$

We denote the images of these mappings as follows:

$$\boldsymbol{A}^M = Im\,(\alpha^M) = \{\alpha_N^M \,\big|\, N \in \boldsymbol{L}^{ch}({}_RM)\},$$

$$\boldsymbol{\Omega}^M = Im\,(\omega^M) = \{\omega_N^M \,\big|\, N \in \boldsymbol{L}^{ch}({}_RM)\}.$$

From the definitions of preradicals α_N^M and ω_N^M immediately follows

Lemma 2.1. *The mappings* α^M *and* ω^M *are isotone, i.e. they preserve the order relation:*

$$N \subseteq K \Rightarrow \alpha_N^M \leq \alpha_K^M, \quad \omega_N^M \leq \omega_K^M. \square$$

We denote by $\mathbf{0}$ and $\mathbf{1}$ the trivial preradicals of R-pr, i.e. $\mathbf{0}(X) = 0$ and $\mathbf{1}(X) = X$, for every $X \in R$-Mod. From the definitions it follows that if $N = 0$, then $\alpha_N^M = \alpha_0^M = \mathbf{0}$ and $\omega_N^M = \omega_0^M = r_M$, where r_M is the *radical* defined by $r_M(X) = \bigcap\limits_{f : X \to M} Ker\, f$ (see Section 1).

In the other extreme case when $N = M$ we have:

a) $\alpha_M^M = r^M$, where r^M is the *idempotent preradical* defined by $r^M(X) = \sum\limits_{f : M \to X} Im\, f$ (see Section 1);

b) $\omega_M^M = \mathbf{1}$.

So we obtain

Lemma 2.2. *For every module $M \in R$-Mod the following relations hold:*

1) $\alpha_0^M = \mathbf{0}, \alpha_M^M = r^M$;
2) $\omega_0^M = r_M, \omega_M^M = \mathbf{1}$;
3) $\boldsymbol{A}^M \subseteq [\mathbf{0},\, r^M] \subseteq R$-pr;
4) $\boldsymbol{\Omega}^M \subseteq [r_M,\, \mathbf{1}] \subseteq R$-pr. □

From the definitions it is clear that if $N, K \in \boldsymbol{L}^{ch}({}_RM)$ and $N \neq K$, then $\alpha_N^M \neq \alpha_K^M$, therefore we have the bijection

$$\boldsymbol{L}^{ch}({}_RM) \longrightarrow \boldsymbol{A}^M, \quad N \to \alpha_N^M.$$

Since $N \subseteq K$ if and only if $\alpha_N^M \leq \alpha_K^M$, the set $\boldsymbol{A}^M\,(\leq)$ can be transformed in a lattice such that for elements $\alpha_N^M,\, \alpha_K^M \in \boldsymbol{A}^M$ the meet is $\alpha_{N\cap K}^M$ and the join is α_{N+K}^M. Hence the indicated bijection becomes the lattice isomorphism: $\boldsymbol{L}^{ch}({}_RM) \cong \boldsymbol{A}^M$.

Similarly, the mapping ω^M determined a bijection from $\boldsymbol{L}^{ch}({}_RM)$ into $\boldsymbol{\Omega}^M$, and the set $\boldsymbol{\Omega}^M$ can be transformed in a lattice such that for $\omega_N^M,\, \omega_K^M \in \boldsymbol{\Omega}^M$ the meet will be $\omega_{N\cap K}$ and the join will be ω_{N+K}. So we have the lattice isomorphism: $\boldsymbol{L}^{ch}({}_RM) \cong \boldsymbol{\Omega}^M$.

From the foregoing it follows that there exists one more possibility to obtain in R-pr a lattice isomorphic to $\boldsymbol{L}^{ch}({}_RM)$. For the fixed module $M \in R$-Mod we define in R-pr the equivalence relation $\cong_M$ as follows:

$$r \cong_M s \Leftrightarrow r(M) = s(M),$$

where $r, s \in R$-pr. Then the lattice R-pr is divided into equivalence classes, which by Lemma 1.3 have the form of intervals $\mathfrak{I}_N^M$. We denote:

$$\boldsymbol{I}^M = R\text{-}pr \big/ \cong_M = \big\{ \mathfrak{I}_N^M = [\alpha_N^M,\, \omega_N^M] \,\big|\, N \in \boldsymbol{L}^{ch}({}_RM) \big\}.$$

On this set the order relation is defined by the rule:

$$\mathfrak{I}_N^M \leq \mathfrak{I}_K^M \Leftrightarrow \alpha_N^M \leq \alpha_K^M \Leftrightarrow \omega_N^M \leq \omega_K^M \Leftrightarrow N \subseteq K,$$

where $N, K \in \boldsymbol{L}^{ch}({}_RM)$. In particular, the least elements of $\boldsymbol{I}^M$ is the interval $[\mathbf{0}, r_M]$ of R-pr, and the greatest element is the interval $[r^M,\, \mathbf{1}]$ (see Lemma 2.2).

