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On some developments in investigation of groups with prescribed properties of generalized normal subgroups

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Dedicated to Professor I.Ya. Subbotin
on the occasion of his 60-th birthday

Abstract. The survey is dedicated to investigation of groups with prescribed properties of generalized normal subgroups. The roots of such investigations lie in the works by R. Dedekind, R. Baer, O.Yu.Schmidt, and S.N. Chernikov. The main goal of this survey is to reflect some important developments in this area.

1. Introduction

One of the main fruitful trends in group theory is defined by investigations of groups with prescribed properties of subgroups. With researches in this area, many important notions such as finiteness conditions, locally nilpotency, locally solubility, ranks, and many others have been introduced. The influence of these ideas is difficult to overestimate. Imposing of some natural restrictions on specifically chosen families of subgroups, we define concrete classes of groups satisfying having these properties. Among many others, the following restrictive properties have been employed by numerous authors: normality, generalized normality, to be abelian, nilpotency, complementability, transitivity, supersolubility, density, the minimal and maximal conditions, restrictions on important characteristics of groups (in particular, on distinct ranks), other finiteness conditions. Topological and linear groups with prescribed restrictions on families of subgroups have been also investigating.

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In this survey, we will consider some important developments in investigation of groups with prescribed properties for subgroups that are related in some way to generalized normality. Of course, the choice of the topics has been determined not only by logic of the subject but also by the authors' tests.

The roots of these investigations lie in the famous classical paper due to R. Dedekind [31], in which he completely described the finite non-abelian groups whose all subgroups are normal (the Hamiltonian groups). Recall that abelian and Hamiltonian groups together form the (named in honor of Dedekind) class of Dedekind groups. Later, R. Baer obtained a description of all infinite and finite Hamiltonian groups [8]. As it has been shown, such groups are direct products of a quaternion group, an elementary abelian 2 – group, and an abelian periodic group with elements of only odd order. Reducing the system of subgroups or weakening the restricted conditions we obtain some classes of groups close to or quite far from the class of Hamiltonian groups. Thus O. Yu. Schmidt has described all groups having only one class of non-normal subgroups [110], and all groups with two classes of non-normal subgroups [111]. As a proof of nowadays actuality of these Schmidt's results, we can mention that these researches were continued in [114] and quite recently have been simply repeated in [15] and [76], and directly generalized in [80]. Some important results due to B. Huppert, N. Ito, J. Szep, Z. Janko, Z., J. Buckly, and many others (see respectively [38, 44, 45, 46, 47, 5]) are closely related to this.

In infinite groups, this approach was transformed in the following way: the groups whose family of non-normal subgroups is in some sense very small became the objects of consideration. Thus, S.N.Chernikov studied groups whose all infinite subgroups are normal, groups whose all non-abelian subgroups are normal [25], [26], and groups whose infinite abelain subgroups are normal [27]. F.N. Liman described periodic groups whose all non-cyclic subgroups normal and groups with all abelain non-cyclic subgroups are normal [72], [73], [74]. G.M. Romalis and N.F. Sesekin investigated some groups in which every non-normal subgroup satisfies a given condition. In particular, in the papers [88], [89], [90], they considered groups in which non-normal subgroups are abelian (so called the metahamiltonian groups). The detailed description of metahamiltonian groups was obtained in a series of papers due to N.F. Kuzenny and N.N. Semko (see the book [57]). L.A. Kurdachenko, J. Otal, Russo and Vinchenci recently investigated the class of groups whose non-FC-subgroups are normal. Clearly, this class contains the classes of metahamiltonian and minimal non-FC-groups (i.e. the non-FC-groups with all proper subgroups being FC-groups [53]).

Of course, this list could be significantly extended since it is not possible to reflect all developments in this are in one survey. So we will focus on some interesting specific aspects of these studies that have not been reflected in surveys yet.

2. The Baer norm of a group and the quasicentralizer conditions

Starting with the concept of Dedekind groups we naturally come to the following generalization of the concept of the center of a group, namely to the *Baer norm of a group,* which is defined in the following equivalent ways:

It is the intersection of normalizers of all its subgroups,
it is the intersection of normalizers of all cyclic subgroups.

We denote the Baer norm of a group G by $\mathbf{BN}(G)$. Clearly, $\mathbf{BN}(G)$ is a Dedekind group, therefore $\mathbf{BN}(\mathbf{BN}(G)) = \mathbf{BN}(G)$.

R.Baer introduced and during 1934-1956 investigated this concept in the cycle of his works [9, 10, 11, 12, 13, 14]. In particular, he proved that even though the norm of a group includes the group's center, but in the case of periodic groups, it could be Hamiltonian. He also observed that for a group with the identity center, the norm is also identity [14]. L. Wos and E. Shenkman [117], [91] established that the second hypercenter of a group includes the norm of this group.

Let A be a subgroup of a group G and let $\mathfrak{L}(A)$ be a family of all subgroups of A. Put $QC_G(A) = \bigcap_{H \in \mathfrak{L}(A)} N_G(H)$. This subgroup is called *a norm of the subgroup* A in the group G (see, for example, [116]). Among others, this concept plays an important role in the investigation of normally factorized groups (see, for example, [118]). The norm of a subgroup A is also called the invariator or the quasicentralizer of A. We will use the last term in this survey.

We say that a subgroup A is *quasicentral* in a group G if every subgroup of A is normal in G. The concept of quasicental subgroup became important in relation with investigations of groups with transitivity of normality. These studies take a special central place in research of groups with prescribed properties of normal subgroups. A group G is said to be *a* $T-$*group* if every subnormal subgroup of G is normal. A group G is said to be a $\bar{T}$-*group*, if every subgroup of G is a $T-$group. It should be noted that $T-$groups are the relatively old objects of investigation (see, for example, [16, 36, 1, 83, 40, 39, 109]). The structure of finite soluble $T-$groups has been described by W. Gaschütz [36]. In particular, he found that every finite soluble $T-$group is a $\bar{T}-$group. Observe that a finite $\bar{T}-$group is metabelian. Infinite soluble $T-$groups and infinite $\bar{T}$-groups have been considered by D.J.S. Robinson [83]. Locally soluble $\bar{T}$-group G has the following structure.

2.1. D.J.S. Robinson [83]. *Let G be a locally soluble $\bar{T}$ – group.*

(i) *If G is not periodic, then G is abelian.*

(ii) *If G is periodic and L is the locally nilpotent residual of G, then G satisfies the following conditions:*

a) *G/L is a Dedekind group;*

b) $\pi(L) \cap \pi(G/L) = \varnothing$;

c) $2 \notin \pi(L)$;

d) *and every subgroup of L is $G-$invariant.*

In particular, if $L \neq \langle 1 \rangle$, then $L = [L, G]$.

It easily follows that every subgroup of a derived subgroup of a soluble T-group is normal in the group. In another words, the derived subgroup of a soluble T-group is quasicentral in a group. A group G is called a $KI-$group, if every subgroup of $[G, G]$ is $G-$invariant (I.Ya. Subbotin [92]). It is obvious that KI-groups are direct and quite wide generalizations of soluble T-groups. For example, any group with cyclic derived subgroup is a KI -group. It is interesting to admit that KI-groups have abelian derived subgroups. As in the study of finite T-groups, it was not difficult to describe finite non-nilpotent KI-groups.

