Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 9 **(2010)**. Number 2. pp. 1 – 10

A note about splittings of groups and commensurability under a cohomological point of view

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Communicated by Komarnytskyj M.Ya.

Abstract. Let G be a group, let S be a subgroup with infinite index in G and let $\mathcal{F}_SG$ be a certain $\mathbb{Z}_2G$-module. In this paper, using the cohomological invariant $E(G, S, \mathcal{F}_SG)$ or simply $\tilde{E}(G, S)$ (defined in [2]), we analyze some results about splittings of group G over a commensurable with S subgroup which are related with the algebraic obstruction "$\mathrm{sing}_G(S)$" defined by Kropholler and Roller ([8]. We conclude that $\tilde{E}(G, S)$ can substitute the obstruction "$\mathrm{sing}_G(S)$" in more general way. We also analyze splittings of groups in the case, when G and S satisfy certain duality conditions.

Introduction

Let (G, S) be a group pair, where G is a group and S is a subgroup of G. Consider the power set $\mathcal{P}G$ of G and the set $\mathcal{F}G$ of the finite subsets of G. Under boolean addition "+", $\mathcal{P}G$ is an addtive group and has a natural structure of left $\mathbb{Z}_2G$-module. It is easy to see that $\mathcal{P}G \simeq Coind_{\{1\}}^G \mathbb{Z}_2$ (denoted by $\overline{\mathbb{Z}_2G}$) and $\mathcal{F}G \simeq Ind_{\{1\}}^G \mathbb{Z}_2 \simeq \mathbb{Z}_2G$. Let $\mathcal{F}_SG := \{B \subset G \mid B \subset F.S$ for some finite subset F of $G\}$. Clearly, $\mathcal{F}_SG$ is a $\mathbb{Z}_2G$-submodule of $\mathcal{P}G$. Consider the $\mathbb{Z}_2G$-module $Ind_S^G \overline{\mathbb{Z}_2S} = \mathbb{Z}_2G \otimes_{\mathbb{Z}_2S} \overline{\mathbb{Z}_2S}$ with the

The authors participate of Project Number 04/10229-6 - supported by FAPESP. The first author is also partially supported by MEC-SESu
2000 Mathematics Subject Classification: *20J05, 20J06; 20E06.*
Key words and phrases: *Splittings of groups, cohomology of groups, commensurability.*

natural G-action of the induced module ($g.(g_1 \otimes m) = gg_1 \otimes m$). It is true that the modules $\mathbb{Z}_2 G \otimes_{\mathbb{Z}_2 S} \overline{\mathbb{Z}_2 S}$ and $\mathcal{F}_S G$ are $\mathbb{Z}_2 G$-isomorphic.

Let $\text{res}^G_{S,\mathcal{F}_S G} : H^1(G; \mathcal{F}_S G) \to H^1(S; \mathcal{F}_S G)$ be the restriction map, we denote it simply by res^G_S. When $[G : S] = \infty$, we can define $\tilde{E}(G, S) := 1 + \dim_{\mathbb{Z}_2} \text{Ker}(\text{res}^G_S)$.

As we have stated in [1], $\tilde{E}(G, S)$ is an algebraic invariant of the category $\mathcal{C}$ which objects are the group pairs (G, S) with $[G : S] = \infty$, and which morphisms are maps $\psi : ((G, S), \mathcal{F}_S G)) \to ((L, R), \mathcal{F}_S G))$ consisting of a homomorphism $\alpha : G \to L$ with $\alpha(S) \subset R$ and a homomorphism $\phi : \mathcal{F}_S G \to \mathcal{F}_S G$ such that $\phi(\alpha(g).x) = g.\phi(x)$ for all $x \in \mathcal{F}_S G$.

Some properties of $\tilde{E}(G, S)$ and its relation to the invariant end $\tilde{e}(G, S)$ defined by Kropholler and Roller in [9] were studied in [2] and [3].

Now, suppose that $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$ with generator u. Then the "obstruction" $\text{sing}_G(S)$ is defined by $\text{sing}_G(S) := \text{res}^G_S(u)$ (see [8]). When G and S are finitely generated and $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$, there is necessary and (under some additional hypotheses) sufficient condition for G to split over a commensurable with S subgroup. This condition is that $\text{sing}_G(S)$ is zero ([8]).

