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A generalization of groups with many almost normal subgroups

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Dedicated to Professor I. Ya. Subbotin on the occasion of his 60-th birthday

Abstract. A subgroup H of a group G is called almost normal in G if it has finitely many conjugates in G. A classic result of B. H. Neumann informs us that $|G : \mathbf{Z}(G)|$ is finite if and only if each H is almost normal in G. Starting from this result, we investigate the structure of a group in which each non-finitely generated subgroup satisfies a property, which is weaker to be almost normal.

1. Anti-$\mathfrak{X}C$-Groups

In this paper $\mathfrak{X}$ denotes an arbitrary class of groups which is closed with respect to forming subgroups and quotients, $\mathfrak{F}$ is the class of all finite groups, $\mathfrak{F}_\pi$ is the class of all finite π-groups (π set of primes), $\check{\mathfrak{C}}$ is the class of all Chernikov groups, $\mathfrak{PF}$ is the class of all polycyclic-by-finite groups, $\mathfrak{S}_2\mathfrak{F}$ is the class of all (soluble minimax)-by-finite groups. Given a positive integer r, we recall that the operator L, defined by

(1.1) $L\mathfrak{X} = \{G \mid \langle g_1, g_2, \ldots, g_r\rangle \in \mathfrak{X}, \forall g_1, g_2, \ldots g_r \in G\}$,

from $\mathfrak{X}$ to $\mathfrak{X}$ is called *local operator* for $\mathfrak{X}$. See [12, §C, p.54]. We recall that the operator H, which associates to $\mathfrak{X}$ the class of *hyper-$\mathfrak{X}$-groups*

This paper is dedicated to the memory of my father and to the future of my brother.

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is called *extension operator*. See [12, §E, p.60]. The notation follows [11, 12, 13, 16].

A subgroup H of a group G is called *almost normal* in G if H has finitely many conjugates in G, that is, if $|G : \mathbf{N}_G(H)|$ is finite. Neumann's Theorem [16, Chapter 4, Vol.I, p.127] shows that G has each H which is almost normal in G if and only if $G/\mathbf{Z}(G) \in \mathfrak{F}$. We have $\mathbf{N}_G(Cl_G(H)) = core_G(\mathbf{N}_G(H)) = \bigcap_{x \in G} \mathbf{N}_G(H)^x = \bigcap_{x \in G} \mathbf{N}_G(H^x)$, where $Cl_G(H)$ is the set of conjugates of H in G. $|G : \mathbf{N}_G(H)| = |Cl_G(H)|$ is finite if and only if $G/core_G(\mathbf{N}_G(H)) \in \mathfrak{F}$. In [8, 9] G has $\mathfrak{F}$-*classes of conjugate subgroups*, if $G/core_G(\mathbf{N}_G(H)) \in \mathfrak{F}$ for each H in G. Thus Neumann's Theorem can be reformulated, stating that G has $G/core_G(\mathbf{N}_G(H)) \in \mathfrak{F}$ for each H in G if and only if $G/\mathbf{Z}(G) \in \mathfrak{F}$. See [9, Introduction]. More generally, G has $\mathfrak{X}$-*classes of conjugate subgroups*, if $G/core_G(\mathbf{N}_G(H)) \in \mathfrak{X}$ for each H in G. [9, Main Theorem] describes groups having $\check{\mathfrak{C}}$-*classes of conjugate subgroups*. [8, Main Theorem] describes those having $\mathfrak{P}\mathfrak{F}$-*classes of conjugate subgroups*.

Recall that $\mathbf{Z}_{\mathfrak{X}}(G) = \{x \in G \mid G/\mathbf{C}_G(\langle x \rangle^G) \in \mathfrak{X}\}$ is a characteristic subgroup of G, called $\mathfrak{X}C$-*center* of G. See [12, Definition B.1, Proposition B.2]. G is called $\mathfrak{X}C$-*group* if it coincides with its $\mathfrak{X}C$-*center*. $\mathfrak{F}C$-*groups*, $\check{\mathfrak{C}}C$-*groups*, $(\mathfrak{P}\mathfrak{F})C$-*groups* and $(\mathfrak{S}_2\mathfrak{F})C$-*groups* are well–known and described in [4, 7, 11, 12, 13, 15].

