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RESEARCH ARTICLE

The groups whose cyclic subgroups are either ascendant or almost self-normalizing

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Abstract. The main result of this paper shows a description of locally finite groups, whose cyclic subgroups are either almost self-normalizing or ascendant. Also, we obtained some natural corollaries of the above situation.

Introduction

The subgroups of a group G are connected with some natural families of subgroups. One of them is the following. Let H be a subgroup of a group G. We construct an ascending series

$$\langle 1\rangle = H_0 \leqslant H_1 \leqslant \dots H_\alpha \leqslant H_{\alpha+1} \leqslant \dots H_\gamma \leqslant G,$$

where $H_1 = H$, $H_2 = N_G(H_1) = N_G(H)$, $H_{\alpha+1} = N_G(H_\alpha)$ for every ordinal $\alpha < \gamma$, $H_\lambda = \bigcup\limits_{\mu<\lambda} H_\mu$ for every limit ordinal $\lambda < \gamma$, and $N_G(H_\gamma) = H_\gamma$. This chain is called the *upper normalized chain* of H in G. Here the two natural types of subgroups appear. If $H_\gamma = G$, then a subgroup H is called *ascendant* in G. If $H_\gamma = H$ (that is $N_G(H) = H$), then a subgroup H is called *self-normalizing* in G. Thus, every subgroup of a group is naturally connected with the two types of subgroups: an ascendant and a self-normalizing subgroups. The presence of a large family of ascendant subgroups has a strong influence on the group structure. For example, if

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every subgroup of a group G is ascendant, then G is locally nilpotent [16]. Moreover, if every cyclic subgroup of a group G is ascendant, then G is locally nilpotent [6, Theorem 2]. More precisely, the subgroup $\mathbf{Gru}(G)$ of an arbitrary group G, generated by all ascendant cyclic subgroups of G, is locally nilpotent. This subgroup is called the *Gruenberg radical* of G. Every finitely generated subgroup of $\mathbf{Gru}(G)$ is ascendant in G and nilpotent [6, Theorem 2]. A group G is said to be a *Gruenberg group*, if $G = \mathbf{Gru}(G)$.

L.A. Kurdachenko and H. Smith [13] have considered the groups, whose subgroups are either subnormal or self-normalizing. A natural generalization of this paper was an article [12]. In [12] L.A. Kurdachenko et al. considered the groups, whose finitely generated subgroups are either ascendant or self-normalizing. From their results it follows that locally finite groups, whose cyclic subgroups are either ascendant or self-normalizing, have the same structure. Here we discuss a more general situation.

We remark that the groups, in which some family of subgroups divides into two types of subgroups, which often have the opposite properties, considered by other authors (see, for example, [15], [17]).

Let H be a subgroup of a group G. Then H is called *almost self-normalizing* in G, if H has finite index in $N_G(H)$.

In this paper we consider the groups whose cyclic subgroups are either almost self-normalizing or ascendant. The main result is the following

Theorem A. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Suppose that $G \neq \mathbf{Gru}(G)$. Then the following assertions hold:*

(i) *a factor-group $G/\mathbf{Gru}(G)$ is finite;*

(ii) *$G = Q \leftthreetimes R$, where Q is a normal Sylow σ'-subgroup of G, R is a Sylow σ-subgroup of G, $\sigma = \Pi(G/\mathbf{Gru}(G))$;*

(iii) *R is a Chernikov subgroup;*

(iv) *$\mathbf{Gru}(G) = C_R(Q) \times Q$;*

(v) *if $g \notin \mathbf{Gru}(G)$, then $C_G(g)$ is finite;*

(vi) *$\mathbf{Gru}(G)$ is nilpotent-by-finite.*

We obtained the following additional information about the structure of a factor-group $G/\mathbf{Gru}(G)$.

Corollary A1. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is infinite. Then*

(i) *if $p \in \sigma$ and $p \neq 2$, then Sylow p-subgroup of F is cyclic;*
(ii) *Sylow 2-subgroup of F is cyclic or a generalized quaternion group;*
(iii) *every subgroup of order pq of F, $p, q \in \sigma$, is cyclic.*

Let G be a Chernikov group and D be the divisible part of G. Put $\mathbf{Sp}(G) = \Pi(D)$.

Corollary A2. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is finite and $\mathbf{Sp}(G) = \{p\}$ for some prime $p \in \sigma$. Then*
(i) *if q is a prime and $q \notin \{2, p\}$, then Sylow q-subgroup of F is cyclic;*
(ii) *if $p \neq 2$, then Sylow 2-subgroup of F is cyclic or generalized quaternion group.*

Corollary A3. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Let $G \neq \mathbf{Gru}(G)$, $F = G/\mathbf{Gru}(G)$ and $\sigma = \Pi(F)$. Suppose that the Sylow σ'-subgroup of G is finite and $|\mathbf{Sp}(G)| \geqslant 2$. Then*
(i) *if $q \in \sigma$ is a prime and $q \neq 2$, then Sylow q-subgroup of F is cyclic;*
(ii) *if $2 \in \sigma$, then Sylow 2-subgroup of F is cyclic or generalized quaternion group.*

An important special case of the ascendant subgroups are the subnormal subgroups. A subnormal subgroup is exactly an ascendant subgroup having finite upper normalized chain. From Theorem A we can obtain the description of locally finite groups whose cyclic subgroups are either almost self-normalizing or subnormal.

The subgroup $\mathbf{B}(G)$, generated by all cyclic subnormal subgroups of G, is called the *Baer radical* of G. Every finitely generated subgroup of $\mathbf{B}(G)$ is subnormal in G and nilpotent (see, for example, [14, Theorem 2.5.1]), so that a subgroup $\mathbf{B}(G)$ is locally nilpotent. A group G is said to be a *Baer group*, if $G = \mathbf{B}(G)$.