By the definitions it follows that the set $\boldsymbol{I}^M\,(\leq)$ can be transformed into a lattice by the operations:

$$\mathfrak{I}_N^M \wedge \mathfrak{I}_K^M = \mathfrak{I}_{N\cap K}^M, \quad \mathfrak{I}_N^M \vee \mathfrak{I}_K^M = \mathfrak{I}_{N+K}^M.$$

Thus the mapping $N \to \mathfrak{I}_N^M$ defines a bijection which becomes the lattice isomorphism: $\boldsymbol{L}^{ch}({}_RM) \cong \boldsymbol{I}^M$.

Totalizing the previous considerations we have

Proposition 2.3. *For every module $M \in R$-Mod the following lattices are isomorphic:*

$$\boldsymbol{L}^{ch}({}_RM),\ \boldsymbol{A}^M, \boldsymbol{\Omega}^M,\ \boldsymbol{I}^M = R\text{-pr}/\cong_M .\square$$

We remark that for elements of $\boldsymbol{A}^M$ and $\boldsymbol{\Omega}^M$ besides the lattice operations defined above, we have the operations „ $\wedge$" and „ $\vee$" in the lattice R-pr, the results of which not necessarily belong to $\boldsymbol{A}^M$ or $\boldsymbol{\Omega}^M$. Now we will compare these operations between them.

For the lattice $\boldsymbol{A}^M$ we have

Proposition 2.4. *Let M be an arbitrary R-module. For every characteristic submodules $N, K \in \boldsymbol{L}^{ch}({}_RM)$ we have the relation:*

$$\alpha^M_{N+K} = \alpha^M_N \vee \alpha^M_K,$$

i.e. the join in $\boldsymbol{A}^M$ coincides with the join in R-pr. Furthermore, $\alpha^M_{N\cap K} \leq \alpha^M_N \wedge \alpha^M_K$ and $\alpha^M_N \wedge \alpha^M_K \in \mathfrak{I}^M_{N\cap K}$.

Proof. For submodules $N, K \in \boldsymbol{L}^{ch}({}_RM)$ by definitions we have:

$$(\alpha^M_N \vee \alpha^M_K)\,(M) = \alpha^M_N(M) + \alpha^M_K(M) = N + K,$$

hence $\alpha^M_N \vee \alpha^M_K \in \mathfrak{I}^M_{N+K} = [\alpha^M_{N+K}\,, \omega^M_{N+K}]$ and so $\alpha^M_N \vee \alpha^M_K \geq \alpha^M_{N+K}$.

On the other hand, since the mapping α^M is isotone, we have $\alpha^M_N \vee \alpha^M_K \leq \alpha^M_{N+K}$ and so we obtain $\alpha^M_N \vee \alpha^M_K = \alpha^M_{N+K}$. Also by the fact that α^M is isotone it follows $\alpha^M_{N\cap K} \leq \alpha^M_N \wedge \alpha^M_K$. Since $(\alpha^M_N \wedge \alpha^M_K)\,(M) = \alpha^M_N(M) \cap \alpha^M_K(M) = N \cap K$, we obtain that $\alpha^M_N \wedge \alpha^M_K \in \mathfrak{I}^M_{N\cap K}$. $\square$

Now we study the same question for the lattice $\boldsymbol{\Omega}^M$.

Proposition 2.5. *For every characteristic submodules $N, K \in \boldsymbol{L}^{ch}({}_RM)$ is true the relation:*

$$\omega^M_{N\cap K} = \omega^M_N \wedge \omega^M_K,$$

i.e. the meet in $\boldsymbol{\Omega}^M$ coincides with the meet in R-pr. Furthermore, $\omega^M_{N+K} \geq \omega^M_N \vee \omega^M_K$ and $\omega^M_N \vee \omega^M_K \in \mathfrak{I}^M_{N+K}$.

Proof. Since $(\omega^M_N \wedge \omega^M_K)\,(M) = \omega^M_N(M) \cap \omega^M_K(M) = N \cap K$, we have $\omega^M_N \wedge \omega^M_K \in \mathfrak{I}^M_{N\cap K} = [\alpha^M_{N\cap K}, \omega^M_{N\cap K}]$, therefore $\omega^M_{N\cap K} \geq \omega^M_N \wedge \omega^M_K$. The inverse inclusion follows from isotony of ω^M, which implies also the last statement of proposition. $\square$

Example. If ${}_RM$ is *ch-simple*, i.e. $\boldsymbol{L}^{ch}({}_RM) = \{0, M\}$, then $\boldsymbol{I}^M = \{\mathfrak{I}^M_0, \mathfrak{I}^M_M\}$, where $\mathfrak{I}^M_0 = [\boldsymbol{0}, r_M]$, $\mathfrak{I}^M_M = [r^M, \boldsymbol{1}]$, and *$R$-pr* $= \mathfrak{I}^M_0 \cup \mathfrak{I}^M_M$, $\boldsymbol{A}^M = \{\boldsymbol{0}, r^M\}$, $\boldsymbol{\Omega}^M = \{r^M, \boldsymbol{1}\}$.