The purpose of this paper is to analyze some results about splittings of a group G over a commensurable with S subgroup obtained, via $\text{sing}_G(S)$, by Kropholler and Roller (given in [8]), in terms of the invariant $\tilde{E}(G, S)$. We show that $\tilde{E}(G, S)$ can replace, under less hypotheses, the obstruction $\text{sing}_G(S)$. Initially we recall some definitions and results.

1. Some results about splittings of groups

Definition 1. **(i)** *Let the groups H and K be given by presentations $H =< ger(H); rel(H) >$, $K =< ger(K); rel(K) >$, where ger denotes a set generators and rel a set of defining relations for each group. Suppose that $S \subset H$ and $T \subset K$ are subgroups with a given isomorphism $\theta : S \xrightarrow{\sim} T$. Then, the free product $H *_S K$, of H and K with amalgamated subgroup $S \simeq T$, is given by $H *_S K :=< ger(H), ger(K); rel(H), rel(K), s = \theta(s), \forall s \in S >$.*
(ii) *Let H be a group, let S and T be subgroups of H with a given isomorphism $\sigma : S \to T$. The HNN-group (or HNN extension) over base group H, with respect to $\sigma : S \simeq T$ and stable letter p, is given by $H*_{S,\sigma} =< ger(H), p; rel(H), psp^{-1} = \sigma(s), \forall s \in S >$.*

Definition 2. *A group G splits over a subgroup S if either G is a HNN-group $H*_{S,\sigma}$ for some subgroup H containing S and some monomorphism σ from S to H, or G is an amalgamated free product $H *_S K$ with $H \neq S \simeq T \neq K$.*

Definition 3 ([11])**.** *Let G be a group and let $\mathcal{P}G$ be the power set of G. Consider the following submodules of $\mathcal{P}G$: $\mathcal{F}G$ which consists of the finite subsets of G and $\mathcal{Q}G := \{A \in \mathcal{P}G : \ \forall g \in G, A + gA \in \mathcal{F}G\}$. The number of ends of G is defined by $e(G) := \dim_{\mathbb{Z}_2}(\mathcal{Q}G/\mathcal{F}G) = \dim_{\mathbb{Z}_2}(\mathcal{P}G/\mathcal{F}G)^G$.*

Example 1. ***(a)*** We have $G = \mathbb{Z} * \mathbb{Z} = \mathbb{Z} *_{\{1\}} \mathbb{Z}$, and $e(\mathbb{Z} * \mathbb{Z}) = \infty$;
(b) $\mathbb{Z}_2 * \mathbb{Z}_2 = \mathbb{Z}_2 *_{\{1\}} \mathbb{Z}_2$, and $e(\mathbb{Z}_2 * \mathbb{Z}_2) = 2$.
(c)$\mathbb{Z} = \{1\} *_{\{1\},\, id} =< \{1\}, p, psp^{-1} = s, \forall s \in \{1\} >=< p >$, and $e(\mathbb{Z}) = 2$.

Remark 1. It is known that $e(G)$ can take only the values $0, 1, 2$ or ∞ ([11], p.176). So, if $e(G) \geq 2$, then $e(G) = 2$ or ∞.

Many important results about splittings of groups, involving the classical end $e(G)$, were proved in [12] and [13] by Stallings. In the following result (see [13]), Stallings gave a complete characterization for finitely generated groups which split over some finite subgroup.

Theorem 1. *If G is a finitely generated group, then $e(G) \geq 2$ if and only if G splits over a finite subgroup.* □

We note that $e(\mathbb{Z} \oplus \mathbb{Z}) = 1$ and so $\mathbb{Z} \oplus \mathbb{Z}$ does not split over a finite subgroup, but $\mathbb{Z} \oplus \mathbb{Z}$ splits over a infinite subgroup since $\mathbb{Z} \oplus \mathbb{Z} = \langle a \rangle \oplus \langle b \rangle = \langle a, b; a.b = b.a \rangle = \langle a, b; b^{-1}.a.b = \sigma(a) \rangle = H *_{H,\, id} =$ "$\mathbb{Z} *_{\mathbb{Z},\, id}$" is a HNN-group, where $H = \langle a \rangle \simeq \mathbb{Z}$, b is the stable letter, $S = T = H$ and $\sigma = id : S \to T$.