Let G be a group and A be an abelian normal subgroup of G. Then A is said to be *G-quasifinite* if A is infinite, but every proper G-invariant subgroup of A is finite.

Theorem B. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or subnormal. Suppose that $G \neq \mathbf{B}(G)$. Then the following assertions hold:*
(i) *a factor-group $G/\mathbf{B}(G)$ is finite;*

(ii) *$G = Q \leftthreetimes R$, where Q is a normal Sylow σ'-subgroup of G, R is a Sylow σ-subgroup of G, $\sigma = \Pi(G/\mathbf{B}(G))$;*
(iii) *R is a Chernikov subgroup;*
(iv) *$\mathbf{B}(G) = C_R(Q) \times Q$;*
(v) *if $g \notin \mathbf{B}(G)$, then $C_G(g)$ is finite;*
(vi) *$\mathbf{B}(G)$ includes a finite G-invariant σ-subgroup K such that*

$$\mathbf{B}(G)/K = QK/K \times U_1/K \times \ldots \times U_k/K,$$

where U_j/K is a G-quasifinite divisible Chernikov p_j-subgroup, $p_j \in \sigma$, $1 \leqslant j \leqslant k$;
(vii) *$\mathbf{B}(G)$ is nilpotent.*

Trivially, every normal subgroup is a special case of subnormal subgroup, and we come to

Corollary B1. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or normal. Suppose that G is a not Dedekind group. Then the following assertions hold:*
(i) *a factor-group $G/\mathbf{B}(G)$ is finite cyclic;*
(ii) *if $g \notin \mathbf{B}(G)$, then $C_G(g)$ is finite;*
(iii) *every subgroup of $\mathbf{B}(G)$ is G-invariant, in particular, $\mathbf{B}(G)$ is a Dedekind group;*
(iv) *Sylow 2-subgroup of $\mathbf{B}(G)$ is Chernikov, moreover, if this Sylow 2-subgroup is infinite, then $\mathbf{B}(G)$ is abelian and $G/\mathbf{B}(G)$ has order 2.*

1. Preliminaries and lemmas

Lemma 1. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). If H is a subgroup of G, then every cyclic subgroup of H is either almost self-normalizing or ascendant (respectively, subnormal).*

Proof. Let C be a cyclic subgroup of H and suppose that C is not ascendant (respectively, subnormal) in H. Then C can not be ascendant (respectively, subnormal) in G. It follows that the index $|N_G(C) : C|$ is finite. An inclusion $N_H(C) \leqslant N_G(C)$ shows that the index $|N_H(C) : C|$ is finite. □

Lemma 2. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). If A is an infinite periodic abelian subgroup of G, then the Gruenberg radical (respectively, Baer radical) of G includes A.*

Proof. Indeed, for each element $x \in A$ we have $A \leqslant C_G(x)$, which follows that the index $|N_G(\langle x \rangle) : \langle x \rangle|$ is infinite. Thus, $x \in \mathbf{Gru}(G)$ (respectively, $x \in \mathbf{B}(G)$). Hence $A \leqslant \mathbf{Gru}(G)$ (respectively, $A \leqslant \mathbf{B}(G)$). $\square$

Lemma 3. *Let L be a locally nilpotent periodic subgroup of G. If L is not Chernikov, then the centralizer of every element of L is infinite.*

Proof. Suppose first that the set $\Pi(L)$ is infinite. Let g be an arbitrary element of L and $|g| = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$, where $p_1, \ldots, p_s$ are primes, $p_j \neq p_m$ whenever $j \neq m$. Since $\Pi(L)$ is infinite, the set $\pi = \Pi(L) \setminus \{p_1, \ldots, p_s\}$ is infinite. Then the Sylow π-subgroup L_π of L is infinite. The fact, that L is locally nilpotent, implies the inclusion $L_\pi \leqslant C_G(g)$, which follows that $C_G(g)$ is infinite.

Suppose now that the set $\Pi(L)$ is finite. Since L is not Chernikov, there exists a prime p such that the Sylow p-subgroup P of L is not Chernikov. Let x be an arbitrary element of L. If x is a p'-element, then $P \leqslant C_G(x)$, which follows again that $C_G(x)$ is infinite. Assume that $x \in P$. Since P is not Chernikov, P includes an $\langle x \rangle$-invariant abelian subgroup A, which is not Chernikov [21]. Then its lower layer $H = \Omega_1(A)$ is an infinite elementary abelian subgroup. Clearly H is an $\langle x \rangle$-invariant subgroup. Let $1 \neq b_1 \in H$. Put $K_1 = \langle b_1 \rangle^{\langle x \rangle}$, then K_1 is a finite $\langle x \rangle$-invariant subgroup. Since H is elementary abelian, H includes a subgroup B_1 such that $H = K_1 \times B_1$. We note that the index $|H : B_1|$ is finite. Then the index $|H : B_1^y|$ is also finite for every element $y \in \langle x \rangle$. Since an element x has finite order, a family $\{B_1^y | y \in \langle x \rangle\}$ is finite. Then the intersection $C_1 = \bigcap_{y \in \langle x \rangle} B_1^y$ has finite index in H. In particular, C_1 is infinite. By such choice C_1 is $\langle x \rangle$-invariant and $K_1 \cap C_1 = \langle 1 \rangle$. Let $1 \neq b_2 \in C_1$ and $K_2 = \langle b_2 \rangle^{\langle x \rangle}$, then K_2 also is a finite $\langle x \rangle$-invariant subgroup such that $K_1 \cap K_2 = \langle 1 \rangle$. Since H is elementary abelian, H includes a subgroup B_2 such that $H = K_1K_2 \times B_2$. Using the similar arguments and ordinary induction, we construct the family $\{K_n | n \in \mathbb{N}\}$ of finite $\langle x \rangle$-invariant subgroups such that $K_1 \cdot \ldots \cdot K_m \cap K_{m+1} = \langle 1 \rangle$ for every $m \in \mathbb{N}$. It follows that $\langle K_n | n \in \mathbb{N} \rangle = \mathbf{Dr}_{n \in \mathbb{N}} K_n$.