If G has $\mathfrak{F}$-classes of conjugate subgroups, then it is an $\mathfrak{F}C$-group. From [9, Lemma 2.3], if G has $\check{\mathfrak{C}}$-classes of conjugate subgroups, then it is a $\check{\mathfrak{C}}C$-group. From [8, Corollary 2.7], if G has $\mathfrak{P}\mathfrak{F}$-classes of conjugate subgroups, then it is a $(\mathfrak{P}\mathfrak{F})C$-group. From [17, Lemma 2.4], if G has $\mathfrak{S}_2\mathfrak{F}$-classes of conjugate subgroups, then it is an $(\mathfrak{S}_2\mathfrak{F})C$-group. The next lemma allows us to generalize these facts.

Lemma 1.1. *Assume that $\mathfrak{F}\mathfrak{X} = \mathfrak{X}$. If G has $\mathfrak{X}$-classes of conjugate subgroups, then $\mathbf{Z}_{\mathfrak{X}}(G) = G$.*

Proof. Let $g \in G$. $G/H \in \mathfrak{X}$, where $H = core_G(\mathbf{N}_G(\langle g \rangle))$. Let $H_1 = \mathbf{C}_H(\langle g \rangle)$ and $H_2 = core_G(H_1) = \mathbf{C}_H(\langle g \rangle^G)$. It is enough to prove $G/H_2 \in \mathfrak{X}$. Of course, $H \geq \mathbf{N}_H(\langle g \rangle)$. Conversely, an element of $\mathbf{N}_H(\langle g \rangle)$ is an element of G, fixing $\langle g \rangle^x = \langle g^x \rangle$ by conjugation for every $x \in G$, again fixing $\langle g \rangle$ by conjugation. If $x = 1$, then we get the elements of H and so $H \leq \mathbf{N}_H(\langle g \rangle)$. Then $H/H_1 = \mathbf{N}_H(\langle g \rangle)/\mathbf{C}_H(\langle g \rangle)$ is isomorphic to a subgroup of the automorphism group of $\langle g \rangle$ and so it is finite. The same is true if we consider H_1/H_2 and $\mathbf{N}_G(\langle g \rangle)/\mathbf{C}_G(\langle g \rangle)$. Therefore, G/H_2 is an extension of the finite group H_1/H_2 by the finite group H/H_1 by $G/H \in \mathfrak{X}$. From $(\mathfrak{F}\mathfrak{F})\mathfrak{X} = \mathfrak{F}\mathfrak{X} = \mathfrak{X}$, $G/H_2 \in \mathfrak{X}$. □

We recall that $\mathfrak{X}$ is called *Dietzmann class*, if for every group G and $x \in G$, the following implication is true:

(1.2) if $x \in \mathbf{Z}_{\mathfrak{X}}(G)$ and $\langle x \rangle \in \mathfrak{X}$, then $\langle x \rangle^G \in \mathfrak{X}$,

See [12, Definitions B.1 and B.6]. Dietzmann classes are studied in [11, 12, 13]. $\mathfrak{F}C$-*groups* form a Dietzmann class [12, Proposition D.3, b)]. In particular, this is true for periodic $(\mathfrak{P}\mathfrak{F})C$-*groups*, which are obviously $\mathfrak{F}C$-*groups*. Note that $\mathfrak{F}$ is a Dietzmann class [12, Proposition B.7, b)], but $\mathfrak{P}\mathfrak{F}$ is not a Dietzmann class [12, Example B.8, c)]. Unfortunately, it is not known whether $(\mathfrak{P}\mathfrak{F})C$-groups, $\check{\mathfrak{C}}C$-groups or $(\mathfrak{S}_2\mathfrak{F})C$-groups form a Dietzmann class. See [4, 7, 11, 12, 13, 15]. But, they extend locally the class of $\mathfrak{F}C$-*groups*. Therefore, the next result is significant.

Theorem 1.2 (see [12], Theorem E.3)**.** *If $\mathfrak{F}_\pi \subseteq \mathfrak{X} \subseteq L\mathfrak{F}_\pi$, then $(H\mathfrak{X})C$ is a Dietzmann class.*

From Lemma 1.1, if $\mathfrak{X} = \mathfrak{F}$, then $\mathfrak{F}C$ is a Dietzmann class. From Lemma 1.1 and Theorem 1.2, if $\mathfrak{F}_\pi \subseteq \mathfrak{X} \subseteq L\mathfrak{F}_\pi$, then $(H\mathfrak{X})C$ is a Dietzmann class. Therefore, it is meaningful to ask whether we may weaken the Neumann's Theorem, looking at the following property for G:

(1.3) if H is non-finitely generated, then $G/core_G(\mathbf{N}_G(H)) \in \mathfrak{X}$, where $\mathfrak{F}_\pi \subseteq \mathfrak{X} \subseteq L\mathfrak{F}_\pi$.