The classical end $e(G)$ was generalized for pairs of groups (G, S) by Houghton in [7] and Scott (using another terminology) in [10]. Following the terminology from Scott, the number of ends of the pair (G, S) is given by $e(G, S) := \dim_{\mathbb{Z}_2}(\mathcal{P}(G/S)/\mathcal{F}(G/S))^G$.

Remark 2. Scott in [10] has proved many results about splittings of groups. He tried to generalize Theorem 1, due to Stallings, for groups which split over infinite subgroups. He showed that "If G splits over a subgroup S, then $e(G, S) \geq 2$" (see [10], Lemma 1.8). The converse of this result is false in general. In fact, Scott tried to prove the following result: "$e(G, S) \geq 2$ if and only if G splits over some finite extension of S," but this is also false in general. The main result obtained by Scott was:

Theorem 2 ([10], Theorem 4.1)**.** *If G and S are finitely generated groups and for any $g \in G - S$ there is a subgroup G_1 of finite index in G such that G_1 contains S but not g, then $e(G, S) \geq 2$ if and only if G has a subgroup T of finite index in G such that T contains S and T splits over S.* □

In [8], Kropholler and Roller studied the splitting of a group G over a commensurable with S subgroup which we will see in the next section.

Here we recall the definition of commensurability.

Definition 4. *Two subgroups S and T of a group G are said to be commensurable if and only if $[S : S \cap T] < \infty$ and $[T : S \cap T] < \infty$.*

Example 2. It is clear that if S is a subgroup of T with $[T : S] < \infty$, then T is commensurable with S.

2. The obstruction $\mathbf{sing_G(S)}$ and $\mathbf{\tilde{E}(G,S)}$

In this section we analyze some results obtained by Kropholler and Roller in [8], about the obstruction $\mathrm{sing}_G(S)$, under the point of view of the invariant $\tilde{E}(G, S)$.

We recall that $\mathrm{sing}_G(S)$ was defined when $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$ and we observe that $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$ is equivalent to $\tilde{e}(G, S) = 2$, where $\tilde{e}(G, S)$ denotes the invariant end defined by Kropholler and Roller in [9], which is also a generalization for pairs of groups (G, S) of the classical invariant end $e(G)$. In fact, $\tilde{e}(G, S) = 1 + \dim_{\mathbb{Z}_2} H^1(G; \mathcal{F}_S G)$ if $[G : S] = \infty$ ([9], Lemma 1.2).

Moreover, we can easily verify that $\mathrm{sing}_G(S) = 0$ if and only if $\mathrm{Ker}\,\mathrm{res}^G_S \neq 0$ and we have:

Lemma 1. *If (G, S) is a group pair with $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$, then*
(i) $\mathrm{sing}_G(S) = 0 \Leftrightarrow \tilde{E}(G, S) = 2$,
(ii) $\mathrm{sing}_G(S) \neq 0 \Leftrightarrow \tilde{E}(G, S) = 1$.

Proof. We have $[G : S] = \infty$ since $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$, and $\tilde{E}(G, S) = 1 + \dim \mathrm{Ker}\,\mathrm{res}^G_S$. Then,
(i) $\mathrm{sing}_G(S) = 0 \Leftrightarrow \mathrm{Ker}\,\mathrm{res}^G_S = H^1(G, \mathcal{F}_S G) \simeq \mathbb{Z}_2 \Leftrightarrow \tilde{E}(G, S) = 2$.
(ii) $\mathrm{sing}_G(S) \neq 0 \Leftrightarrow \mathrm{Ker}\,\mathrm{res}^G_S = 0 \Leftrightarrow \tilde{E}(G, S) = 1$. □

The following result presents a necessary condition for G to split over a commensurable with S subgroup, which was proved in [8], and that can be adapted to the invariant $\tilde{E}(G, S)$, by means of the last lemma.