2.2. (I.Ya. Subbotin [92]). *Let G be a finite non–nilpotent group. Then G is a $KI-$group if and only if G satisfies the following conditions:*

(i) $G = L \rtimes H$ where L is abelian and H is nilpotent;

(ii) $[H, H]$ is normal in G;

(iii) $\pi(L) \cap \pi([H, H]) = \varnothing$;

(iv) every subgroup of L is G – invariant.

However, since all groups with a cyclic derived subgroups and all groups of nilpotency class ≤ 2 are KI–groups, there is impossible to obtain a detailed description of nilpotent KI–groups. These finite groups have been considered in [93]. Infinite finitely generated $KI-$groups have been studied in [94]. The most general results on KI-groups have been obtained in [98] and [99]. We collect together the main results of these papers.

2.3. *Let G be an arbitrary group with a Chernikov derived subgroup. The group G is a KI-group if and only if the following conditions hold:*

(i) $G = A \rtimes B$ where A is an abelian Chernikov subgroup, B is a hypercentral group;

(ii) every subgroup of A is $G-$invariant;

(iii) $[B, B]$ *is normal in* G*;*

(iv) $\pi(A) \cap \pi(B') = \varnothing$.

If A *is a hypercentral rezidula of* G*,then all complements to* A *in* G *are conjugate.*

2.4. *Let* G *be an arbitrary group with a non-periodic derived subgroup whose periodic part is Chernikov. The group* G *is a* KI*-group if and only if* G *is either a nilpotent group of class* 2*, or* G *satisfied the following conditions*:

(i) $G = A \leftthreetimes B$ *where* A *is an abelian Chernikov subgroup,* B *is a* KI*-group;*

(ii) every subgroup of A is G-invariant;

(iii) $2 \notin \pi(A)$;

(iv) $[B, B]$ is normal in G;

(v) $[B, B] = C \times D$ *where* C *is an abelian torsion-free group,* D *is an abelian Chernikov* 2*-group or an identity;*

(vi) B/C *is hypercentral;*

(vii) $|G : C_G(G')| \leq 2$;

(viii) $(G' \cap \zeta(G))^2 = \langle 1 \rangle$;

(vii) any two distinct involutions from $[G, G]$ *are reciprocal.*

2.5. *Let* G *be a hypercentral* KI*-group.*

(1) If $[G, G]$ *is non-priodic, then* G *is nilpotent of class* 2*.*

(2) Suppose that $[G, G]$ *is periodic and denote by* D_p *the Sylow* p*-subgroup of* $[G, G]$. Then

(i) if the orders of elements of $D_p \cap \zeta(G)$ *are not bounded, then* $D_p \leq \zeta(G)$;

(ii) if $\exp(D_p) = p^n = \exp(D_p \cap \zeta(G))$*, then* $D_p \leq \zeta(G)$;

(iii) if $\exp(D_p) = p^n$, $\exp(D_p \cap \zeta(G)) = p^m$,$n > m$*, and* $p \neq 2$*, then* $G/C_G(D_p)$ *is a cyclic group of order* p^{n-m};

(iv) *if* $\exp(D_2) = 2^n$, $\exp(D_p \cap \zeta(G)) = 2^m$*, and* $n > m$*, then* $G/C_G(D_2)$ *is a subgroup of a direct product of a group of order* 2 *and a cyclic group of order* 2^{n-m};

(v) if the orders of elements of D_p *are not bounded and* $\exp(D_p \cap \zeta(G)) = p^m$*, then* $G/C_G(D_p)$ *is a subgroup of the multiplicative group of* p*-adic numbers of the form* $\alpha_\tau = 1 + \sum\limits_{i \geq m} x_i^\tau p^i$ *where* $0 \leq x_i^\tau < p, i \geq 2$*, and there exists such* α_τ *that* $x_m^\tau \neq 0$*. In particular, if* $p \neq 2$ *or* $p = 2$ *and* $m > 1$*, then* $G/C_G(D_p)$ *is torsion free; if* $p = 2$ *and* $m = 1$*, then* $\mathbf{Tor}(G/C_G(D_2))$ *is a subgroup of order* 2*.*

In [99], the following results concerning $KI-p$-groups have been also obtained.

2.6. *Let* G *be a* $p-$*group where* p *is a prime.*

(1) Suppose that the orders of elements of $[G,G]$ *are not bounded. Then* G *is a* $KI-$ *group if and only if* G *either is a nilpotent group of class at most* 2*, or* $G/C_G([G,G]) = \langle xC_G([G,G])\rangle$ *is a group of order* 2*. In the last case,* $p = 2$ *and* $c^x = c^{-1}$ *for each* $c \in C_G([G,G])$*.*

(2) Suppose that $\exp([G,G]) = p^n$*,* $\exp([G,G] \cap \zeta(G)) = p^m, n > m$*. Then* G *is a* $KI-$ *group if and only if the following conditions hold:*

(i) $G/C_G([G,G])$ *is a finite abelian group;*

(ii) if $p \neq 2$*, then* $G/C_G([G,G])$ *is a cyclic group of order* p^{n-m}*;*

(iii) if $p = 2$*, then* $G/C_G([G,G])$ *is a subgroup of a direct product of a group of order* 2 *and a cyclic group of order* 2^{n-m}*;*

(iv) for every coset $C_G([G,G])x$ *there is a positive integer* t_x *such that* $y^x = y^{s_x}$ *where* $s_x = 1 + t_x p^m$*. Moreover, there is a coset* $C_G([G,G])z$ *such that* $(t_z, p) = 1$*.*

In this connection, it is interesting to note that V. P. Shunkov considered groups that decomposed into *uniform product of Sylow* $p-$*subgroups* [115]. Following Shunkov, we say that a group G *is decomposed into uniform product of subgroups* $H_\lambda, \lambda \in \Lambda$, if $\langle x\rangle \langle y\rangle = \langle y\rangle \langle x\rangle$ for every elements $x \in H_\lambda, y \in H_\mu$, for each pair of indexes λ, $\mu \in \Lambda$. V.P. Shunkov described periodic groups that decomposed into uniform product of its Sylow $p-$ subgroups.

2.7. (V.P. Shunkov [115]). *Let* G *be a group. Then* G *is decomposed into uniform product of its Sylow* $p-$*subgroups for all prime* p *if and only if* $G = A \leftthreetimes B$ *where* A *is a normal abelian subgroup,* A *is quasicentral in* G*,* $B = \prod_{p\in\pi(B)} B_p$*,* B_p *is a Sylow* $p-$ *subgroup of* B*,* $p \in \pi(B)$*,* $\pi(A) \cap \pi(B) = \varnothing$*.*

In this setting in [99] it was established that a periodic KI-group G with a Chernikov derived subgroup *is decomposed into uniform product of its Sylow p-subgroups if and only if its hypercentral residual is a Hall subgroup of* G*.*

The following concept of a quisecentral product of groups are natural generalizations of KI-groups [100], [102]. We call a group G *a quasicentral product of a subgroup* A *by a subgroup* B if $G = AB$ and $B \le QC_G(A)$ [104]. This concept is a generalization of a similar concept of a quisicentral extension of a group [100]. As examples of quisicentral products of groups we can mention periodic groups decomposed into uniform product of its Sylow $p-$subgroups [115], soluble T-groups [83], and KI-groups. In [104] the following result has been obtained. This result is especially interesting in view of the question of complementing of the hypercentral residual in $KI-$ and $T-$groups.