G is called *anti-$\mathfrak{X}C$-group* if it satisfies (1.3). *Anti-$\mathfrak{F}C$-groups* were described in [5]. *Anti-$\check{\mathfrak{C}}C$-groups* and *anti-$(\mathfrak{P}\mathfrak{F})C$-groups* were described in [18]. This line of research goes back to [14] and deals with the structure of groups with given properties of a system of subgroups. See [1, 2, 3, 5, 6, 10, 18, 20, 21].

2. Locally Finite Case

We omit the elementary proofs of the next two results.

Lemma 2.1. *Subgroups and quotients of anti-$\mathfrak{X}C$-groups are anti-$\mathfrak{X}C$-groups.*

Lemma 2.2. *If G is an anti-$\mathfrak{X}C$-group and $\mathbf{Z}_{\mathfrak{X}}(G) = G$, then G has $\mathfrak{X}$-classes of conjugate subgroups.*

Lemma 2.3. *Assume that x is an element of the anti-$\mathfrak{X}C$-group G. If $A = Dr_{i \in I} A_i$ is a subgroup of G consisting of $\langle x \rangle$-invariant nontrivial direct factors A_i, $i \in I$, with infinite index set I, then x belongs to $\mathbf{Z}_{\mathfrak{X}}(G)$.*

Proof. This follows by [18, Lemma 3.3, Proof], considering $\mathfrak{X}$ and $\mathbf{Z}_{\mathfrak{X}}(G)$. $\square$

Corollary 2.4. *Assume that G is an anti-$\mathfrak{X}C$-group and $A = Dr_{i\in I}A_i$ is a subgroup of G consisting of infinitely many nontrivial direct factors. Then A is contained in $\mathbf{Z}_{\mathfrak{X}}(G)$.*

Lemma 2.5. *Assume that g is an element of the anti-$\mathfrak{X}C$-group G and $A = Dr_{i\in I}A_i$ is a subgroup of G, with I as in Lemma 2.3. If $g \in \mathbf{N}_G(A)$ and $g^n \in \mathbf{C}_G(A)$ for some positive integer n, then g belongs to $\mathbf{Z}_{\mathfrak{X}}(G)$.*

Proof. This follows by [18, Lemma 3.7, Proof], considering $\mathfrak{X}$ and $\mathbf{Z}_{\mathfrak{X}}(G)$. $\square$

Corollary 2.6. *If the anti-$\mathfrak{X}C$-group G has an abelian torsion subgroup that does not satisfy the minimal condition on its subgroups, then all elements of finite order belong to $\mathbf{Z}_{\mathfrak{X}}(G)$.*

Proof. This follows by [18, Corollary 3.9, Proof], considering $\mathfrak{X}$ and $\mathbf{Z}_{\mathfrak{X}}(G)$. $\square$

Theorem 2.7. *If G is a locally finite anti-$\mathfrak{X}C$-group, then either G has $\mathfrak{X}$-classes of conjugate subgroups or G is a Chernikov group.*

Proof. This follows by [18, Theorem 3.12, Proof], considering $\mathfrak{X}$ and $\mathbf{Z}_{\mathfrak{X}}(G)$. $\square$

Note that Theorem 2.7 improves [18, Theorems 3.11 and 3.12].

