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Hall operators on the set of formations of finite groups

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Abstract. Let π be a nonempty set of primes and let $\mathfrak{F}$ be a saturated formation of all finite soluble π-groups. It is constructed the saturated formation consisting of all finite π-soluble groups whose $\mathfrak{F}$-projectors contain a Hall π-subgroup.

Introduction

In the theory of soluble Fitting classes P. Lockett and P. Hauck considered the classes $\mathcal{L}_\pi(\mathfrak{F})$ and $\mathcal{K}_\pi(\mathfrak{F})$.

Definition 1 ([1, 2]). *Let π be a set of primes and let $\mathfrak{F}$ be a Fitting class of finite soluble groups. Then*

$$\mathcal{L}_\pi(\mathfrak{F}) = (G \in \mathfrak{S} : \text{an } \mathfrak{F}\text{-injector of } G \text{ contains a Hall } \pi\text{-subgroup of } G);$$
$$\mathcal{K}_\pi(\mathfrak{F}) = (G \in \mathfrak{S} : \text{a Hall } \pi\text{-subgroup of } G \text{ belongs to } \mathfrak{F}).$$

In [1] (see also [3, IX, 1.22]) Lockett used the class $\mathcal{L}_\pi(\mathfrak{F})$ to obtain a description of the injectors for a Fitting class product $\mathfrak{F}\mathfrak{G}$. It was proved that $\mathcal{L}_\pi(\mathfrak{F})$ and $\mathcal{K}_\pi(\mathfrak{F})$ are Fitting classes. Furthermore, $\mathcal{K}_\pi(\mathfrak{F}) =$

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$\mathcal{L}_\pi(\mathfrak{F} \cap \mathfrak{S}_\pi)$. Hence we may consider $\mathcal{L}_\pi$ and $\mathcal{K}_\pi$ as operators on the set of all Fitting classes for every π. The class $\mathcal{K}_\pi(\mathfrak{F})$ was introduced by Hauck [2] and has been studied in detail by Brison [4] and Cusack [5]. Moreover, Brison [6, 7] applied $\mathcal{K}_\pi(\mathfrak{F})$ to obtain a description of Hall subgroups radicals.

Analogously one may consider the following operators on the set of all soluble formations.

Definition 2 ([8, 9])**.** *Let π be a set of primes and let $\mathfrak{F}$ be a formation of finite soluble groups. Then*

$\mathcal{L}^\pi(\mathfrak{F}) = (G \in \mathfrak{S}$: *an $\mathfrak{F}$-projector of G contains a Hall π-subgroup of G*);

$\mathcal{K}^\pi(\mathfrak{F}) = (G \in \mathfrak{S}$: *a Hall π-subgroup of G belongs to $\mathfrak{F}$*).

In [9] Blessenohl proved that if $\mathfrak{F}$ is a saturated formation, then $\mathcal{K}^\pi(\mathfrak{F})$ is a saturated formation.

Further L.A. Shemetkov posed the following question in this trend.

Problem (see [10, Problem 19])**.** Let $\mathfrak{F}$ be a saturated formation of finite groups, $\mathcal{C}_\pi(\mathfrak{F})$ be the class of all groups G such that there exist Hall π-subgroups of G in $\mathfrak{F}$ and any two of them are conjugate. Is the class $\mathcal{C}_\pi(\mathfrak{F})$ a saturated formation?

The positive answer of Problem 19 was given by L.M. Slepova [11] in the class of all π-separable groups for some restrictions to $\mathfrak{F}$; in [12] it was shown by E.P. Vdovin, D.O. Revin and L.A. Shemetkov that $\mathcal{C}_\pi(\mathfrak{F})$ is solubly saturated formation for any solubly saturated formation $\mathfrak{F}$. However L.A. Shemetkov and A.F. Vasil'ev [13] proved that in general the class $\mathcal{C}_\pi(\mathfrak{N})$ is not a saturated formation, where $\mathfrak{N}$ is the class of all nilpotent groups.

