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Camina groups with few conjugacy classes

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Abstract. Let G be a finite group having a proper normal subgroup K such that the conjugacy classes outside K coincide with the cosets of K. The subgroup K turns out to be the derived subgroup of G, so the group G is either abelian or Camina. Hence, we propose to classify Camina groups according to the number of conjugacy classes contained in the derived subgroup. We give the complete characterization of Camina groups when the derived subgroup is made up of two or three conjugacy classes, showing that such groups are all Frobenius or extra–special.

Introduction

Let G be a finite group, and K a proper normal subgroup of G. Then there are two natural partitions of G. One partition is given by the conjugacy classes, while the other one by the cosets of K. We assume that these two partitions coincide in $G \setminus K$. In other words, we assume that $cl(x) = xK$ for all $x \in G \setminus K$. This notion was introduced and investigated by the second author in [5], where he proposed to call such a group *Con–Cos*, outlining the coincidence of conjugacy classes with cosets, apart from those contained in K.

The subgroup K turns out to be the derived subgroup of G, so that G is a *Camina group*, unless $K = 1$, and in such a case the group G is abelian. Camina groups were introduced by A.R. Camina in [1], and they have been extensively studied by Macdonald, Chillag, Mann, Scoppola and Dark. For a complete list of papers on Camina groups, we refer the reader to [3].

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If G is abelian, $K = G' = 1$, and there is a complete coincidence of conjugacy classes and cosets in the whole group. If G is a Camina group (i.e., a non–abelian Con–Cos group), the Camina kernel $K = G'$ is a union of a number of conjugacy classes greater than one. In this paper we study the cases when K has 'few' conjugacy classes, precisely when K is the union of two or three conjugacy classes. Our main results are the following.

Theorem I. Let G be a finite group. Then, G is a Camina group and G' is the union of two conjugacy classes if and only if either G is a Frobenius group with Frobenius kernel Z_p^r and Frobenius complement Z_{p^r-1}, or G is an extra–special 2–group.

Theorem II. Let G be a finite group. Then, G is a Camina group and G' is the union of three conjugacy classes if and only if either G is a Frobenius group with Frobenius kernel Z_p^r and Frobenius complement $Z_{(p^r-1)/2}$, or G is an extra–special 3–group, or G is Frobenius with Frobenius kernel Z_3^2 and Frobenius complement Q_8.

In Section 1 we give the basic properties of Con–Cos groups, explain the relations with Camina groups, and recall some fundamental facts about Camina groups. In Sections 2 and 3, we prove, respectively, Theorems I and II. In Section 4 we give some examples of Camina groups with the kernel being the union of a number of conjugacy classes greater than three.

Our notation is as follows. We let 1 denote the identity group, Z_n the cyclic group of order n, D_n the dihedral group of order n, and Q_8 the quaternion group of order 8. If x is an element of a group, we let $cl(x)$ denote the conjugacy class of x. Moreover, when a semidirect product $L \rtimes M$ is Frobenius, this is explicitly mentioned.

1. Con–Cos groups and Camina groups

Definition 1.1. *A finite group G is Con–Cos when there exists a normal subgroup $K < G$ such that $xK = cl(x)$, for all $x \in G \setminus K$.*

Any abelian group G is Con–Cos, since $cl(x) = \{x\}$ for all $x \in G$, and we can take $K = 1$. Some of the smallest examples of non–abelian Con–Cos groups are the dihedral group D_8 and the quaternion group Q_8 with $K = Z_2$, the dihedral groups D_{2n}, for n odd, with $K = Z_n$, the alternating group A_4 with $K = Z_2^2$, and the Frobenius group $Z_5 \rtimes Z_4$ with $K = Z_5$.

Lemma 1.2. *Let G be a finite group, and K a proper normal subgroup of G. Then the following conditions are equivalent:*

- *$xK \subseteq cl(x)$ for all $x \in G \setminus K$;*
- *$K \subseteq \{[x, y] : y \in G\}$, for any $x \in G \setminus K$.*

Moreover, if the above equivalent conditions hold, then $K \subseteq G'$.

Proof. Let $k \in K$, and $x \in G \setminus K$. If $xk \in cl(x)$, then there exists $y \in G$ such that $xk = y^{-1}xy$, hence $k = [x, y]$. Conversely, if there exists $y \in G$ such that $k = [x, y]$, then $xk \in cl(x)$. Moreover, if the equivalent conditions hold for G, then for any $x \in G \setminus K$, the set $\{[x, y] : y \in G\}$ is obviously contained in G', so that $K \subseteq G'$. □

Remark 1.3. The condition $K \subseteq G'$ does not imply that the two equivalent conditions of Lemma 1.2 hold true. For instance, consider the group $G = Z_3 \rtimes Z_4 = \langle a, t : a^3 = t^4 = 1, t^{-1}at = a^2 \rangle$ and take $K = G' = \{1, a, a^2\}$. Then, $t^2 \in Z(G)$, hence $t^2K \not\subseteq cl(t^2)$.