3. Operations in $\boldsymbol{L}^{ch}({}_RM)$ defined by the product in R-pr

The relation between the lattices $\boldsymbol{L}^{ch}({}_RM)$ and R-pr, indicated in Section 2, will be utilized to define some operations in $\boldsymbol{L}^{ch}({}_RM)$ with the help of product and coproduct in R-pr. In this section we consider the operations in $\boldsymbol{L}^{ch}({}_RM)$ which are obtained by the *product* in R-pr.

Since we fix the module $M \in R$-mod, in the rest of this paper for simplicity we will omit the index M in the notations α_N^M, ω_N^M, etc. As was mentioned above (Section 1) the product in R-pr is defined by $(r \cdot s)(M) = r\,(s(M))$ and among the properties we remind that $r \cdot s \leq r \wedge s$ and are true the relations:

$$\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha\Big) \cdot s = \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \cdot s), \quad \Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha\Big) \cdot s = \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \cdot s) \quad \text{(Lemma 1.1).}$$

Definition 1. *For every characteristic submodules K, $N \in \boldsymbol{L}^{ch}({}_RM)$ we define:*

$$K \cdot N = \alpha_K\, \alpha_N(M) = \alpha_K(N),$$

*i.e. $K \cdot N = \sum\limits_{f\,:\,M \to N} f(K)$. The submodule $K \cdot N$ will be called **$\boldsymbol{\alpha}$-product** of submodules K and N in $\boldsymbol{L}^{ch}({}_RM)$.*

Definition 2. *For every characteristic submodules K, $N \in \boldsymbol{L}^{ch}({}_RM)$ we define:*

$$K \odot N = \omega_K\, \omega_N(M) = \omega_K(N),$$

*i.e. $K \odot N = \bigcap\limits_{f:N \to M} f^{-1}(N)$. The submodule $K \odot N$ will be called **$\boldsymbol{\omega}$-product** of submodules K and N in $\boldsymbol{L}^{ch}({}_RM)$.*

From the definitions it is obvious that $K \cdot N$ and $K \odot N$ are characteristic submodules in ${}_RM$. For every $K \in \boldsymbol{L}^{ch}({}_RM)$ we have $\alpha_K \leq \omega_K$, therefore $\alpha_K(N) \subseteq \omega_K(N)$, i.e. $K \cdot N \subseteq K \odot N$. Since the mapping ω^M is isotone, from $N \subseteq M$ it follows:

$$K \odot N = \omega_K(N) \subseteq \omega_K(M) = K,$$

and by Definition 2 $K \odot N = \omega_K(N) \subseteq N$. So we obtain:

$$K \cdot N \subseteq K \odot N \subseteq K \cap N$$

for every submodules K, $N \in \boldsymbol{L}^{ch}({}_RM)$.

Now we consider some particular cases.

a) If $K \cap N = 0$ (for example, if $K = 0$ or $N = 0$), then
$K \cdot N = K \odot N = 0.$

b) If $K = M$, then since $\alpha_M = r^M$ and $\omega_M = \mathbf{1}$ (Lemma 2.2) we have:
$M \cdot N = \alpha_M(N) = r^M(N) = \sum\limits_{f\,:\,M \to N} f(M);$

$$M \odot N = \omega_M(N) = \mathbf{1}(N) = N,$$

for every $N \in \boldsymbol{L}^{ch}({}_R M)$.

c) If $N = M$, then:

$$K \cdot M = \alpha_K(M) = K,$$
$$K \odot M = \omega_K(M) = K,$$

for every $K \in \boldsymbol{L}^{ch}({}_R M)$.

Totalizing these observations we have

Lemma 3.1. 1) *For every submodules K, $N \in \boldsymbol{L}^{ch}({}_R M)$ the following relations are true:*

$$K \cdot N \subseteq K \odot N \subseteq K \cap N;$$

2) *$K \cdot N = K \odot N = 0$, if $K = 0$ or $N = 0$;*

3) *$K \cdot M = K \odot M = K$ for every $K \in \boldsymbol{L}^{ch}({}_R M)$;*

4) *$M \cdot N = r^M(N), M \odot N = N$ for every $N \in \boldsymbol{L}^{ch}({}_R M)$.* □

From Definitions 1 and 2 and since the mappings α^M and ω^M are isotone (Lemma 2.1) we obtain

Lemma 3.2. *The operations „ $\cdot$ ” and „ $\odot$ ” of Definitions 1 and 2 are monotone in both variables:*

$$K_1 \subseteq K_2 \Rightarrow K_1 \cdot N \subseteq K_2 \cdot N,\ \ K_1 \odot N \subseteq K_2 \odot N;$$
$$N_1 \subseteq N_2 \Rightarrow K \cdot N_1 \subseteq K \cdot N_2,\ \ K \odot N_1 \subseteq K \odot N_2.$$ □

Remark. In the paper [5] the product $K \cdot N$ is used for the study of prime modules and prime preradicals.