Wenbin Guo and Baojin Li [14] proved that $\mathcal{K}_\pi(\mathfrak{F})$ is a local Fitting class for every local Fitting class $\mathfrak{F}$. In general N.T. Vorob'ev and V.N. Zagurskii [15] gave the positive answer of Shemetkov's Problem for soluble ω-local Fitting classes.

K. Doerk and T. Hawkes investigated an analog of Problem 19 for the class $\mathcal{L}^\pi(\mathfrak{F})$. It was proved, that if $\mathfrak{F}$ is a solubly saturated formation, then $\mathcal{L}^\pi(\mathfrak{F})$ is a saturated formation (see [8, Bemerkung]). Note that the analog of the above-mentioned problem has the negative answer for soluble Schunck classes (see [8, Beispiel 1]) and soluble Fitting classes (see [3, IX, 3.15]).

A purpose of this paper is to investigate an analog of Shemetkov's Problem for the class $\mathcal{L}^\pi(\mathfrak{F})$, where $\mathfrak{F}$ is the saturated formation of all soluble π-groups.

All groups considered are finite and π-soluble for some fixed nonempty set of primes π. All unexplained notations and terminologies are standard. The reader is refereed to [16], [10] and [3] if necessary.

1. Preliminaries

Recall notation and some definitions used in this paper.

A group class closed under taking homomorphic images and finite subdirect products is called *a formation*.

A group G is said to be π-*soluble* if every chief factor of G is either a p-group for some $p \in \pi$ or a π'-subgroup.

The complementary set of primes, $\mathbb{P} \backslash \pi$, is denoted by π'. $\sigma(G)$ denotes the set of all distinct prime divisors of the order of a group G.

Functions of the form

$$f : \mathbb{P} \to \{\text{formations of groups}\}$$

are called *local satellites* (see [10]). For every local satellite f it is defined the class

$$LF(f) = (G : G \text{ has } f\text{-central chief series}),$$

i.e., for every chief factor H/K of G we have

$$G/C_G(H/K) \in f(p) \text{ for every } p \in \pi(H/K).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = LF(f)$ for a local satellite f, then *the formation* $\mathfrak{F}$ is said to be *saturated* and f is a local satellite of $\mathfrak{F}$.

If $\mathfrak{F}$ is a saturated formation, by [3, IV, 4.3] we have $\mathrm{Char}(\mathfrak{F}) = \sigma(\mathfrak{F})$, where $\sigma(\mathfrak{F}) = \bigcup\{\sigma(G) : G \in \mathfrak{F}\}$.

A satellite F of a formation $\mathfrak{F}$ is called *canonical* if $F(p) \subseteq \mathfrak{F}$, and $F(p) = \mathfrak{N}_p F(p)$ for all $p \in \mathbb{P}$ [17].

Let $\mathfrak{F}$ be a formation. A subgroup H of a group G is called $\mathfrak{F}$-*maximal* in G provided that

(1) $H \in \mathfrak{F}$, and

(2) if $H \leqslant V \leqslant G$ and $V \in \mathfrak{F}$, then $H = V$.

A subgroup H of G is called *an* $\mathfrak{F}$-*projector* of G if HN/N is $\mathfrak{F}$-maximal in G/N for all $N \trianglelefteq G$.

By $\mathrm{Proj}_{\mathfrak{F}} G$ we denote the (possibly empty) set of all $\mathfrak{F}$-projectors of G.

Let $\mathfrak{F}$ be a saturated formation and let $\mathfrak{H}$ be a formation. Following [3, IV, 1.1] we denote the class $(\mathfrak{F}{\downarrow}\mathfrak{H})$ as follows:

$$(\mathfrak{F}{\downarrow}\mathfrak{H}) = (G : \mathrm{Proj}_{\mathfrak{F}} G \subseteq \mathfrak{H}).$$

If $\mathfrak{H} = \varnothing$, then $(\mathfrak{F}\downarrow\mathfrak{H}) = \varnothing$.