Proposition 1 ([8], Lemma 2.4). *Let (G, S) be a group pair with finitely generated S and G. Suppose that $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$. If G splits over a commensurable with S subgroup, then $\tilde{E}(G, S) = 2$.* □

Motivated by this fact and considering the invariant $\tilde{E}(G, S)$ defined without the restriction $H^1(G, \mathcal{F}_S G) \simeq \mathbb{Z}_2$, we believed that it is possible, through the invariant $\tilde{E}(G, S)$, to extend the result of the last proposition, removing the assumption $H^1(G, \mathcal{F}_S G) \simeq \mathbb{Z}_2$. In fact, this is possible (see Theorem 3 bellow), and the proof is similar to that given in [8], uses the following lemmas, which proofs have been adapted to the invariant, without the use of the hypothesis $H^1(G, \mathcal{F}_S G) \simeq \mathbb{Z}_2$.

Lemma 2. *Let (G, S) be a group pair with finitely generated S and G. The following conditions are equivalent:*
(i) $\tilde{E}(G, S) \geq 2$
(ii) *There exists $[B] = B + \mathcal{F}_S G \in (\dfrac{\mathcal{P}G}{\mathcal{F}_S G})^G$ (i.e., $B + gB \in \mathcal{F}_S G, \forall g \in G$) such that $[B] \neq [\emptyset], [B] \neq [G]$ and $SB = B$.* $\square$

Proof. Let $F \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_2$ be a $\mathbb{Z}_2 G$ projective resolution of $\mathbb{Z}_2$. Then $F \to \mathbb{Z}_2$ is a $\mathbb{Z}_2 S$ projective resolution of $\mathbb{Z}_2$, since $\mathbb{Z}_2 G$ is a free $\mathbb{Z}_2 S$-module. Consider the exact sequence

$$0 \longrightarrow \mathcal{F}_S G \overset{k}{\hookrightarrow} PG \longrightarrow \frac{PG}{\mathcal{F}_S G} \longrightarrow 0.$$

We have the following commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(F, \mathcal{F}_S G) \arrow[r] \arrow[d] & \mathrm{Hom}_G(F, PG) \arrow[r] \arrow[d] & \mathrm{Hom}_G(F, \frac{PG}{\mathcal{F}_S G}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_S(F, \mathcal{F}_S G) \arrow[r] & \mathrm{Hom}_S(F, PG) \arrow[r] & \mathrm{Hom}_S(F, \frac{PG}{\mathcal{F}_S G}) \arrow[r] & 0.
\end{tikzcd}$$

Hence, mapping the functor $H^*(-)$, and recalling the definition of cohomology group, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^0(G; \mathcal{F}_S G) \arrow[r] \arrow[d] & H^0(G; PG) \arrow[r] \arrow[d, "i"] & H^0(G; \frac{PG}{\mathcal{F}_S G}) \arrow[r, "\delta"] \arrow[d, "j"] & H^1(G; \mathcal{F}_S G) \arrow[r] \arrow[d, "res_S^G"] & \cdots \\
0 \arrow[r] & H^0(S; \mathcal{F}_S G) \arrow[r] & H^0(S; PG) \arrow[r] & H^0(S; \frac{PG}{\mathcal{F}_S G}) \arrow[r, "\rho"] & H^1(S; \mathcal{F}_S G) \arrow[r] & \cdots
\end{tikzcd}$$

We have in (i) and (ii) that $[G : S] = \infty$, and so $H^0(G; \mathcal{F}_S G) = (Ind_S^G PS)^G = 0$. By Shapiro's lemma $(PG)^G \simeq H^0(G; PG) \simeq \mathbb{Z}_2$ and $H^1(G; PG) = 0$. So we obtain:

$$\begin{tikzcd}
 & 0 \arrow[r] & (PG)^G \arrow[r, "\beta"] \arrow[d, "i"] & (\frac{PG}{\mathcal{F}_S G})^G \arrow[r, "\delta"] \arrow[d, "j"] & H^1(G; \mathcal{F}_S G) \arrow[r] \arrow[d, "res_S^G"] & 0 \\
0 \arrow[r] & (\mathcal{F}_S G)^S \arrow[r, hook] & (PG)^S \arrow[r, "\alpha"] & (\frac{PG}{\mathcal{F}_S G})^S \arrow[r, "\rho"] & H^1(S; \mathcal{F}_S G) \arrow[r] & \cdots
\end{tikzcd}$$