Lemma 1.4. *Let G be a finite group, and K a proper normal subgroup of G. Then the following conditions are equivalent:*

- *$cl(x) \subseteq xK$, for all $x \in G$;*
- *$G' \subseteq K$.*

Proof. Assume the first condition, and consider the factor group G/K. Let $x, y \in G$. Then $(yK)^{-1}(xK)(yK) = (y^{-1}xy)K \subseteq xK$, so that G/K is abelian, and $G' \subseteq K$. Conversely, assume the second condition, and let $x \in G$. For any $y \in G$, we have $[x, y] \in K$, hence $y^{-1}xy \in xK$. □

Remark 1.5. The inclusion $G' \subseteq K$ is not generally an equality. Consider the example of D_8 with $K = Z_4$.

Proposition 1.6. *Let G be a Con–Cos group, and let K be a proper normal subgroup of G satisfying the condition of Definition 1.1. Then $K = G'$, and $G' = \{[x, y] : y \in G\}$, for any $x \in G \setminus G'$. Therefore, a group G is Con–Cos if and only if $G' < G$ and $cl(x) = xG'$ for all $x \in G \setminus G'$.*

Proof. It is a direct consequence of Lemmas 1.2 and 1.4. □

Remark 1.7. It is not true that $K = G'$ implies $cl(x) = xK$ for all $x \in G \setminus K$. A counterexample is the group $Z_3 \rtimes Z_4$ given above.

The previous proposition shows that if G is a non–abelian Con–Cos group, then G is a Camina group. Indeed, a *Camina pair* is a pair (G, K), where G is a finite group and K is a subgroup of G such that $1 < K < G$ and the equivalent conditions of Lemma 1.2 hold. Such a subgroup K is called *Camina kernel.* Moreover, a finite group G is a *Camina group* when

(G, G') is a Camina pair (see [1] and [3]). Therefore, by Proposition 1.6, a Camina group is Con–Cos, and conversely a non–abelian Con–Cos group is a Camina group. We complete these claims in the following proposition.

Proposition 1.8. *Let G be a finite group. The following conditions are equivalent:*

- *G is Con–Cos;*
- *G is either a Camina group, or an abelian group;*
- *$xG' = cl(x)$, for all $x \in G \setminus G'$;*
- *$G' = \{[x, y] : y \in G\}$, for any $x \in G \setminus G'$.*

Therefore, our study of Con–Cos groups converts to that of Camina groups.

Camina proved that if (G, K) is a Camina pair, then one of the following three conditions holds (see [1]):

- G is Frobenius with Frobenius kernel K
- K is a p–group, for some prime p
- G/K is a p–group, for some prime p.

A well known result about Camina pairs and Camina groups is about the center subgroup. We prove here again, since it is of fundamental importance for the proofs in the sequel.

Lemma 1.9. *Let (G, K) be a Camina pair. Then, $Z(G) \subseteq K$. In particular, if G is Camina, then $Z(G) \subseteq G'$.*

Proof. Let $z \in Z(G) \setminus 1$. If $z \notin K$, then we have $zK \subseteq cl(z) = \{z\}$. This would imply $|K| = 1$, which is not possible. □

A wide subclass of Camina groups is given by Frobenius groups with abelian complement. Indeed, if $G = N \rtimes H$ is a Frobenius group with abelian complement, the Frobenius complement H has to be cyclic, by the structure theorem on Frobenius groups (see, e.g., Section 10.5 of [6]).

Proposition 1.10. *Let G be a Frobenius group with Frobenius kernel N and cyclic Frobenius complement H. Then, G is a Camina group with N as Camina kernel and has trivial center.*

Proof. Note first that $G' \subseteq N$, since G/N is abelian. Let $h \in H \setminus 1$. Then $cl(h) \subseteq hN$, by Lemma 1.4. On the other hand, $C_G(h) = H$, since the center of G is trivial. Therefore, $|cl(h)| = |G|/|H| = |N|$, so that $cl(h) = hN$. The same occurs for any other $x \in G \setminus N$, since we can replace H with its conjugate subgroup containing x. □

Another wide subclass of Camina groups is given by extra–special p–groups (for a complete description of such groups, see Section 5.3 of [6]).