2.8. (I.Ya. Subbotin [104]) *Let G be a quicentral product of a periodic hypercentral subgroup A by a periodic hypercentral subgroup B. Let $A = \prod_{p\in\pi(A)} A_p$, where A_p is a Sylow $p-$subgroup of A, $p \in \pi(B)$, and let $\Gamma = \{p \in \pi(A) \mid A_p \cap \zeta(G) = \langle 1\rangle\}$. Then the following assertions hold:*

(i) $C = \prod_{p\in\Gamma} A_p$ is a hypercentral residual of G;

(ii) C is an abelian subgroup and $2 \notin \pi(C)$;

(iii) C has a complement in G.

This result has been generalized in a following way

2.9. *(I.Ya. Subbotin [102]) Let $G = AB$ be a periodic group where A, B be hypercentral subgroups satisfying the following conditions: $A = \prod_{\lambda\in\Lambda} A_\lambda$, and $B \leq QC_G(A_\lambda)$ for each $\lambda \in \Lambda$. Let $\Delta = \{\lambda \in \Lambda \mid A_\lambda \cap \zeta(G) = \langle 1\rangle\}$. Then the following assertions hold:*

(i) $C = \prod_{\lambda\in\Delta} A_\lambda$ is a hypercentral residual of G;

(ii) C is an abelian subgroup and $2 \notin \pi(C)$;

(iii) C has a complement in G.

This result has direct relation to the main results on the groups with all complemented subgroups [42, 20, 21, 22]. In [37] a well known sophisticated construction has been developed. This construction, in particular, allows to construct examples of periodic groups that are non-splitting extensions of its abelian Hall derived subgroup by an uncountable elementary abelian 2-group. In the paper [66], this constraction has been extended on on the non-periodic case. In this paper, the authors, in particularly, constructed examples of non-periodic soluble T-groups and KI-groups with non-complemented periodic hypercentral residual.

The qusicentralizer condition is directly connected to these topic. This is a strong form of the well known normalizer condition which plays one of the main roles in generalized nilpotency. We will say that a group G satisfies *the quisicentralizer condition on the family of subgroups* $\mathfrak{A}$ if $QC_G(A)\backslash A \neq \varnothing$ for each subgroup $A \in \mathfrak{A}$ [95]. It is almost obvious, that the groups with quasicentralizer condition on all subgroups are Dedekind groups. In the cycle of work [95, 96, 97, 101, 103, 105]. I. Ya. Subbotin completely described all groups with the quasicentralizer condition on normal subgroups. Observe, that this groups form a subclass of T-groups. We collect together the main results of these papers.

2.10. *Let G be a periodic soluble group with quaisicentralizer condition on normal subgroups. Then G is a group of one of the following types of groups.*

I. $G = C \times (\prod_{\lambda\in\Lambda} A_\lambda \leftthreetimes \langle b_\lambda\rangle)$

(i) C *is an abelian subgroup;*

(ii) a set Λ *is countable or finite;*

(iii) A_λ *is a non-identity abelian subgroup for each* $\lambda \in \Lambda$*;*

(iv) b_λ *is a* $p_\lambda-$*element where* p_λ *is a prime,* $\lambda \in \Lambda$*;*

(v) $2, p_\mu \notin \pi(A_\lambda)$ *for all* $\mu, \lambda \in \Lambda, \mu \neq \lambda$*;*

(vi) $p_\mu \neq p_\lambda$ *whenever* $\mu \neq \lambda$*;*

(vii) $\pi(A_\lambda) \cap \pi(A_\mu) = \varnothing$ *whenever* $\mu \neq \lambda$*;*

(viii) every subgroup of A_λ *is* $\langle b_\lambda\rangle -$*invariant;*

(ix) $b_\lambda^{p_\lambda} \in \zeta(G)$ *for each* $\lambda \in \Lambda$.

II. $G = G_1 \times G_2$ *where*

(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$*;*

(ii) G_2 *is a group of type I;*

(iii) $G_1 = (D \leftthreetimes \langle t\rangle) \times H$*, where* D *is an abelian subgroup without involution,* $|t| = 2, 4$ *and* H *is an elementary abelian* 2 *– group (*H *can be identity);*

(iv) $d^t = d^{-1}$ *for all* $d \in D$.

III. $G = G_1 \times G_2$ *where*

(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$*;*

(ii) G_2 *is a group of type I;*

(iii) $G_1 = ((D \leftthreetimes \langle t\rangle) \times \langle s, z\rangle \times H$*, where* D *is an abelian subgroup without involution,* $|t| = 2$*,* $\langle z, s\rangle$ *is a quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*

(iv) $d^t = d^{-1}$ *for all* $d \in D$.

IV. $G = G_1 \times G_2$ *where*

(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$*;*

(ii) G_2 *is a group of type I;*

(iii) $G_1 = ((D \times D_0) \leftthreetimes \langle t\rangle) \times H$*, where* D *is an abelian subgroup without involution,* D_0 *is a divisible abelian* $2-$*subgroup,* $|t| = 2$*,* $\langle z, s\rangle$ *is a quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*

(iv) $d^t = d^{-1}$ *for all* $d \in D \times D_0$.

V. $G = G_1 \times G_2$ *where*

(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$*;*

(ii) G_2 *is a group of type I;*

(iii) $G_1 = ((D \times \langle s\rangle) \langle z\rangle) \times H$*, where* D *is an abelian subgroup without involution,* $\langle z, s\rangle$ *is a quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*

(iv) $d^t = d^{-1}$ *for all* $d \in D \times \langle s\rangle$.

VI. $G = G_1 \times G_2$ *where*
(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$;
(ii) G_2 *is a group of type I;*
(iii) $G_1 = ((D \times D_0 \times \langle s\rangle)\langle z\rangle) \times H$, *where* D *is an abelian subgroup without involution,* D_0 *is a divisible abelian* $2-$*subgroup,* $\langle z, s\rangle$ *is a quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*
(iv) $d^t = d^{-1}$ *for all* $d \in D \times D_0 \times \langle s\rangle$.

VII. $G = G_1 \times G_2$ *where*
(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$;
(ii) G_2 *is a group of type I;*
(iii) $G_1 = ((D \times S)\langle z\rangle) \times H$, *where* D *is an abelian subgroup without involution,* S *is a quisicyclic* $2-$*group,* $\langle S, z\rangle$ *is a locally quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*
(iv) $d^t = d^{-1}$ *for all* $d \in D \times S$.