Since $\langle x, K_n \rangle$ is a finite p-subgroup, it is nilpotent. Since K_n is its normal subgroup, $K_n \cap \zeta(\langle x, K_n \rangle) \neq \langle 1 \rangle$. Let $1 \neq z_n \in K_n \cap \zeta(\langle x, K_n \rangle)$ and put $Z = \langle z_n | n \in \mathbb{N} \rangle$. An equality $\langle K_n | n \in \mathbb{N} \rangle = \mathbf{Dr}_{n \in \mathbb{N}} K_n$ implies that Z is an infinite elementary abelian subgroup. By its choice $Z \leqslant C_G(\langle x \rangle)$. It follows that $C_G(x)$ is infinite. $\square$

Corollary 1. *Let L be an infinite periodic nilpotent subgroup of G. Then the centralizer of every element of L is infinite.*

Proof. If L is a not Chernikov subgroup, then result follows from Lemma 3. Therefore, suppose that L is a Chernikov subgroup. Since L is nilpotent, every subgroup of L is subnormal. In particular, L is a Baer group. Then L is central-by-finite [8, Corollary 1 to Lemma 4]. In particular, $C_L(x)$ is infinite for each element $x \in L$. □

Corollary 2. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that L is a locally nilpotent periodic subgroup of G. If L is not Chernikov, then the Gruenberg radical (respectively, Baer radical) of G includes L.*

Proof. In fact, by Lemma 3 $C_G(g)$ is infinite for each element $g \in L$, which follows that the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite and hence $g \in \mathbf{Gru}(G)$ (respectively, $g \in \mathbf{B}(G)$). □

Corollary 3. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that p is a prime such that the Sylow p-subgroup P of G is not Chernikov. Then the Gruenberg radical (respectively, Baer radical) of G includes P. In particular, P is normal in G.*

Corollary 4. *Let G be a group whose cyclic subgroups are either almost self-normalizing or ascendant (respectively, subnormal). Suppose that L is an infinite periodic nilpotent subgroup of G. Then the Gruenberg radical (respectively, Baer radical) of G includes L.*

Proof. In fact, by Corollary 1 $C_G(g)$ is infinite for each element $g \in L$, which follows that the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite and hence $g \in \mathbf{Gru}(G)$ (respectively, $g \in \mathbf{B}(G)$). □

Let G be a Chernikov group. Denote by $\mathbf{D}(G)$ the maximal normal divisible abelian subgroup of G. A subgroup $\mathbf{D}(G)$ is called a *divisible part* of G.

Lemma 4. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Then Sylow p-subgroup of $G/\mathbf{Gru}(G)$ is finite for every prime p. Moreover, for every $p \in \Pi(G/\mathbf{Gru}(G))$ every Sylow p-subgroup of G is Chernikov.*

Proof. Put $B = \mathbf{Gru}(G)$. Let $p \in \Pi(G/B)$ and let P/B be the Sylow p-subgroup of G/B. Let $B \neq xB \in P/B$. Without loss of generality we can assume that x is an p-element. Let B_p be the Sylow p-subgroup of B. Since B is locally nilpotent, B_p is G-invariant. Then a product $\langle x \rangle B_p$ is a p-subgroup. Suppose that B_p is not Chernikov. Let C be the Sylow p-subgroup of G, including $\langle x \rangle B_p$. Then C is a not Chernikov subgroup, and Corollary 3 proves that $C \leqslant B$. In this case $x \in B$, what contradicts to the choice of element x. This contradiction shows that B_p is a Chernikov subgroup.

Suppose now that P/B is not Chernikov. Then P/B includes an infinite elementary abelian subgroup A/B [2, Theorem 8]. Without loss of generality we can assume that A/B is countable. Then A/B has an ascending series of finite subgroups

$$A_1/B \leqslant A_2/B \leqslant \ldots \leqslant A_n/B \leqslant \ldots$$

such that $A/B = \bigcup\limits_{n \in \mathbb{N}} A_n/B$. Since A_1/B is finite, A_1 includes a finite subgroup K_1 such that $A_1 = K_1B$. Choose in K_1 Sylow p-subgroup S_1. Since K_1 is finite, $S_1(B \cap K_1)/(B \cap K_1)$ is a Sylow p-subgroup of $K_1/(B \cap K_1)$. On the other hand, $K_1/(B \cap K_1) \cong K_1B/B$ is a p-group. It follows that $S_1(B \cap K_1)/(B \cap K_1) = K_1/(B \cap K_1)$, or $S_1(B \cap K_1) = K_1$. In turn out it follows that $A_1 = S_1B$. Choose in A_2 a finite subgroup K_2 such that $K_1 \leqslant K_2$ and $A_2 = K_2B$. Let S_2 be the Sylow p-subgroup of K_2, including S_1. Using the above arguments, we can prove that $A_2 = S_2B$. Using similarly arguments and ordinary induction, we construct an ascending series

$$S_1 \leqslant S_2 \leqslant \ldots \leqslant S_n \leqslant \ldots$$

of finite p-subgroups such that $A = \Big(\bigcup\limits_{n \in \mathbb{N}} S_n\Big)B$. Put $S = \bigcup\limits_{n \in \mathbb{N}} S_n$, then S is a p-subgroup and isomorphism $S/(S \cap B) \cong SB/B = A/B$ shows that S is not Chernikov. Since B_p is a normal p-subgroup, then SB_p is a p-subgroup. Let D be the Sylow p-subgroup of G, including SB_p. The fact that S is not Chernikov, implies that D is not Chernikov. Corollary 3 proves that $D \leqslant B$. In this case $S \leqslant B$ and therefore $SB = A \leqslant B$, what contradicts to the choice of A. This contradiction shows that P/B is a Chernikov subgroup.