In continuation we will investigate the concordance of introduced operations „ $\cdot$ ” and „ $\odot$ ” in $\boldsymbol{L}^{ch}({}_R M)$ with the lattice operations „ $\cap$ ” and „ $+$ ” in this lattice.

For the operation „ $\cdot$ ” of $\boldsymbol{L}^{ch}({}_R M)$ we have

Proposition 3.3. *For every submodules K_1, K_2, $N \in \boldsymbol{L}^{ch}({}_R M)$ the following relation is true:*

$$(K_1 + K_2) \cdot N = (K_1 \cdot N) + (K_2 \cdot N),$$

i.e. the α-product is left distributive with respect to the sum of characteristic submodules.

Proof. By Proposition 2.4 $\alpha_{K_1+K_2} = \alpha_{K_1} \vee \alpha_{K_2}$, therefore

$$(K_1 + K_2) \cdot N = \alpha_{K_1+K_2}(N) = (\alpha_{K_1} \vee \alpha_{K_2})(N) =$$
$$= \alpha_{K_1}(N) + \alpha_{K_2}(N) = (K_1 \cdot N) + (K_2 \cdot N).$$

□

A similar result takes place for the operation „ $\odot$ ” of $\boldsymbol{L}^{ch}({}_RM)$.

Proposition 3.4. *For every submodules K_1, K_2, $N \in \boldsymbol{L}^{ch}({}_RM)$ the following relation is true:*

$$(K_1 \cap K_2) \odot N = (K_1 \odot N) \cap (K_2 \odot N),$$

i.e. the ω-product is left distributive with respect to the intersection of characteristic submodules.

Proof. From Proposition 2.5 it follows $\omega_{K_1 \cap K_2} = \omega_{K_1} \wedge \omega_{K_2}$, hence

$$(K_1 \cap K_2) \odot N = \omega_{K_1 \cap K_2}(N) = (\omega_{K_1} \wedge \omega_{K_2})(N) =$$

$$= \omega_{K_1}(N) \cap \omega_{K_2}(N) = (K_1 \odot N) \cap (K_2 \odot N).$$ □

As to the other possible relations of such types, from Lemma 3.2 follows

Proposition 3.5. *In the lattice $\boldsymbol{L}^{ch}({}_RM)$ the following inclusions are true:*

1) $K \cdot (N_1 + N_2) \supseteq (K \cdot N_1) + (K \cdot N_2)$;
2) $K \odot (N_1 + N_2) \supseteq (K \odot N_1) + (K \odot N_2)$;
3) $K \cdot (N_1 \cap N_2) \subseteq (K \cdot N_1) \cap (K \cdot N_2)$;
4) $K \odot (N_1 \cap N_2) \subseteq (K \odot N_1) \cap (K \odot N_2)$;
5) $(K_1 \cap K_2) \cdot N \subseteq (K_1 \cdot N) \cap (K_2 \cdot N)$;
6) $(K_1 + K_2) \odot N \supseteq (K_1 \odot N) + (K_2 \odot N)$. □

4. Operations in $\boldsymbol{L}^{ch}({}_RM)$ defined by the coproduct in R-pr

By analogy with the previous case now we will use the coproduct in R-pr to define two operations in $\boldsymbol{L}^{ch}({}_RM)$. As we mentioned in Section 1, the coproduct $(r : s)$ in R-pr is defined by $[(r : s)(X)] \,/\, r(X) = s\left(X \,/\, r(X)\right)$ for every $X \in R$-Mod. It is known that $(r : s) \geq r + s$ and the following relations hold:

$$\Big(r : \Big(\bigwedge_{\alpha \in \mathfrak{A}} s_\alpha\Big)\Big) = \bigwedge_{\alpha \in \mathfrak{A}} (r : s_\alpha), \quad \Big(r : \Big(\bigvee_{\alpha \in \mathfrak{A}} s_\alpha\Big)\Big) = \bigvee_{\alpha \in \mathfrak{A}} (r : s_\alpha) \quad \text{(Lemma 1.1)}.$$

As before we fix the module ${}_RM$ and consider the lattice $\boldsymbol{L}^{ch}({}_RM)$ of characteristic submodules of ${}_RM$.

Definition 3. *For every submodules N, $K \in \boldsymbol{L}^{ch}({}_RM)$ we define:*

$$(N : K) = (\alpha_N : \alpha_K)(M),$$

i.e. $(N : K) \,/\, N = \alpha_K(M \,/\, N) = \sum\limits_{f \,:\, M \to M/N} f(K)$, *or* $(N : K) = \pi^{-1}\left(\alpha_K(M \,/\, N)\right)$, *where $\pi : M \to M \,/\, N$ is the natural* *epimorphism. The submodule* $(N : K)$ *will be called* **α-coproduct** *of submodules* N *and* K *in* $\boldsymbol{L}^{ch}({}_RM)$.