VIII. $G = G_1 \times G_2$ *where*
(i) $\pi(G_1) \cap \pi(G_2) = \varnothing$;
(ii) G_2 *is a group of type I;*
(iii) $G_1 = ((D \times D_0 \times S)\langle z\rangle) \times H$, *where* D *is an abelian subgroup without involution,* D_0 *is a divisible abelian* $2-$*subgroup,* S *is a quisicyclic* $2-$*group,* $\langle S, z\rangle$ *is a locally quaternion group, and* H *is an elementary abelian* $2-$*group (*H *can be identity);*
(iv) $d^t = d^{-1}$ *for all* $d \in D \times D_0 \times S$.

Conversely, every of groups of the types I – VIII satisfies the quasicentralizer condition on normal subgroups.

2.11. *Let* G *be a non-periodic soluble group with the quasicentralizer condition on normal subgroups. Then* G *is a group of one of the following types of groups.*
I. G *is an abelian group;*
II. $G = C \times B$ *where* B *is an elementary abelian* $2-$*group (*B *can be identity), and* $C = D \leftthreetimes \langle t\rangle$, *where* D *is a non-periodic abelian subgroup such that* $D = D^2$, $|t| = 2, 4$, *and* $d^t = d^{-1}$ *for all* $d \in D$.
III. $G = C \times B$ *where* B *is an elementary abelian* $2-$*group (*B *can be identity), and* $C = D \leftthreetimes < \langle s, z\rangle$, *where* D *is a non-periodic abelian subgroup such that* $D = D^2$, $\langle s, z\rangle$ *is a quaternion group,* $\langle s\rangle \in C_G(D)$ *and* $d^z = d^{-1}$ *for all* $d \in D$.
IV. $G = C \times B$ *where* B *is an elementary abelian* $2-$*group (*B *can be identity), and* $C = D \leftthreetimes < \langle S, z\rangle$, *where* D *is a non-periodic abelian subgroup such that* $D = D^2$, S *is a quasicyclic* $2-$*group,* $\langle S, z\rangle$ *is a locally quaternion group,* $S \in C_G(D)$ *and* $d^z = d^{-1}$ *for all* $d \in D$.

Conversely, every of groups of the types I – IV satisfies the quasicentralizer condition on normal subgroups.

2.12. *Let G be a periodic non–soluble group with the quasicentralizer condition on normal subgroups. Then $G = Zr_{\lambda \in \Lambda} G_\lambda$ is a direct product with joint center of groups G_λ, and the following conditions hold.*

(i) there exists an index $\mu \in \Lambda$ such that $[G_\mu, G_\mu]$ is abelian;

(ii) for every $\lambda \in \Lambda \setminus \{\mu\}$ the factor-group $G_\lambda/\zeta(G)$ is a simple non-abelian group;

(iii) if D_μ is the Sylow 2′-subgroup of G_μ, then $\pi(D_\mu) \cap \pi(\langle G_\lambda \mid \lambda \in \Lambda \setminus \{\mu\}\rangle) = \varnothing$;

(iv) if $D_\mu \neq G_\mu$, then the subgroup $\langle G_\lambda \mid \lambda \in \Lambda \setminus \{\mu\}\rangle$ has no elements of order 4.

Conversely, the group having the above properties satisfies the quasicentralizer condition on normal subgroups.

2.13. *Let G be a non-periodic and non-soluble group with the quasicentralizer condition on normal subgroups. Then G is a central extension of an abelian group by a group decomposed in a direct product of simple non – abelian groups.*

Conversely, every such group satisfies the quasicentralizer condition on normal subgroups.

In [107], [108] some types of groups with the quasicentralizer conditions on non-abelian normal subgroups have been described.

3. Generalized normality and arrangement of subgroups

Let G be a group and G_0 its subgroup. A subgroup H is called *intermediate to* G_0 if $G_0 \leq H \leq G$ [2]. Z.I. Borevich and his students studied the lattices of all subgroups intermediate to a fixed subgroup G_0. They generalized the theorem on homomorphisms on some non-normal subgroups (see [18, 17, 7, 34, 2]). The following definition belongs to Z.I. Borevich [2].

Let G be a group and G_0 a subgroup of G. We say that a family $\{G_\lambda \mid \lambda \in \Lambda\}$ of intermediate to G_0 subgroups is a fan for G_0 if for each intermediate subgroup H there exists an unique index $\lambda(H) \in \Lambda$ such that $G_{\lambda(H)} \leq H \leq N_G(G_{\lambda(H)})$. The factor-groups $N_G(G_{\lambda(H)})/G_{\lambda(H)}$ are called sections of this fan.

Each normal subgroup G_0 of G is a fan subgroup in G. In this case, the fan consists of this subgroup G_0 only and has the unique section G/G_0. An opposite example of fan subgroups is provided by a subgroup G_0 whose every intermediate subgroup H is self-normalizing in G, i. e.

$N_G(H) = H$. In this setting, the fan is the set of all intermediate to G_0 subgroups, and all sections are trivial.

We obtain a natural generalization of a Dedekind group if we suppose that in a group G there exists a fixed subgroup $M(G)$ such that every non-normal subgroup $D \leq G$ has a fan $\{D, M(G)\}$. In the case $M(G) = G$, the following theorem has been obtained in [71] (note that in this setting every non-normal subgroup D is normal in every proper subgroup $H \geq D$ of G).

3.1. *Let G be a locally graded group and suppose that each its subgroup D has a fan $\{D, G\}$. Then G is a group of one of the following types:*

(i) G is a Dedekind group;

(ii) G is a finite non-abelian group whose proper subgroups are abelian;

(iii) $G = Q \leftthreetimes C$, where Q is a quaternion group, C is a cyclic $p-$subgroup, where p is an odd prime;

(iv) G is a generalized quaternion group of order 16.

It is important to note that A.Yu. Olshanskii [79, Theorems 28.1, 31.8] has constructed examples of infinite groups with two generators whose proper subgroups are abelian (even cyclic). These groups are not locally graded.

A subgroup H of a group G *is abnormal in* G if $g \in \langle H, H^g \rangle$ for each element g of G. Abnormal subgroups have been introduced in the paper [43] due to P. Hall, while the term "abnormal subgroup" itself belongs to R. Carter [23]. Abnormal subgroups are antipodes to normal subgroups. If G_0 is abnormal subgroup and H an intermediate subgroup to G_0, then H is self-normalizing. In this setting, the set of all intermediate to G_0 subgroups is a fan for G_0. Hence every abnormal subgroup is a fan subgroup.

A subgroup H of a group G is said to be *pronormal* in G if for every $g \in G$ the subgroups H and H^g conjugate in the subgroup $\langle H, H^g \rangle$. These subgroups have been introduced by P. Hall [42]. Such important subgroups of finite (soluble) groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal. Based on these concepts J. Rose considered balanced chains of subgroups in a group and contranormal subgroups [86]. Later, Z.I. Borevich and his students [2] introduced some generalizations of the mentioned subgroups, namely polynormal, paranormal, weakly pronormal, and weakly abnormal subgroups. All these definitions (except of the Carter subgroups) have no limitation of finiteness.

Let D be a fan subgroup in G. If the family of all intermediate to D subgroups satisfies the minimal condition, then there exists a unique fan for D [2]. In particular, the uniqueness of the fan holds for finite groups. Simple examples show that this statement is not true for infinite groups. However some subgroups such as pronormal and abnormal subgroups always have a unique fan [2]. These subgroups play an important role in arrangement of subgroups [18, 17, 7, 34, 2].