If $RB \supseteq A$, then it is said that A/B *covered* by R.

The symbols G_π, $\mathfrak{S}^\pi$, $\mathfrak{E}_{\pi'}$, $\mathfrak{E}_\pi$ and $\mathfrak{N}_p$ denote, respectively, a Hall π-subgroup of a group G, the class of all π-soluble groups, the class of all π'-groups, the class of all π-groups and the class of all p-groups.

We need some lemmas to prove the main result.

Lemma 1 ([18, Lemma 1.2, Lemma 1.3]). *Let $\mathfrak{F} = LF(F)$ be the formation of all soluble π-groups. Then the following statements hold:*

(1) $\mathfrak{F} = LF(m)$, where

$$m(p) = (\mathfrak{F}\downarrow F(p)) \text{ for all } p \in \mathbb{P}.$$

(2) If V is an $\mathfrak{F}$-projector of a group G, then:

(a) V covers every m-central chief factor of G.

(b) Every chief factor of G covered of the subgroup V is m-central.

Lemma 2 ([10, Theorem 15.7]). *Let $\mathfrak{F}$ be a saturated formation and G be a group having $\sigma(\mathfrak{F})$-soluble $\mathfrak{F}$-residual. Then G has $\mathfrak{F}$-projectors and any two of them are conjugate.*

2. The proof of Theorem

First we prove

Lemma 3. *Let $\mathfrak{F}$ be a saturated formation of all soluble π-groups. Then the following statements hold:*

(1) The class $\mathcal{L}^\pi(\mathfrak{F})$ is a formation.

(2) $\mathfrak{E}_{\pi'}\mathcal{L}^\pi(\mathfrak{F}) = \mathcal{L}^\pi(\mathfrak{F})$.

Proof. (1) If $\pi = \varnothing$, then $\mathcal{L}^\varnothing(\mathfrak{F}) = \mathfrak{S}^\pi$; if $\pi = \mathbb{P}$, then $\mathcal{L}^\mathbb{P}(\mathfrak{F}) = \mathfrak{F}$. We have saturated formations $\mathfrak{S}^\pi$ and $\mathfrak{F}$, and hence the result. Now suppose $\varnothing \subset \pi \subset \mathbb{P}$. Since a formation $\mathfrak{F}$ is saturated, by [3, IV, 4.3] we have $\mathrm{Char}(\mathfrak{F}) = \sigma(\mathfrak{F})$.

Since $\sigma(\mathfrak{F}) \subseteq \pi$, a π-soluble group G is $\sigma(\mathfrak{F})$-soluble. Consequently, the subgroup $G^\mathfrak{F}$ of G is $\sigma(\mathfrak{F})$-soluble.

Let $G \in \mathcal{L}^\pi(\mathfrak{F})$, let $K \lhd G$ and let F be an $\mathfrak{F}$-projector of G. Then there exists a Hall π-subgroup G_π of G such that $G_\pi \subseteq F$.

By [10, Lemma 15.2] and [10, Lemma 15.1], we see that $G_\pi K/K$ is a Hall π-subgroup of G/K and FK/K is an $\mathfrak{F}$-projector of G/K. Therefore $G/K \in \mathcal{L}^\pi(\mathfrak{F})$.

Let K_1 and K_2 be normal subgroups of G such that $K_1 \cap K_2 = 1$ and let $G/K_1 \in \mathcal{L}^\pi(\mathfrak{F})$ and $G/K_2 \in \mathcal{L}^\pi(\mathfrak{F})$. Then $G_\pi K_1/K_1 \subseteq FK_1/K_1$ and $G_\pi K_2/K_2 \subseteq FK_2/K_2$, where $G_\pi K_1/K_1$ is a Hall π-subgroup of G/K_1

and $G_\pi K_2/K_2$ is a Hall π-subgroup of G/K_2, FK_1/K_1 is an $\mathfrak{F}$-projector of G/K_1 and FK_2/K_2 is an $\mathfrak{F}$-projector of G/K_2.