Definition 4. *For every submodules* $N, K \in \boldsymbol{L}^{ch}({}_RM)$ *we define:*

$$(N \odot K) = (\omega_N : \omega_K)(M),$$

i.e. $(N \odot K) / N = \omega_K(M / N) = \bigcap_{f : M/N \to M} f^{-1}(K)$, *or*

$(N \odot K) = \pi^{-1}\big(\omega_K(M / N)\big)$, *where* $\pi : M \to M / N$ *is the natural epimorphism. The submodule* $(N \odot K)$ *will be called* **ω-coproduct** *of submodules* N *and* K *in* $\boldsymbol{L}^{ch}({}_RM)$.

Obviously $(N : K), (N \odot K) \in \boldsymbol{L}^{ch}({}_RM)$ and since $\alpha_K \leq \omega_K$ we have $\alpha_K(M / N) \subseteq \omega_K(M / N)$, so $(N : K) \subseteq (N \odot K)$. Moreover, from Definition 3 it follows that if we distinguish among all R-morphism $f : M \to M / N$ the natural epimorphism $\pi : M \to M / N$, then we have:

$$\alpha_K(M / N) = \sum_{f : M \to M/N} f(K) \supseteq \pi(K) = (K + N) / N,$$

therefore $(N \odot K) = \pi^{-1}\big(\alpha_K(M / N)\big) \supseteq K + N$. So we have:

$$N + K \subseteq (N : K) \subseteq (N \odot K)$$

for every $N, K \in \boldsymbol{L}^{ch}({}_RM)$.

We consider the defined operations for some extremal cases.

a) If $N + K = M$ (for example, $N = M$ or $K = M$), then $(N : K) = (N \odot K) = M$. So we have: $(M : K) = M, (M \odot K) = M,\ (N : M) = M, (N \odot M) = M$ for every $K, N \in \boldsymbol{L}^{ch}({}_RM)$.

b) If $N = 0$, then
$(0 : K) = \pi^{-1}\big(\alpha_K(M / 0)\big) = \alpha_K(M) = K$;
$(0 \odot K) = \pi^{-1}\big(\omega_K(M / 0)\big) = \omega_K(M) = K$.

c) If $K = 0$, then since $\alpha_0 = \mathbf{0}$ and $\omega_0 = r_M$ (Lemma 2.2) we obtain:
$(N : 0) = \pi^{-1}\big(\alpha_0(M / N)\big) = \pi^{-1}\big(\mathbf{0}(M / N)\big) = \pi^{-1}(0) = N$;
$(N \odot 0) = \pi^{-1}\big(\omega_0(M / N)\big) = \pi^{-1}\big(r_M(M / N)\big)$, i.e.
$(N \odot 0) / N = r_M(M / N)$.

Unifying these remarks we have

Lemma 4.1. 1) *For every* $N, K \in \boldsymbol{L}^{ch}({}_RM)$ *the following relations hold:*

$$N + K \subseteq (N : K) \subseteq (N \odot K);$$

2) $(N : K) = (N \odot K) = M$, *if* $N = M$ *or* $K = M$;

3) $(0 : K) = (0 \odot K) = K$ *for every* $K \in \boldsymbol{L}^{ch}({}_RM)$;

4) $(N \odot 0) / N = r_M(M / N)$ *for every* $N \in \boldsymbol{L}^{ch}({}_RM)$. □

From Definitions 3 and 4 follows

Lemma 4.2. *The operations „:” and „$\odot$” in $\boldsymbol{L}^{ch}({}_RM)$ are monotone in both variables:*

$$N_1 \subseteq N_2 \Rightarrow (N_1 : K) \subseteq (N_2 : K), \quad (N_1 \odot K) \subseteq (N_2 \odot K);$$

$$K_1 \subseteq K_2 \Rightarrow (N : K_1) \subseteq (N : K_2), \quad (N \odot K_1) \subseteq (N \odot K_2). \quad \square$$

Remark. In the paper [6] the submodule $(N \odot K)$ is used for the definition of coprime submodule and for the study of coprime preradicals.

Similarly to Propositions 3.3 and 3.4 for the α-coproduct and ω-coproduct some properties of distributivity can be shown.

Proposition 4.3. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}^{ch}({}_RM)$ the following relation hold:*

$$(N : (K_1 + K_2)) = (N : K_1) + (N : K_2),$$

i.e. the α-coproduct is right distributive with respect to the sum of characteristic submodules.