T.A. Peng has considered finite groups whose all subgroups are pronormal. He proved the following result.

3.2. (T.A. Peng [81]) *Let G be a finite soluble group. Then every subgroup of G is pronormal is and only if G is a $T-$group.*

However, in the infinite case, as the following theorem shows, the situation is much more sophisticated.

3.3. (N.F. Kuzennyi and I.Ya. Subbotin [67]). *Let G be a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent.*

1. Every cyclic subgroup of G is pronormal in G.

2. G is a soluble T-group.

Infinite groups whose subgroups are pronormal have been considered in [65]. The authors completely described such infinite locally soluble non-periodic and infinite locally graded periodic groups. The main result of this paper is the following interesting theorem.

3.4 (N.F. Kuzennyi and I.Ya. Subbotin [65]) *Let G be a group whose all subgroups are pronormal, and L be a locally nilpotent residual of G.*

(i) If G is periodic and locally graded, then G is a soluble $\bar{T}-$ group, in which L complements every Sylow $\pi(G/L)-$ subgroup.

(ii) If G is non-periodic and locally soluble, then G is abelian.

Conversely, if G has a such structure, then every subgroup of G is pronormal in G.

In the paper [87], the assertion (ii) has been extended to non-periodic locally graded groups proving that in this case such groups still to be abelian.

N.F. Kuzennyi and I. Ya. Subbotin also completely described locally graded periodic groups in which all primary subgroups are pronormal [69], and infinite locally soluble groups in which all infinite subgroups are pronormal [68]. They proved that in the infinite case, the class of groups whose subgroups are pronormal is a proper subclass of $\bar{T}$-groups. Moreover, it is also a proper subclass of the class of groups whose primary subgroups are pronormal. However, it is important to admit that the pronormality condition for all subgroups can be weakened to the pronormality for only abelian subgroups [70].

Let G be a group and D be its subgroup. An intermediate subgroup F, $D \leq F \leq G$, is called *a complete intermediate subgroup* if the normal closure D^F of D in F coincides with F.

A subgroup D is called *a polynormal* subgroup in a group G if for any $x \in G$ the subgroup $D^{<x>} = \langle D^x \mid x \in \langle x \rangle \rangle$ is a complete intermediate subgroup [2].

From the fan point of view, these concepts could be characterized in the following way [2].

3.5. (M.S. Ba, Z. I. Borevich Z. I. [2])

(i) D is polynormal in group G if and only if it is a fan subgroup and all complete intermediate subgroups form its fan.

(ii) D is abnormal in group G if and only if

a) D is a fan subgroup and its fan basis consists of all intermediate subgroups, and

b) any two intermediate conjugate subgroups coincide.

(iii) D is pronormal in G if and only if

a) D is a fan subgroup and its fan basis consists of D and all subgroups of group G, which strictly contain the normalizer $N_G(D)$; and

b) any such two conjugate subgroups coincide.

The subgroups mentioned above and their generalizations are very useful in finite group theory. In infinite groups, these subgroups gain some properties they cannot posses in the finite case. For example, it is well-known that every finite p–group has no proper abnormal subgroups. Nevertheless, A.Yu. Olshanskii has constructed a series of examples of infinite finitely generated p–groups saturated with abnormal subgroups. Specifically, for a sufficiently large prime p there exists an infinite p-group G whose all proper subgroups have prime order p [79, Theorem 28.1].

In finite soluble groups, abnormality is tightly bounded to self-normalizing. For example, D. Taunt has shown that *a subgroup H of a finite soluble group G is abnormal if and only if every intermediate subgroup for H coincides with its normalizer in G*; that is, such a subgroup is self-normalizing (see, for example, [84, 9.2.11]).

The following theorem extends this result to the radical groups [62].

3.6. (L.A. Kurdachenko, I.Ya. Subbotin [62]) *Let G be a radical group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

As corollaries we obtain

3.7 (F. de Giovanni, G. Vincenzi [35]) *Let G be a hyperabelian group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

3.8. (M.S. Ba, Z. I. Borevich Z. I. [2]) *Let G be a soluble group and let H be a subgroup of G. Then H is abnormal in G if and only if every intermediate subgroup for H is self-normalizing.*

Recall that a subgroup H of a group G is said to have *the Frattini property,* if given two intermediate subgroups K and L for H such that $K \trianglelefteq L$, we have $L \leq N_G(H)K$ (in this case, it is also said that H is *weakly pronormal* in G). It is not hard to see that every pronormal subgroup has Frattini property.

3.9. (T. A. Peng [82]) *Let G be a finite soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has a Frattini property.*

This Peng's characterization of pronormal subgroups could be extended in the following way.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is said to be *a hyper–$\mathfrak{X}$–group* if G has an ascending series of normal subgroups whose factors are $\mathfrak{X}$–groups.

Recall that a group G is an $N-$group if $H \neq N_G(H)$ for each subgroup H of G.

3.10. (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [49]) *Let G be a hyper–N–group. Then a subgroup H of G is pronormal in G if and only if H is weakly pronormal in G.*

As corollaries we obtain

3.11. (F. de Giovanni, G. Vincenzi [35]) *Let G be a hyperabelian group and let H be a subgroup of G. Then H is pronormal in G if and only if H is weakly pronormal in G* [35].

3.12. *Let G be a soluble group and let H be a subgroup of G. Then H is pronormal in G if and only if H is weakly pronormal in G.*

Carter subgroups are important case of abnormal subgroups. These subgroups have been introduced by R. Carter [23] as the self-normalizing nilpotent subgroups of a finite group. Some attempts of extending the definition of a Carter subgroup to infinite groups were made by S.E. Stonehewer [112, 113], A.D. Gardiner, B. Hartley and M.J. Tomkinson [33], and M.R. Dixon [30]. In [62], this concept have been extended to the class of nilpotent-by-hypercentral (not necessary periodic) groups.

We may define a Carter subgroup of a finite metanilpotent group as a minimal abnormal subgroup. The first logical step here is to consider those groups whose locally nilpotent residual is nilpotent.

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an *artinian-by-$\mathfrak{X}$-group* if G has a normal subgroup H such that $G/H \in \mathfrak{X}$ and H satisfies $\mathbf{Min} - G$.

3.13. (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [49]) *Let G be an artinian–by–hypercentral group and suppose that its locally nilpotent residual K is nilpotent.*

(i) G has a minimal abnormal subgroup L. Moreover, L is maximal hypercentral subgroup, and it includes the upper hypercenter of G. In particular, $G = KL$.

(ii) Two minimal abnormal subgroups of G are conjugate.

Thus, given an artinian–by–hypercentral group G with a nilpotent hypercentral residual, *a subgroup L is called a Carter subgroup of a group G if L is a hypercentral abnormal subgroup of G or equivalently, if H is a minimal abnormal subgroup of G.*

A Carter subgroup of a finite soluble group can be also characterized as *a covering subgroup for the formation of nilpotent groups.* In the paper [62], this characterization was extended to the class of artinian–by–hypercentral groups with a nilpotent locally nilpotent residual.