Therefore $G_\pi K_1 \subseteq FK_1$ and $G_\pi K_2 \subseteq FK_2$. Hence $G_\pi K_1 \cap G_\pi K_2 \subseteq FK_1 \cap FK_2$. By [18, Lemma 1.4] and [10, Theorem 15.2] we have $G_\pi(K_1 \cap K_2) \subseteq F(K_1 \cap K_2)$, i.e., $G_\pi \subseteq F$. Thus $G \in \mathcal{L}^\pi(\mathfrak{F})$. This proves (1).

(2) Inclusion $\mathcal{L}^\pi(\mathfrak{F}) \subseteq \mathfrak{E}_{\pi'}\mathcal{L}^\pi(\mathfrak{F})$ is obvious. We show that $\mathfrak{E}_{\pi'}\mathcal{L}^\pi(\mathfrak{F}) \subseteq \mathcal{L}^\pi(\mathfrak{F})$. Let $G \in \mathfrak{E}_{\pi'}\mathcal{L}^\pi(\mathfrak{F})$. Then $G^{\mathcal{L}^\pi(\mathfrak{F})} \in \mathfrak{E}_{\pi'}$ and $G/G^{\mathcal{L}^\pi(\mathfrak{F})} \in \mathcal{L}^\pi(\mathfrak{F})$.

Let G_π be a Hall π-subgroup of G and let F be an $\mathfrak{F}$-projector of G. By [10, Lemma 15.2] and [10, Lemma 15.1], we see, $G_\pi G^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})}$ is a Hall π-subgroup of $G/G^{\mathcal{L}^\pi(\mathfrak{F})}$ and $FG^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})}$ is an $\mathfrak{F}$-projector of $G/G^{\mathcal{L}^\pi(\mathfrak{F})}$. Therefore

$$G_\pi G^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})} \subseteq F^x G^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})}.$$

By [10, Lemma 15.1], $F^x G^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})}$ is an $\mathfrak{F}$-projector of $G/G^{\mathcal{L}^\pi(\mathfrak{F})}$, where $x \in G/G^{\mathcal{L}^\pi(\mathfrak{F})}$. Consequently,

$$|G/G^{\mathcal{L}^\pi(\mathfrak{F})} : F^x G^{\mathcal{L}^\pi(\mathfrak{F})}/G^{\mathcal{L}^\pi(\mathfrak{F})}| = \frac{|G|}{|F^x G^{\mathcal{L}^\pi(\mathfrak{F})}|} =$$

$$\frac{|G||F \cap G^{\mathcal{L}^\pi(\mathfrak{F})}|}{|F||G^{\mathcal{L}^\pi(\mathfrak{F})}|} = \frac{|G|}{|F||G^{\mathcal{L}^\pi(\mathfrak{F})}|}$$

is a π'-number. Since $|G^{\mathcal{L}^\pi(\mathfrak{F})}|$ is a π'-number, $|G : F|$ is a π'-number. Thus a Hall π-subgroup G_π of G is contained in the $\mathfrak{F}$-projector F of G. Hence $G \in \mathcal{L}^\pi(\mathfrak{F})$. The lemma is proved. □

The following theorem shows that if $\mathfrak{F}$ is a saturated formation, then the formation $\mathcal{L}^\pi(\mathfrak{F})$ is saturated.

Theorem. *Let $\mathfrak{F} = LF(F)$ be the formation of all soluble π-groups. Then $\mathcal{L}^\pi(\mathfrak{F}) = LF(f)$ for a local satellite f such that*

$$f(p) = \begin{cases} (\mathfrak{F}{\downarrow}F(p)), & \text{if } p \in \pi, \\ \mathfrak{S}^\pi, & \text{if } p \in \pi'. \end{cases}$$

Proof. If $\pi = \varnothing$, then $\mathcal{L}^\varnothing(\mathfrak{F}) = \mathfrak{S}^\pi$; if $\pi = \mathbb{P}$, then $\mathcal{L}^\mathbb{P}(\mathfrak{F}) = \mathfrak{F}$. We have saturated formations $\mathfrak{S}^\pi$ and $\mathfrak{F}$, and hence the result.