Suppose now that (i) is true. If $\tilde{E}(G,S) = 1 + \dim \operatorname{Ker} \operatorname{res}_S^G \geq 2$, there exists $u \in H^1(G; \mathcal{F}_S G)$, $u \neq 0$ such that $\operatorname{res}_S^G u = 0$. Since δ is surjective, there exists $[B_0] \in (\frac{PG}{\mathcal{F}_S G})^G$ such that $u = \delta[B_0]$, with $[B_0] \notin \operatorname{Im} \beta = \{[\emptyset], [G]\}$ since $\delta[B_0] \neq 0$ and Im $\beta = \operatorname{Ker} \delta$. Using the commutativity of the diagram, we obtain $\rho(j[B_0]) = (\operatorname{res}_S^G \circ \delta)[B_0] = \operatorname{res}_S^G u = 0$. Hence $[B_0] = j[B_0] \in \operatorname{Ker} \rho = \operatorname{Im} \alpha$, and therefore there exists $B \in (PG)^S$ such that $[B] = \alpha(B) = [B_0]$. So we have $SB = B$, $[B] \in (\frac{PG}{\mathcal{F}_S G})^G$ and $[B] \notin \{[\emptyset], [G]\}$ (since $[B] = [B_0]$), which proves (ii).

Conversely, assuming (ii), consider $[B] \in (\frac{PG}{\mathcal{F}_S G})^G$ such that $[B] \neq [\emptyset], [B] \neq [G]$ and $SB = B$. Thus $[B] \notin \operatorname{Im} \beta = \operatorname{Ker} \delta$ and therefore $u := \delta([B]) \neq 0$, with $B \in (PG)^S$. By the commutativity of the diagram we obtain $\operatorname{res}_S^G u = \operatorname{res}_S^G(\delta[B]) = (\rho \circ j)([B]) = \rho([B]) = \rho(\alpha(B)) = 0$. Therefore, $\operatorname{Ker} \operatorname{res}_S^G \neq 0$ and so $\tilde{E}(G,S) \geq 2$. $\square$

Lemma 3. *Let S and T be subgroups of G. If T is commensurable with S then $\tilde{E}(G,S) \geq 2$ if and only if $\tilde{E}(G,T) \geq 2$.*

Proof. Initially we prove that, if H and K are subgroups of G, with $K \leq H \leq G$ and $[H:K] = n < \infty$, then $\tilde{E}(G,K) \geq 2$ implies $\tilde{E}(G,H) \geq 2$.

In fact, if $\tilde{E}(G,K) \geq 2$, then there exists, by Lemma 2, $B \subset G$ such that $B + gB \in \mathcal{F}_K G, \forall g \in G, [B] \neq [\emptyset], [B] \neq [G]$ and $KB = B$. Since $[H:K] < \infty$ we have that $\mathcal{F}_K G = \mathcal{F}_H G$. Thus

$$B + gB \in \mathcal{F}_H G, \forall g \in G. \tag{1}$$

Let $H_0 = \{h_1, \ldots, h_n\}$ be a set of representatives for the left cosets hK, $h \in H$. We have

$$\begin{aligned} B + H_0 B &= B + (h_1 B \cup \ldots \cup h_n B) \subset (B + h_1 B) \cup \ldots \cup (B + h_n B) \\ &\subset F_1 H \cup \ldots \cup F_n H, \text{ [with } F_i \in FG, i = 1, \ldots, n \text{ by (1)]} \\ &= (F_1 \cup \ldots \cup F_n) H \end{aligned}$$

Therefore $B + H_0 B \in \mathcal{F}_H G$ and so $[B] = [H_0 B]$. Consider $B_0 := H_0 B$. Hence:

(a) $B_0 + gB_0 \in \mathcal{F}_H G, \forall g \in G$, because

$$\begin{aligned} B_0 + gB_0 &= H_0 B + gH_0 B = (H_0 B + B) + (B + gH_0 B) \\ &= (H_0 B + B) + B + g(h_1 B \cup \ldots \cup h_n B) \\ &= (H_0 B + B) + B + (gh_1 B \cup \ldots \cup gh_n B) \\ &\subset (B + H_0 B) + (B + gh_1 B) + \ldots + (B + gh_n B) \in \mathcal{F}_H G, \end{aligned}$$

where the last affirmation is consequence of (1).