Let Q be the Sylow p-subgroup of B, then $B = B_p \times Q$. Since B_p and P/B are Chernikov, P/Q likewise is a Chernikov group. In particular, it is countable. Then P includes a p-subgroup R such that $P = QR$ (see, for example, [3, Theorem 2.4.5]). Denote by W the divisible part of R. Since W is abelian and infinite, Lemma 2 shows that $W \leqslant B$. An inclusion $Q \leqslant B$ implies that $WQ \leqslant B$. In turn out, it follows that P/B is finite. $\square$

Let G be a group. Recall that a subgroup H of a group G is called *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element g of G.

Lemma 5. *Let G be an infinite locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Then the factor-group $G/\mathbf{Gru}(G)$ is finite. Moreover, if $\pi = \Pi(G) \setminus \Pi(G/\mathbf{Gru}(G))$ then $G = Q \leftthreetimes R$, where Q is a Sylow π-subgroup of G and R is a Chernikov subgroup.*

Proof. Put $B = \mathbf{Gru}(G)$. Suppose that G/B is infinite. Then G/B includes an infinite abelian subgroup A/B [10]. Since the Sylow p-subgroups of G/B are finite for each prime p by Lemma 4, the set $\Pi(A/B)$ is infinite. Let $p \in \Pi(A/B)$ and let P/B be the Sylow p-subgroup of A/B. By this choice P/B is non-identity, i.e. B does not include P. Lemma 4 shows that P/B is finite. Being almost locally nilpotent, P has a Carter subgroup C, that is maximal locally nilpotent self-normalizing subgroup. We remark also that all Carter subgroup of P are conjugate and abnormal [20, Theorem 2.1 and Corollary 2.2]. Since C is abnormal in P, CB is also abnormal. Then CB/B is abnormal in P/B. On the other hand, P/B is abelian, which follows that $CB/B = P/B$.

Let a be an arbitrary element of A. Then C^a is maximal locally nilpotent self-normalizing subgroup of P. As we noted above, the subgroups C and C^a are conjugate in P, that is there exists an element x of P such that $C^a = C^x$. It follows that $ax^{-1} \in N_A(C)$, which follows the equality $A = PN_A(C)$. Take into account an equality $P = CB$, we obtain that $A = BN_A(C)$. If we suppose that a subgroup C is not Chernikov, then the fact that C is locally nilpotent together with Corollary 1 imply the inclusion $C \leqslant B$. But this contradicts to the choice of P. This contradiction shows that C is a Chernikov subgroup. The isomorphism

$$A/B \cong BN_A(C)/B \cong N_A(C)/(B \cap N_A(C))$$

shows that $N_A(C)/(B \cap N_A(C))$ has infinite set $\Pi(N_A(C)/(B \cap N_A(C)))$ and is abelian. Since C is abnormal in P, $N_P(C) = C$. It follows that

$$B \cap N_A(C) = B \cap P \cap N_A(C) = B \cap N_P(C) = B \cap C.$$

Together with $C/(B \cap C) \cong CB/B = P/B$ and the fact, that P/B is finite, it follows that $N_A(C)/C$ is an abelian group with infinite set

$\Pi(N_A(C)/C)$. Put $N_A(C) = K$. Since $K/C_K(C)$ is a Chernikov group (see, for example, [3, Theorem 1.5.16]), $C_K(C)C/C$ is an abelian group with infinite set $\Pi(C_K(C)C/C)$. Let S/C be the Sylow σ'-subgroup of $C_K(C)C/C$, where $\sigma = \Pi(C)$. Clearly S/C is countable, so that $S = C \leftthreetimes U$ (see, for example, [3, Theorem 2.4.5]), where U is a Sylow σ'-subgroup of S. An inclusion $C \leqslant C_K(C)$ implies that $S = C \times U$. The fact that U is an infinite abelian subgroup implies that a subgroup $C_S(y)$ is infinite for every element $y \in C$. In turn out, it implies that a cyclic subgroup $\langle y \rangle$ is ascendant in G. Hence $y \in B$. In other words, $C \leqslant B$. But in this case the equality $P = CB$ implies that $P = B$, which contradicts to the choice of P. This contradiction proves that G/B is finite.

In particular, a set π is finite. Let Q be a Sylow π'-subgroup of G. The choice of π shows that $Q \leqslant B$, so that Q is normal in G. Using Lemma 4 we obtain that for each prime $p \in \pi$ every Sylow p-subgroup of G is Chernikov. It follows that B/Q is countable. Take in account that G/B is finite, we obtain that G/Q is countable. Then $G = Q \leftthreetimes R$, where R is a Sylow π-subgroup of G (see, for example, [3, Theorem 2.4.5]). Clearly R is a Chernikov subgroup. □

Lemma 6. *Let G be a group and P be a normal Chernikov divisible p-subgroup of G such that $G/C_G(P)$ is finite. Then P includes a finite G-invariant subgroup F such that $P/F = U_1/F \times \ldots \times U_k/F$, where U_j/F is a G-quasifinite subgroup, $1 \leqslant j \leqslant k$.*

Proof. Let

$$\mathfrak{Q} = \{Q \mid Q \text{ is an infinite } G\text{-invariant subgroup of } P\}.$$