Following J.S. Rose [86], a subgroup H of a group G is called *contranormal*, if $H^G = G$. Abnormal subgroups are contranormal. If H is a pronormal subgroup and $H \leq L$, then $N_L(H)$ is abnormal in L.

A subgroup H of a group G is called *nearly pronormal* if $N_L(H)$ is contranormal in L for every subgroup L including H.

In the paper [55], the groups whose subgroups are nearly pronormal have been considered.

3.14. (L.A. Kurdachenko, A. Russo, G. Vincenzi [55]) *Let G be a locally radical group.*

(i) If every cyclic subgroup of G is nearly pronormal, then G is a $\bar{T}$–group.

(ii) If every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.

If G is a finite group, then for each subgroup H there is a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots . \leq H_{n-1} \leq H_n = G$$

such that H_j is maximal in H_{j+1}, $0 \leq j \leq n-1$. Generalizing this, J. Rose has arrived at the *balanced chain* connecting a subgroup H to a group G, that is, a chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots . \leq H_{n-1} \leq H_n = G$$

such that for each j, $0 \leq j \leq n-1$, either H_j is normal in H_{j+1}, or H_j is abnormal in H_{j+1}; the number n is *the length* of this chain. He refers appropriately to two consecutive subgroups $H_j \leq H_{j+1}$ as forming a *normal link* or an *abnormal link* of this chain [85]. In a finite group, every subgroup can be connected to the group by some balanced chain.

It is natural to consider the case when all of these balanced chains are short, i.e. their lengths are bounded by a small number. If these lengths are ≤ 1, then every subgroup is either normal or abnormal in a group. Such finite groups were studied in [32]. Infinite groups of this kind and some of their generalizations were described in [106] and [28]. Moreover, in the last paper have been considered the groups, whose subgroups are either abnormal or subnormal. More general situation was considered in a paper of L.A. Kurdachenko and H. Smith [58] . They considered the groups, whose subgroups are either self – normalizing or subnormal.

Observe that in the groups in which the normalizer of any subgroup is abnormal and in the groups in which every subgroup is abnormal in its normal closure, the mentioned lengths are ≤ 2. It is logical to choose these groups as the subject for investigation.

It is interesting to observe that if G is a soluble $\overline{T}$-group, then every subgroup of G is abnormal in its normal closure. As we mentioned above, for any pronormal subgroup H of a group G, the normalizer $N_G(H)$ is an abnormal subgroup of G. So the subgroups having abnormal normalizers make a generalization of pronormal subgroups. There are examples showing that this generalization is non-trivial.

The article [56] initiated the study of groups whose subgroups are connected to a group by balanced chains of length at most 2. As we recently mentioned, such groups are naturally related to the T-groups.

3.15. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [56]) *Let G be a radical group. Then G is a $\overline{T}$-group if and only if every cyclic subgroup of G is abnormal in its normal closure.*

3.16. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [56]) *Let G be a periodic soluble group. Then G is a$\overline{T}$–group if and only if its locally nilpotent residual L is abelian and the normalizer of each cyclic subgroup of G is abnormal in G.*

The following result from [56] is a new interesting and useful characterization of groups with all pronormal subgroups.

3.17. (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [56]) *Let G be a periodic soluble group. Then every subgroup of G is pronormal if and only if its locally nilpotent residual L is abelian and the normalizer of every subgroup of G is abnormal in G.*

For the non-periodic case, there exist non-periodic non-abelian groups in which normalizers of all subgroups are abnormal [56]. On the other hand, the non-periodic locally soluble groups in which all subgroups are pronormal are abelian [65]. So, in the non-periodic case we cannot count on a characterization, similar to above. However, we have the following result.

3.18 (L.A. Kurdachenko, A. Russo, I.Ya. Subbotin, G. Vincenzi [56]) *Let G be a non-periodic group with the abelian locally nilpotent residual L. If the normalizer of every cyclic subgroup is abnormal and for each prime $p \in \Pi(L)$ the Sylow p-subgroup of L is bounded, then G is abelian.*

Following [59], we will call normal and abnormal subgroups U-normal (from "union" and "U-turn"). Finite groups with only U-normal subgroups have been considered in [32]. Locally soluble (in the periodic case locally graded) infinite groups with U-subgroups have been studied in [106]. In [59], the groups with all U-normal subgroups and the groups with transitivity of U-normality were completely described.

Next natural question regarding the structure of *groups whose U-normal subgroups form a lattice.* These groups are denoted as *#U-groups* [64]. It is easy to see that the groups with no abnormal subgroups are #U-groups. In particular, all locally-nilpotent groups have this property [67].

Observe that a union of any two U-normal subgroups is U-normal. However, the similar assertion is obviously false for intersections.

It is easy to see that in a soluble group an abnormal subgroup R is exactly the subgroup that is contranormal in all subgroups containing R [28]. The condition "every contranormal subgroup is abnormal" (the CA-property) is an amplification of the transitivity of abnormality (the TA-property). Some simple examples show that the class of TA-groups is wider then the class of CA-groups and does not coincide with the class of #U-groups.

A description of soluble CA-groups having #U-property were obtained in [64].

4. Generalized normality and criteria of generalized nilpotency

The following well-known characterizations of finite nilpotent groups are tightly bound to abnormal and pronormal subgroups.

A finite group G is nilpotent if and only if G has no proper abnormal subgroups.

A finite group G is nilpotent if and only if its every pronormal subgroup is normal.

Note that since the normalizer of a pronormal subgroup is abnormal, *the absence of abnormal subgroups is equivalent to the normality of all pronormal subgroups.*

One part of these criteria is still true for infinite groups.

4.1. (N.F. Kuzenny, I.Ya. Subbotin [68]) *Let G be a locally nilpotent group. Then G has no proper abnormal subgroups and every pronormal subgroup of G is normal.*

However, we do not know whether or not the converse to this result holds.

In the paper [48], the following generalization of a well-known nilpotency criterion was obtained.

Let G be a group, A a normal subgroup of G. We say that A satisfies the condition **Max**$-G$ *(respectively* **Min**$-G$*) if A satisfies the maximal (respectively the minimal) condition for G–invariant subgroups. A group G is said to be a generalized minimax group, if it has a finite series of normal subgroups every factor of which is abelian and either satisfies* **Max**$-G$ *or* **Min**$-G$.

Every soluble minimax group is obviously generalized minimax. However, the class of generalized minimax groups is significantly wider than the class of soluble minimax groups.

In the paper [48], the first generalization of the mentioned nilpotency criterion was obtained.

4.2. (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [48]) *Let G be a soluble generalized minimax group. If every pronormal subgroup of G is normal (or, what is equivalent, G has no proper abnormal subgroups), then G is hypercentral.*

Let G be a group. Then the set

$$FC(G) = \{x \in G \mid x^G \text{ is finite}\}$$

is a characteristic subgroup of G which is called *the FC–center of G*. Note that a group G is an FC–group if and only if $G = FC(G)$. Starting from the FC–center, we construct *the upper FC–central series of a group G*

$$\langle 1\rangle = C_0 \leq C_1 \leq \ \cdots \ \leq C_\alpha \leq C_{\alpha+1} \leq \ \cdots C_\gamma$$

where $C_1 = FC(G)$, $C_{\alpha+1}/C_\alpha = FC(G/C_\alpha)$ for all $\alpha < \gamma$, and $FC(G/C_\gamma) = \langle 1\rangle$.