Proposition 1.11. *Let G be an extra–special p–group, with p prime. Then, G is a Camina group with Camina kernel equal to the center subgroup.*

Proof. Let $K = G'$. Since G is an extra–special p–group, we have $K = Z(G) = Z_p$. Then $|K| = p$, and K contains exactly p conjugacy classes. For any $x \in G \setminus K$, we have $cl(x) \subseteq xK$, by Lemma 1.4. On the other hand, it is easy to see that if $|G| = p^{2n+1}$ then $|C_G(x)| = p^{2n}$. Hence $|cl(x)| = p$, and $cl(x) = xK$. □

Let G be a Con–Cos group, that is an abelian group or a Camina group. Consider the number n of conjugacy classes contained in $K = G'$. Then, $n = 1$ if and only if G is abelian. Otherwise, G is a Camina group, and we propose the following definition.

Definition 1.12. *A n–Camina group, where $n \geq 2$, is a Camina group G with G' being the union of n conjugacy classes.*

Taking into account this definition, we can briefly say that Theorems I and II give the classification, respectively, of 2–Camina and 3–Camina groups.

2. Camina kernels with two conjugacy classes

Let G be a 2–Camina group, so that $G' = 1 \cup cl(a)$, for some $a \in G$ with $a \notin 1$. Then, it is easy to see that the subgroup G' is an abelian elementary group. On the other hand, we remark that groups with a normal subgroup which is the union of two conjugacy classes have been thoroughly studied by Shahryari and Shahabi in [8].

We separate the proof of Theorem I in the following two propositions, according to whether the center of G is trivial or not.

Proposition 2.1. *Let G be a 2–Camina group with $Z(G) = 1$. Then, G is a Frobenius group $Z_p^r \rtimes Z_{p^r-1}$, for some prime p and some $r \geq 1$. Conversely, such Frobenius groups are 2–Camina.*

Proof. Since G' is the unique minimal normal subgroup of G, by Theorem 2.1 (a) of [8], or by Lemma 12.4 of [4], G is Frobenius with elementary abelian kernel G'. Moreover, the complement H is abelian, since it is isomorphic to G/G', hence it is cyclic. Therefore, $G' = Z_p^r$ for some prime p and some $r \geq 1$, and $H = Z_h$ for some $h \geq 2$. As $G' \setminus 1$ is a single conjugacy class, the action of H on G' by conjugation has to be transitive, so that $|H| = p^r - 1$.

Conversely, it is easy to see that Frobenius group $Z_p^r \rtimes Z_{p^r-1}$ are 2–Camina. □

Proposition 2.2. *Let G be 2–Camina and $Z(G) \neq 1$. Then $G' = Z(G) = Z_2$, and G is an extra–special 2–group. Conversely, any extra–special 2–group is a 2–Camina group.*

Proof. Let $a \in Z(G) \setminus 1$. Then $a \in G'$, by Lemma 1.9, and $cl(a) = \{a\}$. Hence, $G' = \{1, a\}$, and $G' = Z(G) = Z_2$. Now, let $B = G/G'$. We have just to show that B is a 2–group. Let $x, y \in G$. We have $[x, y^2] = [x, y]^2 = 1$, so that y^2 commutes with all the elements of G (see also [6]). Hence $y^2 \in Z(G) = G'$, for all $y \in G$.

Conversely, a straight verification shows that any extra–special 2–group is 2–Camina. □

3. Camina kernels with three conjugacy classes

The groups with a normal subgroup which is the union of three conjugacy classes have been deeply studied by Shahryari and Shahabi in [7]. In the following proposition, we extract the relevant results we need to our aims.

Proposition 3.1. *Let G be a group with a normal subgroup K which is the union of three conjugacy classes. Then, there are only three possible cases:*

1. *K is an abelian elementary group of odd order, i.e. $K = Z_p^r$, for some odd prime p and some integer $r \geq 2$, the two conjugacy classes different from 1 have the same size, and K is a minimal normal subgroup of G, or*

2. *K is a metabelian p–group, i.e. $K'' = 1$, and K' is elementary abelian, $K' = Z_p^s$, for some prime p and some integer $s \geq 1$, or*

3. *K is Frobenius with Frobenius kernel K', which is elementary abelian, $K' = Z_p^s$, and Frobenius complement of prime order, Z_q, for some primes p and q and some integer $s \geq 1$, with $q \mid p^s - 1$.*

Proof. See Cases A, B, and C of [7]. □

We first consider the possible order of the center of a 3–Camina group.