Since P is Chernikov, it satisfies the minimal condition on subgroups, and therefore $\mathfrak{Q}$ has a minimal element, say V_1. Clearly V_1 is G-quasifinite. Then V_1 must be divisible and hence V_1 has a direct complement in P (see, for example, [4, Theorem 21.2]). It follows that P includes a G-invariant subgroup R_1 such that $P = V_1R_1$ and the intersection $V_1 \cap R_1$ is finite (see, for example, [11, Corollary 5.11]). Put $F_1 = V_1 \cap R_1$, then $P/F_1 = V_1/F_1 \times R_1/F_1$. Clearly V_1/F_1 is G-quasifinite. Now we choose in R_1/F_1 a G-invariant G-quasifinite subgroup V_2/F_1. Using the above arguments, we have found a G-invariant subgroup R_2/F_1 such that $R_1/F_1 = (V_2/F_1)(R_2/F_1)$ and the intersection $V_2/F_1 \cap R_2/F_1 = F_2/F_1$ is finite. By such choice we have

$$P/F_2 = (V_1F_2/F_2) \times V_2/F_2 \times R_2/F_2.$$

The both factors V_1F_2/F_2 and V_2/F_2 are G-quasifinite. Using the similar arguments, after finitely many steps we prove the required result. $\square$

Lemma 7. *Let G be a locally finite group whose cyclic subgroups are either almost self-normalizing or ascendant. Suppose that $G = Q \leftthreetimes F$, where $Q = \mathbf{Gru}(G)$, $\Pi(Q) \cap \Pi(F) = \varnothing$ and F is a finite subgroup. If Q is infinite, then*

(i) *Q is nilpotent-by-finite;*
(ii) *$C_F(Q) = \langle 1 \rangle$;*
(iii) *if $g \notin Q$, then $C_G(g)$ is finite;*
(iv) *if $p \in \Pi(F)$ and $p \neq 2$, then Sylow p-subgroup of F is cyclic;*
(v) *Sylow 2-subgroup of F is cyclic or generalized quaternion group;*
(vi) *every subgroup of order pq of F, $p, q \in \Pi(F)$, is cyclic.*

Proof. Choose an arbitrary element $g \notin Q$ and suppose that $C_G(g)$ is infinite. Since g has finite order, the index $|C_G(g) : \langle g \rangle|$ is infinite. In turn out, it implies $\langle g \rangle$ has infinite index in its normalizer, which show that $\langle g \rangle$ is ascendant in G. This contradiction shows that $C_G(g)$ is finite.

Let $y \in C_F(Q)$, then $Q \leqslant C_G(y)$, in particular, $C_G(y)$ is infinite. As we have seen above in this case $y \in \mathbf{Gru}(G) = Q$, i.e. $y \in Q \cap F = \langle 1 \rangle$.

Let $p \in \Pi(F)$ and g be an element of F, having order p. By above proved $C_Q(g)$ is finite. Then Q is nilpotent-by-finite [19, Theorem 1.2].

Suppose first that Q is not Chernikov. Then Q includes an F-invariant abelian subgroup A, which is not Chernikov [21]. Let y be an arbitrary element of F. The equality $\Pi(Q) \cap \Pi(F) = \varnothing$ implies that $A = C_A(y) \times [A, y]$ [1, Proposition 2.12]. By above proved $C_A(y)$ is finite, so that $[A, y]$ has finite index in A. It is valid for every element $y \in F$. Therefore, the finiteness of F implies that a subgroup $C = \bigcap_{y \in F} [A, y]$ has finite index in A. By its choice $C_C(y) = \langle 1 \rangle$ for every element $y \in F$. Put $E = \bigcap_{y \in F} C^y$, then E has finite index in A (in particular, E is infinite), E is F-invariant and $C_E(y) = \langle 1 \rangle$ for every element $y \in F$. Since F is finite, we can choose in E minimal F-invariant subgroup V. Then $C_V(y) = \langle 1 \rangle$ for every element $y \in F$. Therefore, F satisfies the conditions (iv)-(vi) by [9, Satz V.8.15].

Suppose now that Q is a Chernikov subgroup. Denote by D the divisible part of Q. The equality $\Pi(Q) \cap \Pi(F) = \varnothing$ implies again that $D = C_D(y) \times [D, y]$ [1, Proposition 2.12] for every element $y \in F$. If we suppose that $C_D(y) \neq \langle 1 \rangle$, then $C_D(y)$ must be infinite, and we obtain a contradiction with condition (iii). This contradiction shows that $C_D(y) = \langle 1 \rangle$ for every element $y \in F$. Again choose in D a minimal

F-invariant subgroup W. Then $C_W(y) = \langle 1 \rangle$ for every element $y \in F$. Therefore, F satisfies the conditions (iv)-(vi) by [9, Satz V.8.15]. $\square$

Lemma 8. *Let G be a Chernikov group and P be a divisible part of G. Suppose that P is a p-subgroup and $C_G(g)$ is finite for each p'-element g. Then the following assertions hold:*

(i) *if q is a prime and $q \notin \{2, p\}$, then Sylow q-subgroup of $G/\mathbf{Gru}(G)$ is cyclic;*

(ii) *if $p \neq 2$, then Sylow 2-subgroup of $G/\mathbf{Gru}(G)$ is cyclic or generalized quaternion group.*

Proof. Put $C = O_p(G)$. Our conditions yields that $\mathbf{Gru}(G) = C$. Let q be a prime, $q \neq p$, and Q/C be a Sylow q-subgroup of G/C. Since C is a p-subgroup, $Q = C \leftthreetimes R$, where R is a Sylow q-subgroup of Q (see, for example, [3, Theorem 2.4.5]). Choose in R an arbitrary abelian subgroup A. Let

$$\mathfrak{S} = \{S |\ S \text{ is an infinite } A\text{-invariant subgroup of } P\}.$$