Lemma 2.8. *Assume that $\mathfrak{X}$ is residually closed. If G has $\mathfrak{X}$-classes of conjugate subgroups, then $G \in \mathfrak{N}_2\mathfrak{X}$, where $\mathfrak{N}_2$ is the class of nilpotent groups of class at most 2.*

Proof. Let $\mathbf{N}(G) = \bigcap_{H\leq G} \mathbf{N}_G(H)$ be the norm of G. $\mathbf{N}(G) \leq \mathbf{Z}_2(G)$ from a result of Schenkman [19, Corollary 1.5.3]. Since G has $\mathfrak{X}$-classes of conjugate subgroups, $G/\mathbf{N}(G)$ is residually $\mathfrak{X}$ and so $G/\mathbf{N}(G) \in \mathfrak{X}$. This gives as claimed. $\square$

Corollary 2.9. *Assume that $\mathfrak{X}$ is residually closed. If G is a locally finite anti-$\mathfrak{X}C$-group, then either $G \in \mathfrak{N}_2\mathfrak{X}$ or G is a Chernikov group.*

Proof. This follows by Theorem 2.7 and Lemma 2.8. $\square$

3. Locally Nilpotent Case

Recall that G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for every prime p (see [16, Chapter 10, vol.II]). Following [5, 16, 20], a soluble-by-finite group G is an $\mathfrak{S}_1$-*group* if it has finite abelian section rank and the set of prime divisors of orders of elements of G is finite.

Theorem 3.1. *Assume that $\mathfrak{X}$ is residually closed. Let G be an anti-$\mathfrak{X}C$-group having an ascending series whose factors are either locally nilpotent or locally finite. Then either G has $\mathfrak{X}$-classes of conjugate subgroups or is a soluble-by-finite $\mathfrak{S}_1$-group or has a normal soluble $\mathfrak{S}_1$-subgroup K such that $G/K \in \mathfrak{X}$.*

Proof. G has an ascending normal series whose factors are either locally nilpotent or locally finite by [16, Theorem 2.31]. Let K be the largest radical normal subgroup of G. From Lemma 2.1 and Corollary 2.9, the largest locally finite normal subgroup T/K of G/K is either a Chernikov group or in $\mathfrak{N}_2\mathfrak{X}$.

In the first case, if H/T is a locally nilpotent normal subgroup of G/T, then $\mathbf{C}_{H/K}(T/K)$ is a locally nilpotent normal subgroup of G/K, so $\mathbf{C}_{H/K}(T/K)$ is trivial and H/K is a Chernikov group. Then $T = G$ and so G has a normal radical subgroup K such that T/K is a Chernikov group (in this situation G is said to be a radical-by-Chernikov group).

In the second case, $T/K = (N/K)(L/K)$, where $N/K \in \mathfrak{N}_2$ is a normal subgroup of T/K such that $(T/K)/(N/K) \in \mathfrak{X}$. If N/K is nontrivial, then there exists a nontrivial element $xK \in N/K$ such that $\langle xK\rangle^G = \langle x\rangle^G K/K$ is a nilpotent normal subgroup of G/K contained in T/K. Since G/K has no nontrivial locally nilpotent normal subgroups, we get to a contradiction. Therefore N/K is trivial and $T/K \in \mathfrak{X}$. Then we may deduce as above that G has a normal radical subgroup K such that $T/K \in \mathfrak{X}$ (in this situation G is said to be a radical-by-$\mathfrak{X}$ group).

Assume that G has $\mathfrak{X}$-classes of conjugate subgroups. Then every abelian subgroup of G has finite total rank by Corollary 2.4. A result of Charin [16, Theorem 6.36] implies that K is a soluble $\mathfrak{S}_1$-group. We conclude that G has a normal soluble $\mathfrak{S}_1$-subgroup K such that G/K is a Chernikov group. Therefore G is an extension of a soluble $\mathfrak{S}_1$-group by an abelian group with *min* by a finite group. An abelian group with *min* is clearly an $\mathfrak{S}_1$-group and the class of $\mathfrak{S}_1$-groups is closed with respect to extensions of two of its members (see [16, Chapter 10]). Therefore G is a soluble-by-finite $\mathfrak{S}_1$-group. The remaining case is that G has a normal soluble $\mathfrak{S}_1$-subgroup K such that $G/K \in \mathfrak{X}$. □

Note that Theorem 3.1 improves [18, Theorems 4.1 and 4.2].

Corollary 3.2. *Assume that $\mathfrak{X}$ is residually closed. Let G be an anti-$\mathfrak{X}C$-group having an ascending series whose factors are either locally nilpotent or locally finite. Then either $G \in \mathfrak{N}_2\mathfrak{X}$ or G is a soluble-by-finite $\mathfrak{S}_1$-group or G has a normal soluble $\mathfrak{S}_1$-subgroup K such that $G/K \in \mathfrak{X}$.*

Proof. This follows by Theorem 3.1 and Corollary 2.9. □

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