The term C_α is called *the α–FC–hypercenter of G,* while the last term C_γ of this series is called *the upper FC–hypercenter of G*. If $C_\gamma = G$, then the group G is called *FC–hypercentral,* and, if γ is finite, then G is called *FC–nilpotent.*

The following criteria of hypercentrality have been obtained in [54].

4.3. (L.A. Kurdachenko, A. Russo, G. Vincenzi [54]) *Let G be a group whose pronormal subgroups are normal. Then every $FC-$*hypercenter *of G having finite number is hypercentral.*

Let G be an FC–nilpotent group. If all pronormal subgroups in G are normal, then G is hypercentral.

Let G be a group whose pronormal subgroups are normal. Suppose that H be an FC- hypercenter of G having finite number. If C is a normal subgroup of G such that $C \geq H$ and C/H is hypercentral, then C is hypercentral.

For periodic groups, the above results were obtained in [59].

Observe that abnormal subgroups are an important particular case of contranormal subgroups: abnormal subgroups are exactly the subgroups that are contranormal in each subgroup containing them. On the other hand, abnormal subgroups are a particular case of pronormal subgroups. Pronormal subgroups are connected to contranormal subgroups in the following way. If H is a pronormal subgroup of a group G and $H \leq K$, then its normalizer $N_K(H)$ in K is an abnormal and hence contranormal subgroup of K.

Starting from the normal closure of H, we can construct the *normal closure series of H in G*

$$H^G = H_0 \geq H_1 \geq ...H_\alpha \geq H_{\alpha+1} \geq ...H_\gamma$$

by the following rule: $H_{\alpha+1} = H^{H_\alpha}$ for every $\alpha < \gamma$, $H_\lambda = \bigcap_{\mu<\gamma} H_\mu$ for a limit ordinal λ. The term H_α of this series is called the *α-th normal closure of H in* G and will be denoted by $H^{G,\alpha}$. The last term H_γ of this series is called *the lower normal closure of H in G* and will be denoted by $H^{G,\infty}$. Observe that every subgroup H is contranormal in its lower normal closure.

In finite groups, the subgroup $H^{G,\infty}$ is called *the subnormal closure* of H in G. The rationale for this is the following. In a finite group G, the normal closure series of every subgroup H is finite, and $H^{G,\infty}$ is the smallest subnormal subgroup of G containing H. A subgroup H *is called descendant in* G if H coincides with its lower normal closure $H^{G,\infty}$. An important particular case of descendant subgroups are subnormal subgroups. A subnormal subgroup is exactly a descending subgroup having finite normal closure series. These subgroups strongly affect structure of a group. For example, it is not hard to prove that if every subgroup of a locally (soluble-by-finite) group is descendant, then this group is locally nilpotent. If every subgroup of a group G is subnormal, then, by a remarkable result due to W. Möhres [75], G is soluble. Subnormal subgroups have been studied very thoroughly for quite a long period of time. We are not going to consider this topic here since it has been excellently presented in the survey of C. Casolo [19]. However, we need to admit that, with the exception of subnormal subgroups, we have no significant information regarding descendant subgroups. The next results connect the conditions of generalized nilpotency to descendant subgroups.

4.4. (L.A. Kurdachenko, I.Ya. Subbotin [61]) *Let G be a group, every subgroup of which is descendant. If G is FC–hypercentral, then G is hypercentral.*

4.5. (L.A. Kurdachenko, I.Ya. Subbotin [62]) *Let G be a generalized minimax group. Then every subgroup of G is descendant if and only if G is nilpotent.*

If every subgroup of a group G is descendant, then G does not include proper contranormal subgroups. The study of groups without contranormal subgroups is the next logical step. We observe that every non-normal maximal subgroup of an arbitrary group is contranormal. Since a finite group whose maximal subgroups are normal is nilpotent, we come to the following criterion of nilpotency of finite groups in terms of contranormal subgroups:

A finite group G is nilpotent if and only if G does not include proper contranormal subgroups.

The question on existing of an analog of this criterion for infinite groups is very natural. However, in general, the absence of contranormal subgroups does not imply nilpotency. In fact, there exist non-nilpotent groups all subgroups of which are subnormal. The first such example has been constructed by H. Heineken and I.J. Mohamed [41]. Nevertheless, for some classes of infinite groups the absence of contranormal subgroups does imply nilpotency of a group. The groups without proper contranormal subgroups have been considered in papers [51, 52]. We show the main results of these articles.

4.6. *Let G be group and H be a normal soluble-by-finite subgroup such that the factor-group G/H is nilpotent. Suppose that H satisfies* **Min**$-G$. *If G has no proper contranormal subgroups, then G is nilpotent. In particular, if soluble-by-finite group G without proper contranormal subgroups satisfies the minimal condition on normal subgroups, then G is nilpotent.*

4.7. *Let G be a group and H be a normal Chernikov subgroup. Suppose that G/H is nilpotent. If G has no proper contranormal subgroups, then G is nilpotent. In particular, a Chernikov group without proper contranormal subgroups is nilpotent.*

4.8. *Let G be group and C be a normal subgroup such that the factor-group G/C is nilpotent. Suppose that C has a finite series of $G-$invariant subgroups*

$$\langle 1\rangle = C_0 \leq C_1 \leq ... \leq C_n = C$$

whose factors $C_j/C_{j-1}, 1 \leq j \leq n$, satisfy one of the following conditions:
(i) C_j/C_{j-1} is finite;
(ii) C_j/C_{j-1} is hyperabelian and minimax;
(iii) C_j/C_{j-1} is hyperabelian and finitely generated;
(iv) C_j/C_{j-1} is abelian and satisfies **Min** $- G$.
If G has no proper contranormal subgroups, then G is nilpotent.

Let G be a group and let A be an infinite normal abelian subgroup of G. We say that A is a $G-$*quasifinite subgroup* if every proper $G-$invariant subgroup of A is finite. This means that either A includes a proper finite $G-$invariant subgroup B such that A/B is $G-$simple, or A is an union of all finite proper $G-$invariant subgroups.

4.9. *Suppose that a group G includes a normal subgroup C such that the factor-group G/C is nilpotent. Suppose that C has a finite series of $G-$invariant subgroups*

$$\langle 1\rangle = C_0 \leq C_1 \leq ... \leq C_n = C$$

whose factors $C_j/C_{j-1}, 1 \leq j \leq n$, satisfy one of the following conditions:
(i) C_j/C_{j-1} is finite;
(ii) C_j/C_{j-1} is hyperabelian and minimax;

(iii) C_j/C_{j-1} is hyperabelian and finitely generated;
(iv) C_j/C_{j-1} is abelian and $G-$quasifinite.
If G has no proper contranormal subgroups, then G is nilpotent.