Proof. By Proposition 2.4 we have $\alpha_{K_1+K_2} = \alpha_{K_1} \vee \alpha_{K_2}$ and from Lemma 1.1 it follows:

$$\big(\alpha_N : (\alpha_{K_1} \vee \alpha_{K_2})\big) = (\alpha_N : \alpha_{K_1}) \vee (\alpha_N : \alpha_{K_2}).$$

Therefore:

$$(N : (K_1 + K_2)) = (\alpha_N : \alpha_{K_1+K_2})(M) = \big(\alpha_N : (\alpha_{K_1} \vee \alpha_{K_2})\big)(M) =$$

$$= [(\alpha_N : \alpha_{K_1}) \vee (\alpha_N : \alpha_{K_2})](M) = (\alpha_N : \alpha_{K_1})(M) + (\alpha_N : \alpha_{K_2})(M) =$$

$$= (N : K_1) + (N : K_2). \qquad \square$$

Proposition 4.4. *For every submodules N, K_1, $K_2 \in \boldsymbol{L}^{ch}({}_RM)$ the following relation holds:*

$$(N \odot (K_1 \cap K_2)) = (N \odot K_1) \cap (N \odot K_2),$$

i.e. the ω-coproduct is right distributive with respect to the intersection of characteristic submodules.

Proof. Applying Proposition 2.5 we have $\omega_{K_1 \cap K_2} = \omega_{K_1} \wedge \omega_{K_2}$ and by Lemma 1.1 $\big(\omega_N : (\omega_{K_1} \wedge \omega_{K_2})\big) = (\omega_N : \omega_{K_1}) \wedge (\omega_N : \omega_{K_2})$. Consequently

$$(N : (K_1 \cap K_2)) = (\omega_N : \omega_{K_1 \cap K_2})(M) = [\omega_N : (\omega_{K_1} \wedge \omega_{K_2})](M) =$$

$$= [(\omega_N : \omega_{K_1}) \wedge (\omega_N : \omega_{K_2})](M) = (\omega_N : \omega_{K_1})(M) \cap (\omega_N : \omega_{K_2})(M) =$$

$$= (N \odot K_1) \cap (N \odot K_2). \qquad \square$$

In the other possible cases we obtain only inclusions, which follows from Lemma 4.2.

Proposition 4.5. *In the lattice* $\boldsymbol{L}^{ch}({}_RM)$ *the following relations hold:*

1) $(N : (K_1 \cap K_2)) \quad \subseteq \quad (N : K_1) \cap (N : K_2)$;
2) $(N \odot (K_1 + K_2)) \quad \supseteq \quad (N \odot K_1) + (N \odot K_2)$;
3) $((N_1 \cap N_2) : K) \quad \subseteq \quad (N_1 : K) \cap (N_2 : K)$;
4) $((N_1 \cap N_2) \odot K) \quad \subseteq \quad (N_1 \odot K) \cap (N_2 \odot K)$;
5) $((N_1 + N_2) : K) \quad \supseteq \quad (N_1 : K) + (N_2 : K)$;
6) $((N_1 + N_2) \odot K) \quad \supseteq \quad (N_1 \odot K) + (N_2 \odot K)$. □

Remark. The Propositions 3.3, 3.4, 4.3, 4.4 are true for arbitrary intersections $\bigcap\limits_{\alpha \in \mathfrak{A}} K_\alpha$ and sums $\sum\limits_{\alpha \in \mathfrak{A}} K_\alpha$ of characteristic submodules.

We complete this section by some remarks on the arrangement (reciprocal position) of some preradicals in R-pr, related by the defined above operations in $\boldsymbol{L}^{ch}({}_RM)$.

1) If $N, K \in \boldsymbol{L}^{ch}({}_RM)$ then we have $\alpha_K\, \alpha_N \in R$-pr, the submodule $K \cdot N = \alpha_K\, \alpha_N(M)$ and corresponding preradical $\alpha_{K \cdot N} \in R$-pr. From definition $\alpha_{K \cdot N} \leq \alpha_K\, \alpha_N$ and these preradicals belong to the equivalence class $\mathfrak{I}_{K \cdot N}$. From the relations $K \cdot N \subseteq K \odot N \subseteq N \cap K$ if follows $\alpha_{K \cdot N} \leq \alpha_{K \odot N} \leq \alpha_{N \cap K}$ and since α^M is isotone we have $\alpha_{N \cap K} \leq \alpha_N \wedge \alpha_K$.

2) Submodules $N, K \in \boldsymbol{L}^{ch}({}_RM)$ define the submodule $K \odot N = \omega_K\, \omega_N(M)$ and the preradical $\omega_{K \odot N} \in R$-pr. We have $\omega_K\, \omega_N, \omega_{K \odot N} \in \mathfrak{I}_{K \odot N}$, so $\omega_{K \odot N} \geq \omega_K\, \omega_N$. From the same relations $K \cdot N \subseteq K \odot N \subseteq K \cap N$ if follows $\omega_{K \cdot N} \leq \omega_{K \odot N} \leq \omega_{K \cap N} = \omega_N \wedge \omega_K$ (by Proposition 2.5).