Since P is Chernikov, it satisfies the minimal condition on subgroups, and therefore $\mathfrak{S}$ has a minimal element, say V. Clearly V is A-quasifinite. In particular, it follows that V is a divisible abelian subgroup. Since A is a p'-subgroup, $V = C_V(y) \times [V, y]$ for each element $y \in A$ [1, Proposition 2.12]. The fact, that A is abelian, implies that the subgroups $C_A(y)$ and $[V, y]$ are A-invariant. Therefore, if we assume that $C_A(y) \neq \langle 1 \rangle$, then $C_A(y)$ must be infinite, and we obtain a contradiction. This contradiction shows that $C_A(y) = \langle 1 \rangle$ for each element $y \in A$. Using Lemma 3.1 of paper [7] we obtain that a subgroup A is cyclic. In particular, $\zeta(R)$ is cyclic. Let $\langle d \rangle = \Omega_1(\zeta(R))$. Suppose that x is an element of R, having order q. If $\langle d, x \rangle \neq \langle d \rangle$, then $\langle d, x \rangle$ is an elementary abelian subgroup of order q^2. But in this case it is not cyclic, and we obtain a contradiction with above proved. This contradiction shows that $\langle d, x \rangle = \langle d \rangle$. In other words, R has only one subgroup of order q. Then R (and hence Q/C) is cyclic, whenever $q \neq 2$, and Q/C is cyclic or generalized quaternion group, whenever $q = 2$ (see, for example, [9, Satz III.8.2]). $\square$

Lemma 9. *Let G be an infinite periodic group whose cyclic subgroups are either almost self-normalizing or ascendant. If $g \notin \mathbf{Gru}(G)$, then $C_G(g)$ is finite.*

Proof. Choose an arbitrary element $g \notin \mathbf{Gru}(G)$ and suppose that $C_G(g)$ is infinite. Since g has finite order, the index $|C_G(g) : \langle g \rangle|$ is infinite. In turn out, it implies that $\langle g\rangle$ has infinite index in its normalizer, which show that $\langle g\rangle$ is ascendant in G. This contradiction shows that $C_G(g)$ is finite. □

Lemma 10. *Let G be an infinite periodic group whose cyclic subgroups are either almost self-normalizing or ascendant. If K is a finite normal subgroup of G, then every cyclic subgroup of G/K is either almost self-normalizing or ascendant. More precisely, for each element $g \notin \mathbf{Gru}(G)$ the centralizer $C_{G/K}(gK)$ is finite.*

Proof. At once we note that $K \leqslant \mathbf{Gru}(G)$. In fact, since K is finite, every element g of K has only finitely many conjugates in G. Then $C_G(g)$ has finite index in G, in particular, $C_G(g)$ is infinite. As we have seen above, in this case the index $|N_G(\langle g\rangle) : \langle g\rangle|$ is infinite, which show that $\langle g\rangle$ is ascendant in G.

Let g be an arbitrary element of $G \setminus \mathbf{Gru}(G)$ and suppose that the index $|N_{G/K}(\langle gK\rangle) : \langle gK\rangle|$ is infinite. Put $V/K = N_{G/K}(\langle gK\rangle)$ and $X = \langle g, K\rangle$. By its choice X is a normal subgroup of infinite subgroup V. Since g has finite order, a subgroup X is finite. It follows that $C_V(X)$ has finite index in V, in particular, it is infinite. An inclusion $C_V(X) \leqslant C_V(g)$ shows that $C_G(g)$ is infinite and we obtain a contradiction. This contradiction shows that a cyclic subgroup $\langle gK\rangle$ has finite index in its normalizer. It follows that $N_{G/K}(\langle gK\rangle)$ is finite and hence $C_{G/K}(gK)$ is finite.

If $g \in \mathbf{Gru}(G)$, then a cyclic subgroup $\langle g\rangle$ is ascendant in G. Therefore, $\langle gK\rangle$ is ascendant in G/K. In other words, every cyclic subgroup of G/K is either almost self-normalizing or ascendant. □

Now we can describe the general structure of locally finite groups, whose cyclic subgroups are either almost self-normalizing or ascendant.

2. The proofs of the main results

Proof of Theorem A. Put $B = \mathbf{Gru}(G)$ and let Q be a Sylow π-subgroup of G, $\pi = \Pi(G) \setminus \sigma$. Lemma 5 shows that G/B is finite and a group G has a semidirect decomposition: $G = Q \leftthreetimes R$, where R is a Chernikov Sylow σ-subgroup of G.

(v) follows from Lemma 9.

Let $y \in C_R(Q)$, then $Q \leqslant C_G(y)$, in particular, $C_G(y)$ is infinite. As we have seen above in this case $y \in \mathbf{Gru}(G)$.

Suppose that Q is infinite and consider a factor-group G/C, where $C = C_R(Q)$ is a Sylow σ-subgroup of B. We have $G/C = QC/C \leftthreetimes R/C$.

Every cyclic subgroup of QC/C is ascendant in G/C. Let $zC \notin QC/C$ and suppose that $C_{G/C}(zC)$ is infinite. Then $C_{QC/C}(zC) = Z/C$ is infinite. We have $Z = D \times C$, where D is a Sylow π-subgroup of Z. The fact, that Z/C is infinite, implies that D is infinite. For every element $d \in D$ we have $[z, d] \in C$. On the other hand, $D \leqslant Q$ and Q is a normal subgroup of G, therefore $[z, d] \in Q$, that is $[z, d] \in C \cap Q = \langle 1 \rangle$. This shows that $D \leqslant C_G(z)$, in particular $C_G(z)$ is infinite. However it contradicts to (v). This contradiction shows that $C_{G/C}(zC)$ is finite for every element $zC \notin QC/C$. It is not hard to prove that in this case a subgroup $\langle zC \rangle$ has finite index in its normalizer. In other words, every cyclic subgroup of G/C is ascendant or almost self-normalizing. Furthermore, $QC/C = \mathbf{Gru}(G/C)$ and we can use Lemma 7. By Lemma 7 QC/C and hence Q is nilpotent-by-finite.