(b) $[B_0] \neq [\emptyset]$ and $[B_0] \neq [G]$ since $[B_0] = [B]$ and $[B] \neq [\emptyset]$ and $[G]$.
(c) $HB_0 = B_0$ since $B_0 \subset HB_0$ and, using that $HH_0 \subset H$ (because $H_0 \subset H$), $H = h_1K\dot{\cup}\ldots\dot{\cup}h_nK = H_0K$, $KB = B$ and $B_0 = H_0B$, we obtain $HB_0 = H(H_0B) \subset HB = H_0KB = H_0B = B_0$. Hence B_0 satisfies Lemma 2(ii) for the group pair (G,H) and so $\tilde{E}(G,H) \geq 2$.

Now, if T is a commensurable with S subgroup of G, then $\tilde{E}(G,S) \leq \tilde{E}(G,S\cap T)$ and $\tilde{E}(G,T) \leq \tilde{E}(G,S\cap T)$ ([2], Proposition 7). Hence $\tilde{E}(G,S) \geq 2$ implies $\tilde{E}(G,S\cap T) \geq 2$ and so, by the initially proved statement, we have $\tilde{E}(G,T) \geq 2$. Similarly, $\tilde{E}(G,T) \geq 2$ implies $\tilde{E}(G,S) \geq 2$. □

Theorem 3. *Let (G,S) be a group pair with finitely generated S and G and $[G,S] = \infty$. If G splits over a commensurable with S subgroup, then $\tilde{E}(G,S) \geq 2$. Or equivalently, if $\tilde{E}(G,S) = 1$, then G does not split over any commensurable with S subgroup.*

Proof. Suppose that G splits over a commensurable with S subgroup T. Then, similarly to the proof of Lemma 2.4 in [8], we obtain a set $B \subset G$ satisfying the condition (ii) of Lemma 2, and so $\tilde{E}(G,T) \geq 2$. □

As a consequence of the Theorem, we have the following result in the duality theory. For concepts and results of duality theory see [4], [5] and [6].

Corollary 1. *If either (G,S) is a duality pair of dimension n over $\mathbb{Z}_2$ (or simply a D^n-pair) with $[G:S] = \infty$, or G is a duality group of dimension n (D^n-group) and the homological dimension $hdS \leq n-2$, then G does not split over any commensurable with S subgroup.*

Proof. This follows from the former theorem and the fact that, under the above hypotheses, $\tilde{E}(G,S) = 1$ (see [2], Proposition 8). □

Example 3. Consider $G =< a > * < b > \simeq \mathbb{Z} * \mathbb{Z}$ and $S =< aba^{-1}b^{-1} >$. We know that (G,S) is a PD^2-pair. So, by the previous corollary, G does not split over any commensurable with S subgroup. In particular, G does not split over any finite extension of S.

Remark 3. Theorem 3 can be considered as an extension of the Kropholler-Roller's result since, in the former example, $H^1(G;\mathcal{F}_SG)$ has infinite dimension (or equivalently, $\tilde{e}(G,S) = \infty$) and therefore the obstruction $\mathrm{sing}_G(S)$ is not defined. Moreover, if G and S are as in Example 3, then G does not split over any commensurable with S subgroup and the invariant end $e(G,S) = \infty > 2$. This example confirms that the Scott's initial idea (see Remark 2) is not really true.

Now, consider a group G with subgroups S and K satisfying the following conditions:
(a) *G is a finitely generated group of cohomological dimension $cdG \leq n$;*
(b) *S is a PD^{n-1}-subgroup of G;*
(c) *$H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$;*
(d) *$cdK \leq (n-1)$ for any subgroup K of G such that $(G:K) = \infty$.*
In [8], §3, the authors have proved the following result considering these hypotheses:

Proposition 2 ([8], Lemma 3.2)**.** *Suppose that $[N_G(S) : S] = \infty$, where $N_G(S)$ denotes the normaliser of S in G. Then G splits over a commensurable with S subgroup if and only if, the obstruction* $\mathrm{sing}_G(S) = 0$. □

We hoped to generalize the last proposition, removing the hypothesis **(c)** and replacing the condition $\mathrm{sing}_G(S) = 0$ by $\tilde{E}(G, S) \geq 2$. However, we prove *(*see next theorem*)* that if $[N_G(S) : S] = \infty$, then the hypothesis **(c)** is a consequence of the others and so can not be removed. We also observe that hypothesis **(b)** can be replaced by **(b')**: *S is a D^{n-1}-subgroup of G*. So we need the following lemma which proof is similar to the one of Lemma 3.1 in [8].