Lemma 3.2. *Let G be a 3–Camina group. Then $|Z(G)| \leq 3$, and if $|Z(G)| = 3$ then $G' = Z(G)$.*

Proof. By Lemma 1.9, $Z(G) \subseteq G'$. If $|Z(G)| = 3$, then G' contains the three conjugacy classes $cl(x)$, for $x \in Z(G)$, hence $G' = Z(G)$. If $|Z(G)| > 3$, then G' would contain more than three conjugacy classes, which is not possible. □

We now prove Theorem II, considering separately the three possible cases of Proposition 3.1.

Proposition 3.3. *Let G be a 3–Camina group, and assume that case (1) of Proposition 3.1 holds for $K = G'$. If $Z(G) = 1$, then G is a Frobenius group $Z_p^r \rtimes Z_{(p^r-1)/2}$, for some odd prime p and some $r \geq 1$; whereas, if $Z(G) \neq 1$, then G is an extra–special 3–group. Conversely, such Frobenius groups and extra–special 3–groups are 3–Camina groups.*

Proof. When $Z(G) = 1$, the derived subgroup G' is necessarily the unique minimal normal subgroup of G. Hence, by Lemma (12.3) of [4], G is Frobenius with Frobenius kernel G' and cyclic Frobenius complement. Since $G' \setminus 1$ contains exactly two conjugacy class of the same size, the order of the complement has to be $(p^r - 1)/2$. Also, this implies that p is odd.

When the center is not trivial, it is elementary abelian, since G' is such. Since its order is odd, by Lemma 3.2 we deduce $Z(G) = Z_3$, and $G' = Z(G)$. Using the same arguments of Proposition 2.2, it follows at once that G is an extra–special 3–group.

The converse claim is easy to verify. □

Before dealing with the cases (2) and (3) of Proposition 3.1, we prove a lemma holding in both cases.

Lemma 3.4. *Let G be a 3–Camina group, with $K = G'$ not abelian. Then K' is a minimal normal subgroup of G, and G/K' is a 2–Camina group. Hence there are only two possible cases:*

1. *G/K' is a Frobenius group $Z_r^t \rtimes Z_{r^t-1}$, for some prime r and some positive integer t, or*
2. *G/K' is an extra–special 2–group.*

Proof. Since K is a union of three conjugacy classes and $K' \neq 1$, then K' is a union of two conjugacy classes, hence it is minimal normal. On the other hand, G/K' is easily seen to be 2–Camina, and the two cases are the only possible ones for 2–Camina groups, by Theorem I. □

Proposition 3.5. *There does not exist any* 3*–Camina group* G*, such that case (2) of Proposition 3.1 holds for* $K = G'$*.*

Proof. Assume that G is 3–Camina, and that case (2) of Proposition 3.1 holds for $K = G'$. Note that $Z(K) \cap K' \neq 1$, since K is a p–group. Moreover, $Z(K) \cap K'$ is a normal subgroup of G contained in K'. By Lemma 3.4, we have $Z(K) \cap K' = K'$, hence $K' \subseteq Z(K) \subseteq K$, which implies $Z(K) = K'$, because K is not abelian and contains exactly three conjugacy classes. This excludes at once case (2) of Lemma 3.4. Indeed, if G/K' were an extra-special 2–group, then K/K' would be a cyclic group (of order two), and, since $K/K' = K/Z(K)$, then K would be abelian.

Therefore, assume that case (1) of Lemma 3.4 holds. Hence, $K' = Z_p^s$, $G/K' = Z_p^t \rtimes Z_{p^t-1}$, and $K/K' = Z_p^t$. So, K is a normal Sylow p–subgroup of G, and G is Frobenius with Frobenius kernel K, by Proposition 1 of [1]. Then, the Frobenius complement is a cyclic group of order $p^t - 1$. If $p \neq 2$, the order of the complement is even, and this implies that K is abelian, which is not the case. Hence we have to assume that $p = 2$. So, G is a Frobenius group $K \rtimes Z_{2^t-1}$, with $|K| = 2^{s+t}$ and $|K'| = 2^s$. Since $K' \rtimes Z_{2^s-1}$ is Frobenius too, $K' = Z(K)$, and $K' \setminus 1$ is a single conjugacy class, then we have $s = t$.