3) Similarly, if $N, K \in \boldsymbol{L}^{ch}({}_RM)$ we have $(N : K) = (\alpha_N : \alpha_K)(M)$ and preradical $\alpha_{(N:K)} \in R$-pr. Since $\alpha_{(N:K)}$, $(\alpha_N : \alpha_K) \in \mathfrak{I}_{(N:K)}$, we obtain $\alpha_{(N:K)} \leq (\alpha_N : \alpha_K)$. Using the relations $N + K \subseteq (N : K) \subseteq (N \odot K)$ and Proposition 2.4, we have $\alpha_N \vee \alpha_K = \alpha_{N+K} \leq \alpha_{(N:K)} \leq \alpha_{(N \odot K)}$.

4) Finally, submodules $N, K \in \boldsymbol{L}^{ch}({}_RM)$ define the submodule $(N \odot K) = (\omega_N : \omega_K)(M)$ and preradical $\omega_{(N \odot K)} \in R$-pr. We have $\omega_{(N \odot K)} \geq (\omega_N : \omega_K) \in \mathfrak{I}_{(N \odot K)}$. From the same relations $N + K \subseteq (N : K) \subseteq (N \odot K)$ if follows $\omega_{N+K} \leq \omega_{(N:K)} \leq \omega_{(N \odot K)}$ and since ω^M is isotone we have $\omega_N \vee \omega_K \leq \omega_{N+K}$.

5. The case ${}_RM = {}_RR$

Now we specify briefly the situation when ${}_RM = {}_RR$, i.e. when $\boldsymbol{L}^{ch}({}_RR)$ is the lattice of two-sided ideals of the ring R. We show the relation between $\boldsymbol{L}^{ch}({}_RR)$ and R-pr, as well as the operations introduced above by the mappings α^M and ω^M, using the product and coproduct in R-pr.

For ideal $I = 0$ we have $\alpha_0 = \mathbf{0}$ and $\omega_0 = r_R$, where $r_R(X) = \bigcap_{f\,:\,X \to R} Ker f$ for every $X \in R$-Mod, i.e. r_R is the radical cogenerated by the module ${}_RR$ (i.e. $\mathcal{P}(r_R) = Cog\,({}_RR)$).

In the other extreme case when $I = {}_RR$ we have $\alpha_R = r^R = \mathbf{1}$, since ${}_RR$ is a generator of R-Mod:

$$\mathcal{R}(r^R) = Gen({}_RR) = R\text{-}Mod.$$

From the other hand, $\omega_R = \mathbf{1}$ and so $\omega_R = \alpha_R$, therefore in the lattice $\boldsymbol{I}^R = R\text{-}pr\big/ \cong_R$ the least element is the interval $[\mathbf{0},\, r_R]$ and the greatest element is the degenerated interval $\mathcal{I}_R$, consisting of one preradical: $\alpha_R = \omega_R = \mathbf{1}$.

Every ideal $I \in \boldsymbol{L}^{ch}({}_RR)$ determines in the lattice $\boldsymbol{I}^R = R\text{-}pr\big/ \cong_R$ the equivalence class $\mathcal{I}_I = [\alpha_I,\, \omega_I]$. We concretize these preradicals. By definition $\alpha_I(X) = \sum_{f:R \to X} f(I)$ for every $X \in R$-Mod. The isomorphism $Hom_R({}_RR,\, {}_RX) \cong_R X$ show that every R-morphism $f : {}_RR \to {}_RX$ has the form $f_x : {}_RR \to {}_RX$, where $x \in X$ and $f_x(r) = r\,x$ for every $r \in R$, so $f_x(I) = Ix$. Thus we obtain:

$$\alpha_I(X) = \sum_{f:R \to X} f(I) = \sum_{x \in X} Ix = IX.$$

In such way α_I coincides with the *coherediitary radical* $r^{(I)}$, defined by the class of modules

$$\mathcal{P}(r^{(I)}) = \{X \in R\text{-}Mod \,\big|\, IX = 0\} \quad \text{(see Section 1)}.$$

From the other hand, the preradical ω_I by definition acts as follows:

$$\omega_I(X) = \bigcap_{f:X \to R} f^{-1}(I) = \{x \in X \,\big|\, f(x) \in I \forall\, f :\, {}_RX \to {}_RR\}$$

for every $X \in R$-Mod.