Lemma 4. *Let (G, S) be a group pair satisfying the conditions* **(a), (b')** *and* **(d)**. *If $[N_G(S) : S] = \infty$ then*
(i) *$[G : N_G(S)] < \infty$, and*
(ii) *$N_G(S)/S$ has an infinite cyclic subgroup of finite index.* □

Now, we can prove the mentioned result.

Theorem 4. *Let (G, S) be a group pair satisfying the conditions* **(a), (b') and (d)**. *Let C' be the dualizing module of S. If $[N_G(S) : S] = \infty$ then G is a D^n-group with dualizing module C such that $Res^G_S C \simeq C'$ and $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$.*

Proof. Under the above hypotheses we have, by the previous lemma, that $N_G(S)/S$ has a subgroup $L/S \simeq \mathbb{Z}$ with finite index such that $[G : L] < \infty$. Consider the short exact sequence $0 \to S \to L \twoheadrightarrow L/S \to 0$. Since S is a D^{n-1}-group with dualizing module C' and L/S is a PD^1-group, then L is a D^n-group with dualizing module $H^n(L; \mathbb{Z}_2 L) \simeq \mathbb{Z}_2 \otimes C' \simeq C'$ (as $\mathbb{Z}_2 L$-modules) ([4], Theorem 9.10). Hence, using that G does not have $\mathbb{Z}_2$-torsion (since $cdG \leq n$) and $[G : L] < \infty$, we conclude that (see [4], Theorem 9.9) G is a D^n-group with dualizing module $C = H^n(G; \mathbb{Z}_2 G)$ with $Res^G_L C \simeq C'$ (as $\mathbb{Z}_2 L$-modules). Thus S is a D^{n-1}-group with dualizing module $C' \simeq Res^G_S C$, where C is the dualizing module of G. Finally, using duality and Shapiro's lemma, we have $H^1(G; \mathcal{F}_S G) \simeq \mathbb{Z}_2$. □

In [8], §5, under the hypothesis that G is a PD^n-group and S is a PD^{n-1}-subgroup, the authors proved the following fact:

Theorem 5 ([8], Theorem A)**.** *Let G be a PD^n-group and S a PD^{n-1}-subgroup. Then G splits over a commensurable with S subgroup if and only if* $\mathrm{sing}_G(S) = 0$. □

Adapting this result to the invariant $\tilde{E}(G, S)$ we have:

Theorem 6. *Let G be a PD^n-group and S a PD^{n-1}-subgroup. Then G splits over a commensurable with S subgroup if and only if $\tilde{E}(G, S) = 2$.*

Proof. This follows from the previous theorem and Lemma 1. □

Example 4. In the two following cases G and S satisfy the hypotheses of the former theorem and $\tilde{E}(G, S) = 2$ ([2] Example 6 (iii) and (vi), respectively). So G splits over a commensurable with S subgroup:
(1) $G = \mathbb{Z}^k$ and $S = \mathbb{Z}^{k-1}$, $k \geq 2$;
(2) $G = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$, where $\theta : \mathbb{Z} \to Aut(\mathbb{Z} \oplus \mathbb{Z})$ is defined by $\theta(c)(a, b) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^c \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2c & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (a, 2ca + b)$, with the operation in G defined by $((a, b), c) + ((a_1, b_1), c_1) = ((a, b) + \theta(c)(a_1, b_1), c + c_1) = (a + a_1, b + b_1 + 2ca_1, c + c_1)$ e $S = \{((a, b), 0); a, b \in \mathbb{Z}\}$.

Using the last result and Theorem 3 we have:

Proposition 3. *Let G be a finitely generated group, T and S subgroups of G with $S \leq T \leq G, [G : T] < \infty$ and $[T : S] = \infty$. If S and T are finitely generated and $\tilde{E}(T, S) = 1$, in particular, if T is a PD^n-group, S a PD^{n-1}-subgroup, and T does not split over a commensurable with S subgroup, then also G does not split over a commensurable with S subgroup.*

Proof. We have $\tilde{E}(G, S) \leq \tilde{E}(T, S) = 1$ ([2], Proposition 7). So the result follows from Theorem 3. □

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Received by the editors: 16.09.2009
and in final form ????.