Therefore, $G = K \rtimes Z_{2^s-1}$, $|K| = 2^{2s}$, $K' = Z_2^s$, and $K/K' = Z_2^s$. Now, consider the order of the elements in K. All the elements in $K' \setminus 1$ have order 2, and if all the elements in $K \setminus K'$ have also order 2, then K would be abelian. Hence, there is an element in $K \setminus K'$ of order different from 2. Since $K \setminus K'$ is a single conjugacy class, all of its elements have order 4. Let $b \in K \setminus K'$. Then $cl(b) = K \setminus K'$, so $|cl(b)| = 2^s(2^s - 1)$, and $|C_G(b)| = 2^s$. Since $K' = Z(K) \subseteq C_G(b)$, we have $C_G(b) = K'$, which is a contradiction, because $b \in C_G(b) \setminus K'$. □

Proposition 3.6. *Let* G *be a* 3*–Camina group, and assume that case (3) of Proposition 3.1 holds for* $K = G'$*. Then* G *is Frobenius with Frobenius kernel* N *and Frobenius complement* Q_8*, where* $N = Z_3^2$*, and* $K = N \rtimes Z_2$*. Conversely, such Frobenius groups are* 3*–Camina.*

Proof. Assume that case (3) of Proposition 3.1 and case (1) of Lemma 3.4 hold. Then, using their notation, we have $K/K' = Z_q = Z_r^t$, so $t = 1$ and $r = q$. Hence, $G/K' = Z_q \rtimes Z_{q-1}$, $|G| = p^s q(q-1)$ and $|K| = p^s q$. Since $p \mid q - 1$, K is a normal Hall subgroup of G, so G is a Frobenius group with Frobenius kernel K, by Proposition 1 of [1]. Moreover, the Frobenius complement turns out to be a cyclic group of order $q-1$, an even integer, so that K is abelian, which is false (note that the case $q=2$ is not possible, because it would imply $K=G$).

By Lemma 3.4, the only remaining case to consider is when G/K' is an extra–special 2–group. In this case we have $q=2$, so K is a Frobenius group $Z_p^s \rtimes Z_2$ with $p \neq 2$. On the other hand, since K' is a normal Hall subgroup, G splits over K', hence $G = Z_p^s \rtimes H$, with H an extra–special 2–group. Note that G/K is a 2–group, G is neither a 2–group (because $Z(G)=1$, being $Z(G) \subseteq Z(K)$ and K Frobenius), nor a Frobenius group with Frobenius kernel K (otherwise, K would be abelian, being the Frobenius complement of even order). Therefore, G has property F2(2), according to [2]. Moreover, H is a Sylow 2–subgroup of class 2. Hence we can apply Theorem 5.1 of [2], deducing that G is a Frobenius group with Frobenius kernel of index 2 in K and Frobenius complement $H=Q_8$. That is, G is the Frobenius group $Z_p^s \rtimes Q_8$. If $a \in K' \setminus 1$, then $cl(a) = K' \setminus 1$, hence $|cl(a)| = p^s - 1$. Therefore $p^s - 1 \mid 8p^s$, which implies $p^s - 1 \mid 8$. Hence the only possible case is $p=3$ and $s=2$, which, indeed, gives a 3–Camina group. □

4. Camina kernels with more conjugacy classes

A classification of n–Camina groups for a fixed $n \geq 4$ seems to be very difficult to obtain. Nevertheless, here we give some examples of Camina groups, for which we know the number of conjugacy classes in the kernel. They are all generalizations of the groups we have encountered in the previous sections.

Example 4.1. Let G be a Frobenius group with abelian Frobenius kernel K of order n and cyclic Frobenius complement H of order h. Then G is a Camina group and the number of conjugacy classes in $K = G'$ is $(n-1)/h + 1$.

Indeed, we know that $h \mid n-1$ and, since K is abelian, the conjugacy classes in K are determined just by the action by conjugation of H. Since this action is fixed–point–free, for any $x \in K \setminus 1$ we have $|cl(x)| = h$.

Example 4.2. Let G be an extra–special p–group. Then, G is a p–Camina group.

Here the result is almost obvious, since $G' = Z(G) = Z_p$.

Example 4.3. Let G be a Frobenius group with Frobenius kernel Z_p^2, with p an odd prime, and Frobenius complement Q_8. Then, G is Camina, being $K = G' = Z_p^2 \rtimes Z_2$, and the number of conjugacy classes in K is $(p^2-1)/8+2$.

Note first that $8 \mid p^2-1$. If $x \in Z_p^2 \setminus 1$, then $|cl(x)| = 8$. Meanwhile, if $x \in K \setminus Z_p^2$, we have $|cl(x)| = p^2$. So, in $K \setminus 1$ there is a single conjugacy class containing all the elements of order 2, while the elements of order p are partitioned in conjugacy classes of size 8. For the structure of such groups, see also Example 1 of [2].

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