The following useful assertions are almost direct consequences of this theorem

4.10. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C has a finite $G-$chief series. If G has no proper contranormal subgroups, then G is nilpotent.*

4.11. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian minimax subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is hyperabelian minimax group without proper contranormal subgroups, then G is nilpotent.*

4.12. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a Chernikov subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is a Chernikov group without proper contranormal subgroups, then G is nilpotent.*

4.13. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian finitely generated subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is hyperabelian finitely generated group without proper contranormal subgroups, then G is nilpotent.*

4.14. *Suppose that the group G includes a normal $G-$minimax subgroup C such that G/C is a nilpotent group of finite section rank. If G has no proper contranormal subgroups, then G is nilpotent.*

5. Transitivity of generalized normality

We mentioned above some important results on transitivity of normality. Transitivity of such important subgroup properties as pronormality, abnormality and other related to them properties have been studied by L.A. Kurdachenko, I.Ya. Subbotin, and J.Otal (see, [60], [59], and [49]).

The groups, in which pronormality is transitive are called TP-groups and the groups in which all subgroups are $TP-$groups are called $\bar{T}P-$groups. The following descriptions of soluble $\bar{T}P-$groups and $TP-$groups have been obtained in [60]

5.1. (L.A. Kurdachenko, I.Ya. Subbotin [60]) *Let G be a locally soluble group. Then G is a $\bar{T}P-$group if and only if G is a $\bar{T}-$group.*

5.2. (L.A. Kurdachenko, I.Ya. Subbotin [60]) *Let G be a periodic soluble group. Then G is a TP-group if and only if $G = A \leftthreetimes (B \times P)$ where*

(i) A and B are abelian $2'$-subgroups in G and P is a 2-subgroup (if P is non-identity);

(ii) $\pi(A) \cap \pi(A) = \varnothing$;

(iii) P is a T-group;

(iv) $[G, G] = A \times [P, P]$;

(v) every subgroup of $[G, G]$ is $G-$invariant;

(vi) A is a complement to every Sylow $\pi(B \times P)-$subgroup of G.

In [60] the authors list all types of *periodic soluble $TP-$groups.*

The following theorem completes a description of soluble $TP-$groups.

5.3. (L.A. Kurdachenko, I.Ya. Subbotin [60]) *Let G be a non-periodic soluble group.*

(i) If $C_G([G, G])$ is non-periodic, then G is a $TP-$ group if and only if G is a $T-$group.

(ii) If $C_G([G, G])$ is periodic, then G is a $TP-$group if and only if G is a hypercentral $T-$group.

In this setting, it is interesting to mention the following, most general yet, result on transitivity of abnormal subgroups.

5.4. (L.A. Kurdachenko, I.Ya. Subbotin [62]) *Let G be a group and suppose that A is a normal subgroup of G such that G/A has no proper abnormal subgroups. If A satisfies the normalizer condition, then abnormality is transitive in G.*

In particular, if G is metanilpotent group, then abnormality is transitive in G.

Recall the following interesting property of pronormal subgroups:

Let G be a group, H, K be the subgroups of G and $H \leq K$. If H is a subnormal and pronormal subgroup in K, then H is normal in K.

We say that a subgroup H of a group G *is transitively normal* if H is normal in every subgroup $K \geq H$ in which H is subnormal [63]. In [78], these subgroups have been introduced under a different name. Namely, a subgroup H of a group G is said to satisfy *the subnormalizer condition* in G if for every subgroup K such that H is normal in K we have $N_G(K) \leq N_G(H)$.

We say that a subgroup H of a group G is *strong transitively normal*, if HA/A is transitively normal for every normal subgroup A of the group G

[63]. Since the homomorphic image of pronormal subgroup is pronormal, we can conclude that every pronormal subgroup is a strong transitively normal subgroup.

5.5. (L.A. Kurdachenko, I.Ya. Subbotin [63]). *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is strong transitively normal in G and R satisfies $Min-H$, then H is a pronormal subgroup of G.*

As direct corollaries we can mention the following results [63].

5.6. *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble Chernikov subgroup R such that G/R is hypercentral. If H is strong transitively normal in G, then H is a pronormal subgroup of G. In particular, if G is a soluble Chernikov group and H is a hypercentral strong transitively normal subgroup of G, then H is pronormal in G.*

5.7. *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is a polynormal subgroup of G and R satisfies* $\mathbf{Min}-H$ *(in particular, if R is Chernikov), then H is pronormal in G.*

The following theorem from [78] directly follows from this.

5.8. *Let G be a soluble finite group, H be a nilpotent subgroup of G. If H is a polynormal subgroup of G, then H is a pronormal subgroup of G.*

A subgroup H is said to be *paranormal* in a group G if H is contranormal in $\langle H, H^g\rangle$ for all elements $g \in G$ (M.S. Ba and Z.I. Borevich [2]). Every pronormal subgroup is paranormal, and every paranormal subgroup is polynormal [2].

5.9. (L.A. Kurdachenko, I.Ya. Subbotin [63]). *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal soluble subgroup R such that G/R is hypercentral. If H is a paranormal subgroup of G and R satisfies $Min-H$ (in particular, if R is a Chernikov group), then H is pronormal in G.*

As a corollary we obtain

5.10. *Let G be a soluble finite group, H be a nilpotent subgroup of G. If H is a paranormal subgroup of G, then H is a pronormal subgroup of G.*

In [82], the following criterion of pronormality of a nilpotent subgroup in a finite group has been established.

5.11. *Let G be a nilpotent-by-abelian finite group, H be a nilpotent subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

The article [63] contains the following useful strong generalization of this criterion on some infinite cases.

5.12.(L.A. Kurdachenko, I.Ya. Subbotin [63]) *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is transitively normal in G and R satisfies $\mathbf{Min}-H$ (in particular, if R is Chernikov), then H is a pronormal subgroup of G.*

As a corollary we obtain

5.13. *Let G be a nilpotent-by-hypercentral Chernikov group, H be a hypercentral subgroup of G. If H is transitively normal in G, then H is a pronormal subgroup of G.*

A subgroup H of a group G is called *weakly normal* if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ (K.H. Müller [77]). We note that every pronormal subgroup is weakly normal [3], every weakly normal subgroup satisfies the subnormalizer condition [3], and hence it is transitively normal in G. Thus from above result we obtain

5.14. (L.A. Kurdachenko, I.Ya. Subbotin [63] *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is weakly normal in G and R satisfies $\mathbf{Min} - H$ (in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

A subgroup H of a group G is called *an $\mathfrak{H}-$ subgroup* if $N_G(H) \cap H^g \leq H$ for all elements $g \in G$ [6]. Note that every $\mathfrak{H}-$ subgroup is transitively normal [6]. Therefore, from above result we obtain

5.15 (L.A. Kurdachenko, I.Ya. Subbotin [63]) *Let G be a group, H be a hypercentral subgroup of G. Suppose that G includes a normal nilpotent subgroup R such that G/R is hypercentral. If H is an $\mathfrak{H}-$ subgroup of G and R satisfies $\mathbf{Min} - H$ (in particular, if R is a Chernikov group), then H is a pronormal subgroup of G.*

Some properties of transitively normal subgroups (under another name) have been considered in the paper [29], which in particular, contains the following result.

5.16. *Let G be an $FC-$group, H be a transitively normal subgroup of G. If H is a $p-$subgroup for some prime p, then H is a pronormal subgroup of G.*

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