If Q is finite, then $\mathbf{Gru}(G)$ is a Chernikov subgroup, in particular, it is abelian-by-finite. □

Now we can obtain some additional information about the structure of factor-group $G/\mathbf{Gru}(G)$.

Proof of Corollary A1. Put $B = \mathbf{Gru}(G)$, then $B = P \times R$, where P is a Sylow p-subgroup of B and R is a Sylow p'-subgroup of B. We have $G = Q \leftthreetimes R$, where R is a Chernikov Sylow π-subgroup of G, $\pi = \Pi(G) \setminus \sigma$. Consider a factor-group G/C, where $C = C_R(Q) = R \cap B$ is a Sylow σ-subgroup of B. We have $G/C = QC/C \leftthreetimes R/C$. As in a proof of Theorem A we can show that every cyclic subgroup of QC/C is ascendant in G/C and every cyclic subgroup of G/C, which does not lie in QC/C, is almost self-normalizing. Furthermore, $QC/C = \mathbf{Gru}(G/C)$, hence we can apply Lemma 7 and prove this result. □

Proof of Corollary A2. Put $B = \mathbf{Gru}(G)$. Then $B = P \times R$, where P is a Sylow p-subgroup of B and R is a Sylow p'-subgroup of B. By our conditions R is finite. Consider a factor-group G/R. Suppose that the Sylow p'-subgroup S/R of $\mathbf{Gru}(G/R)$ is non-identity. Let $R \neq xR \in S/R$. Since Sylow p-subgroup of $\mathbf{Gru}(G/R)$ includes BR/R, $x \notin B$. Lemma 10 shows that $C_{G/R}(xR)$ is finite. On the other hand, since $\mathbf{Gru}(G/R)$ is locally nilpotent, $PR/R \leqslant C_{G/R}(xR)$ and we obtain a contradiction. This contradiction shows that $\mathbf{Gru}(G/R)$ is a p-subgroup. If $xR \notin \mathbf{Gru}(G/R)$ and xR is a p'-element, then $x \notin B$ and $C_G(x)$ is finite by Lemma 9. Using Lemma 10 we obtain that $C_{G/R}(xR)$ is finite. The application of Lemma 8 yields that Sylow q-subgroup of $(G/R)/\mathbf{Gru}(G/R)$ is cyclic whenever $q \notin \{2, p\}$, and Sylow 2-subgroup of $(G/R)/\mathbf{Gru}(G/R)$ is cyclic or generalized quaternion group, if $p \neq 2$. Remain to note that the Sylow q-subgroups of F are isomorphic to Sylow q-subgroups of $(G/R)/\mathbf{Gru}(G/R)$, because $\mathbf{Gru}(G/R)$ is a p-subgroup. □

Proof of Corollary A3. Let $p, r \in \mathbf{Sp}(G)$, $B = \mathbf{Gru}(G)$. Then $B = P \times R \times S$, where P is a Sylow p-subgroup of B, R is a Sylow r-subgroup of B, S is a Sylow $\{p, r\}'$-subgroup of B. Consider a factor-group G/D, where $D = R \times S$. Let $D \neq xD$ and assume that xD is a p'-element. In this case $xD \notin PD/D = B/D$. Let $Z/D = C_{B/D}(xD)$. We have $Z = C \times D$, where C is a Sylow p-subgroup of Z. By choice of C we have $[x, C] \leqslant D$. On the other hand, $C \leqslant P$ and since P is G-invariant, then $[x, C] \leqslant P$. Thus $[x, C] \leqslant D \cap P = \langle 1 \rangle$, which shows that $C_{B/D}(xD) = C_B(x)D/D$. The fact that $x \notin B$ together with Lemma 9 shows that $C_B(x)$ is finite, so that $C_{B/D}(xD)$ is finite. Since G/B is finite, $C_{G/D}(xD)$ is likewise finite.

If we suppose now that $\mathbf{Gru}(G/D)$ is a not p-subgroup, then it contains some p'-element yD. Since $\mathbf{Gru}(G/D)$ is locally nilpotent, $B/D \leqslant C_{G/D}(xD)$, in particular, $C_{G/D}(xD)$ is infinite, which contradicts to above proved. This contradiction proves that $\mathbf{Gru}(G/D)$ is a p-subgroup.

The application of Lemma 8 yields that the Sylow q-subgroup of the factor-group $(G/D)/\mathbf{Gru}(G/D)$ is cyclic whenever $q \notin \{2, p\}$, and Sylow 2-subgroup of $(G/D)/\mathbf{Gru}(G/D)$ is cyclic or generalized quaternion group, if $p \neq 2$. Since $\mathbf{Gru}(G/D)$ is a p-subgroup, the Sylow q-subgroups of F are isomorphic to Sylow q-subgroups of $(G/D)/\mathbf{Gru}(G/D)$.

Consider now a factor-group G/PS. Using the above arguments, we obtain that the Sylow p-subgroup of F is cyclic whenever $p \neq 2$, and Sylow p-subgroup of F is cyclic or generalized quaternion group, if $p = 2$. □

Proof of Theorem B. Put $B = \mathbf{B}(G)$. Repeating almost word to word the proof of Theorem A, we will prove assertions (i)-(v). If Q is infinite, then using arguments of a proof of Lemma 7, we obtain that Q is nilpotent-by-finite. Let W be a nilpotent normal subgroup of Q, having finite index. Then $Q = WH$ for some finite subgroup H. An inclusion $H \leqslant \mathbf{B}(G)$ implies that H is subnormal in G. Then WH is a nilpotent subgroup [8, Lemma 4]. A subgroup $R \cap B$ is Chernikov, and being a Baer group, it is central-by-finite [8, Corollary 1 to Lemma 4]. In particular, $R \cap B$ is nilpotent. Moreover, let D be a divisible part of $R \cap B$. Then $D \leqslant \zeta(R \cap B)$. Let T be a finite subgroup such that $R \cap B = TD$. Clearly T is normal in $R \cap B$, and hence in B. Since G/B is finite, $T^G = U$ is also finite. Then $(R \cap B)/U = UD/U$ is divisible. Therefore, using Lemma 6 we obtain (vi). □

Proof of Corollary B1. Let L be a locally nilpotent radical of G. Since $\mathbf{B}(G) \leqslant L$, G/L is finite by Theorem B. In particular, L is infinite.