Now suppose $\varnothing \subset \pi \subset \mathbb{P}$. Since a formation $\mathfrak{F}$ is saturated, by [3, IV, 4.3] we have $\mathrm{Char}(\mathfrak{F}) = \sigma(\mathfrak{F})$.

So a π-soluble group G is $\sigma(\mathfrak{F})$-soluble. Consequently, the subgroup $G^\mathfrak{F}$ of G is $\sigma(\mathfrak{F})$-soluble.

By Lemma 1 we have $\mathfrak{F} = LF(m)$, where m is a local satellite of $\mathfrak{F}$ such that $m(p) = \mathfrak{F}{\downarrow}F(p)$ for all $p \in \mathbb{P}$.

We show $LF(f) \subseteq \mathcal{L}^\pi(\mathfrak{F})$. Suppose $LF(f) \nsubseteq \mathcal{L}^\pi(\mathfrak{F})$. Let G be a group of minimal order in $LF(f) \backslash \mathcal{L}^\pi(\mathfrak{F})$. Then G is a monolithic group and $K = G^{\mathcal{L}^\pi(\mathfrak{F})}$ is the socle of G. We have $|G/K| < |G|$, so by induction, $G/K \in \mathcal{L}^\pi(\mathfrak{F})$. If T is an $\mathfrak{F}$-projector of G and G_π is a Hall π-subgroup of G, then by the definition $\mathcal{L}^\pi(\mathfrak{F})$, we have $G_\pi K/K \subseteq TK/K$. Hence $G_\pi K \subseteq TK$. Since G is π-soluble, K is either a p-group, where $p \in \pi$ or a normal π'-subgroup.

Let K be a p-group, where $p \in \pi$. Since $G \in LF(f)$,

$$G/C_G(K) \in f(p) = (\mathfrak{F}{\downarrow}F(p)).$$

By Lemma 1, an $\mathfrak{F}$-projector T covers K, i.e., $K \subseteq T$. Therefore $T = TK \supseteq G_\pi K \supseteq G_\pi$. It follows that $G \in \mathcal{L}^\pi(\mathfrak{F})$, a contradiction.

Now let $K \in \mathfrak{E}_{\pi'}$. Lemma 3 implies $G \in \mathfrak{E}_{\pi'}\mathcal{L}^\pi(\mathfrak{F}) = \mathcal{L}^\pi(\mathfrak{F})$, a contradiction.

We prove the converse inclusion, i.e., $\mathcal{L}^\pi(\mathfrak{F}) \subseteq LF(f)$. Suppose $\mathcal{L}^\pi(\mathfrak{F}) \nsubseteq LF(f)$. Let H be a group of minimal order in $\mathcal{L}^\pi(\mathfrak{F}) \backslash LF(f)$. Then H is a monolithic group and $R = H^{LF(f)}$ is the socle of H. Since H is π-soluble, R is either a p-group, where $p \in \pi$ or a normal π'-subgroup.

Let R be a π'-subgroup. By induction, $H/R \in LF(f)$. Consequently, all factors of the chief series $H \supset \ldots \supset R$ are f-central. By assumption, $H/C_H(R) \in \mathfrak{S}^\pi = f(p)$. Hence $H \in LF(f)$, a contradiction.

Now let R be a p-group, where $p \in \pi$. If H_π is a Hall π-subgroup of H and V is an $\mathfrak{F}$-projector of H, then by Chunihin's Theorem [19], we have $R \subseteq H_\pi$. Since $H \in \mathcal{L}^\pi(\mathfrak{F})$, $H_\pi \subseteq V$. Consequently, $R \subseteq V$, i.e., V covers R. Lemma 1 implies that R is m-central chief factor of H. By induction, $H/R \in LF(f)$. Consequently, $H \in LF(f)$. This final contradiction completes the proof. □

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