Now we show what the defined above four operations represent in the case of the lattice $\boldsymbol{L}^{ch}({}_RR)$.

a) *The α-product in $\boldsymbol{L}^{ch}({}_RR)$.*

If $J,\, I \in \boldsymbol{L}^{ch}({}_RR)$ then by definition $J \cdot I = \alpha_J(I) = \sum f(J)$ for all R-morphisms $f :\, {}_RR \to {}_RI$. We apply again the canonical isomorphism ${}_RI \cong Hom_R(R, I)$, representing every $f :\, {}_RR \to {}_RI$ in the form $f_i :\, {}_RR \to {}_RI$, where $i \in I$ and $f_i(r) = r\,i$ for every $r \in R$, so $f_i(J) = Ji$. Therefore

$$J \cdot I = \sum_{i \in I} f_i(J) = \sum_{i \in I} Ji = JI,$$

where JI is the ordinary product of ideals in R. So we have the following *conclusion*: α-product in $\boldsymbol{L}^{ch}({}_R R)$ coincides with the ordinary product of ideals in R.

b) *The ω-product in $\boldsymbol{L}^{ch}({}_R R)$.*

By the definition of operation „ $\odot$" for ideals $J, I \in \boldsymbol{L}^{ch}({}_R R)$ we have:

$$J \odot I = \omega_J({}_R I) = \bigcap_{f\,:\,I \to R} f^{-1}(J) = \{i \in I \,\big|\, f(i) \in J \forall f : {}_R I \to {}_R R\}.$$

By previous results it follows that $JI = J \cdot I \subseteq J \odot I \subseteq J \cap I$ and from Proposition 3.4 we have: $(J_1 \cap J_2) \odot I = (J_1 \odot I) \cap (J_2 \odot I)$. Since the mapping ω^R is isotone we obtain the inclusions similar to relations of Proposition 3.5.

c) *The α-coproduct in $\boldsymbol{L}^{ch}({}_R R)$.*

For ideals $I, J \in \boldsymbol{L}^{ch}({}_R R)$ by definition we have $(I : J) = (\alpha_I : \alpha_J)({}_R R)$, i.e. $(I : J) \,/\, I = \alpha_J(R \,/\, I) = \sum f(J)$ for all $f : {}_R R \to {}_R(R \,/\, I)$. By the isomorphism $Hom_R(R, R \,/\, I)$ we can represent every R-morphism $f : {}_R R \to {}_R(R \,/\, I)$ in the form $f_{x+I} : {}_R R \to {}_R(R \,/\, I)$, where $x+I \in R \,/\, I$ and $f_{x+I}(r) = r(x+I)$ for every $r \in R$. Since $f_{x+I}(J) = J(x+I)$, we obtain:

$$(I : J) \,/\, I = \sum_{x+I \in R/I} f_{x+I}(J) = \sum_{x+I \in R/I} J(x+I) =$$

$$= J(R \,/\, I) = (JR + I) \,/\, I = (J + I) \,/\, I,$$

therefore $(I : J) = I + J$. So the α-coproduct in $\boldsymbol{L}^{ch}({}_R R)$ coincides with the sum of ideals of R.

d) *The ω-coproduct in $\boldsymbol{L}^{ch}({}_R R)$.*

If $I, J \in \boldsymbol{L}^{ch}({}_R R)$ then by definition we have:

$$(I \odot J) = (\omega_I : \omega_J)(R) = \pi^{-1}\Big(\bigcap_{f:R/I \to R} f^{-1}(J)\Big),$$

where $f : {}_R(R \,/\, I) \to {}_R R$ are R-morphism and $\pi : {}_R R \to {}_R(R \,/\, I)$ is the natural epimorphism. In other form:

$$(I \odot J) = \{r \in R \,|\, f(r+I) \in J \forall f : {}_R(R \,/\, I) \to {}_R R\}.$$

From general results we have $I + J = (I : J) \subseteq (I \odot J)$ and $(I \odot (J_1 \cap J_2)) = (I \odot J_1) \cap (I \odot J_2)$.

So in the case ${}_R M = {}_R R$ two operations coincide with product and sum of ideals, having two new operations which can present interest for further investigations.

References

[1] L. Bican, T. Kepka, P. Nemec, *Rings, Modules and Preradicals.* Marcel Dekker, New York, 1982.

[2] B. Stenström, *Rings of Quotients.* Springer Verlag, Berlin, 1975.

[3] A.I. Kashu, *Radicals and Torsions in Modules.* Ştiinţa, Chişinău, 1983 (in Russian).

[4] F. Raggi, J.R. Montes, H. Rincon, R. Fernandes-Alonso, C. Signoret, *The lattice structure of preradicals.* Commun. in Algebra, **30**(3) (2002), pp. 1533-1544.

[5] F. Raggi, J. Rios, H. Rincon, R. Fernandes-Alonso, C. Signoret, *Prime and irreducible preradicals.* J. of Algebra and its Applications, v. 4, N. 4 (2005), pp. 451-466.

[6] F. Raggi, J.R. Montes, R. Wisbauer, *Coprime preradicals and modules.* J. of Pure and Applied Algebra, **200**(2005), pp. 51-89.

[7] A.I. Kashu, *On some bijections between ideals, classes of modules and preradicals of R-Mod.* Bulet. A.Ş.R.M. Matematica, N 2(36), 2001, pp. 101-110.

[8] R. Marcov, *The characteristic submodules determined by pretorsions, torsions and jansian torsions.* Bulet. A.Ş.R.M. Matematica, N 3(28), 1998, pp. 71-80.

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