Suppose first that L is not Chernikov. Then $C_L(g)$ is infinite by Lemma 3. It follows that every cyclic subgroup of L is G-invariant. Then every subgroup of L is G-invariant. In particular, $L = \mathbf{B}(G)$. Furthermore, L is a Dedekind group. Then either L is abelian, or $L = Q \times E \times R$, where Q is a quaternion group, E is elementary abelian 2-subgroup and R is an abelian $2'$-subgroup (see, for example, [14, Theorem 6.1.1]). We note here, that E must be finite. In fact, every non-identity cyclic subgroup of E is G-invariant, and being a subgroup of order 2, lies in the center of G. Hence if we suppose that E is infinite, then $\zeta(G)$ is infinite. But in this case $C_G(g)$ is infinite for each element $g \in G$, so that every cyclic subgroup of G is normal in G, and G must be Dedekind.

Let H be an infinite subgroup of L. If $x \in C_G(H)$, then $H \leqslant C_G(x)$, so that $C_G(x)$ is infinite and $x \in \mathbf{B}(G)$. Hence $C_G(H) \leqslant L$, in particular, $C_G(L) \leqslant L$. The fact, that every subgroup of L is G-invariant, implies that $G/C_G(L)$ is abelian (see, for example, [18, Theorem 1.5.1]).

If Sylow 2-subgroup D of L is infinite, then by above proved L is abelian. If we suppose that $\Omega_1(D)$ is infinite, then using the above arguments, we obtain that G is a Dedekind group. This implies that $\Omega_1(D)$ is finite. Then D is a Chernikov group. Being infinite, D includes a quasicyclic 2-subgroup W. As we have seen above, W is G-invariant and $C_G(W) \leqslant L$. Furthermore, $G/C_G(W)$ is isomorphic to a periodic subgroup of $Aut(W)$. We recall that $Aut(W)$ is isomorphic to the multiplicative group of a ring of integer 2-adic numbers (see, for example, [5, Section 113, Example 3]). Recall also, that a periodic subgroup of the multiplicative group of a ring of integer 2-adic numbers has order 2 (see, for example, [5, Section 128, Example 2]). Thus in this case the factor-group G/L has order 2.

Suppose now that there exists an odd prime p such that Sylow p-subgroup P of L is infinite. By above proved P is abelian. Assume that $\Omega_1(P)$ is infinite. Then again we have an inclusion $C_G(\Omega_1(P)) \leqslant L$. Since every subgroup of $\Omega_1(P)$ is G-invariant, $G/C_G(\Omega_1(P))$ is a cyclic group, whose order divides $p-1$. Hence G/L is a cyclic group, whose order divides $p-1$. Suppose now that $\Omega_1(P)$ is finite. Then P is a Chernikov group. In this case the orders of elements of P are not bounded. Since every subgroup of P is G-invariant, G/P is isomorphic to a periodic subgroup of the multiplicative group of a ring of integer p-adic numbers (see, for example, [18, Theorem 1.5.6]). We recall that a periodic subgroup of the multiplicative group of a ring of integer p-adic numbers is cyclic and its order divides $p-1$ (see, for example, [5, Section 128, Example 2]). Thus in this case the factor-group G/L is cyclic and its order divides $p-1$.

Suppose now that the Sylow p-subgroups of L are finite for all primes p. Since L is not Chernikov subgroup, $\Pi(L)$ is infinite. Let $\sigma = \Pi(G/L)$, then σ is finite by Theorem B. It follows that Sylow σ-subgroup K of L is finite. Theorem B shows that $C_G(g)$ is finite for each $g \notin L$. Lemma 10 implies that $C_{G/K}(gK)$ is finite for each $g \notin L$. Obviously L/K is normal Sylow σ'-subgroup of G/K, so that $G/K = L/K \leftthreetimes S/K$, where S/K is a finite Sylow σ-subgroup of G/K. Since $\Pi(L/K)$ is infinite, we can find in L/K a finite σ'-subgroup R/K such that $R/K \cap C_{G/K}(gK) = \langle 1 \rangle$ for every element $gK \in S/K$ (recall that every subgroup of L/K is G-invariant). Taking into account the fact, that S/K is abelian ($S/K \cong G/L$) and Satz V.8.15 of a book [9], we obtain that every Sylow subgroup of S/K is cyclic, and therefore S/K is cyclic.

Consider now the case when L is a Chernikov subgroup. By Theorem B the Baer radical $\mathbf{B}(G)$ is nilpotent. Corollary 1 shows that every element of $\mathbf{B}(G)$ has infinite centralizers. Then every cyclic subgroup of $\mathbf{B}(G)$ is G-invariant, and therefore every subgroup of $\mathbf{B}(G)$ is G-invariant. As above we can shows that $\mathbf{B}(G)$ includes a centralizer of each its infinite subgroup. Since $G/\mathbf{B}(G)$ is finite, $\mathbf{B}(G)$ is infinite. Being Chernikov, $\mathbf{B}(G)$ includes a quasicyclic p-subgroup for some prime p. Using the above arguments, we obtain that $G/\mathbf{B}(G)$ is cyclic and its order divides $p-1$. Furthermore, if $p = 2$, then $\mathbf{B}(G)$ is abelian and $G/\mathbf{B}(G)$ has order 2. $\square$

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