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RESEARCH ARTICLE

# Lattices of classes of groupoids with one-sided quasigroup conditions

**Jan Gałuszka**

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Abstract. It is shown that two classes of groupoids satisfying certain one-sided quasigroup conditions, namely the classes of one-sided torsion groupoids and of one-sided finite exponent groupoids, are complete lattices, both isomorphic to the lattice of Steinitz numbers with the divisibility relation.

## 1. Introduction

Recall that Steiner quasigroups can be defined as groupoids which are idempotent commutative and satisfy $xy^2 = x$ where $xy^2$ denotes the term $(xy)y$. So-called 'power conditions' (like $(*^n)$ and $({}^m*)$ in Section 1.2) are quite often applied in group and groupoid theory (see for example [4]). Groupoids satisfying at least one of $(*^n)$ and $({}^m*)$ are one-sided quasigroups (the terminology is briefly recalled in Section 1.1). In [6] it is shown among other things that the varieties of groupoids defined by $(*^n)$ (as well as by $({}^m*)$) form a lattice isomorphic to the lattice of positive integers with the divisibility relation (see [6, Theorem 13], recalled in this paper as Theorem 4). By a slight generalization of conditions $(*^n)$ and $({}^m*)$ (see $(*^s)$ and $(*^t_s)$ below) we obtain a collection of new classes of groupoids which are also one-sided quasigroups. To describe the algebraic structure formed by these classes we compare it to the lattice structure which arises from an idea of E. Steinitz (see [13, p. 250]). This idea —

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recalled briefly in Section 1.3 — allows us to treat the lattice of positive integers with the divisibility relation as a sublattice of a certain complete lattice. Today this idea is known as Steinitz numbers ([2, 12]) (or supernatural or surnatural numbers ([11])). These numbers have found applications in various parts of algebra, especially in field theory, group theory and some related areas. Examples of such applications can be found in [2, 7, 9, 10, 11, 12]. In the present paper it is proved that the classes of one-sided torsion groupoids and of one-sided finite exponent groupoids (the definitions are recalled in Section 1.2) form lattices both isomorphic to the lattice of Steinitz numbers with the divisibility relation. This main result is presented in Section 2 (Theorem 7).

Parts of the results of this paper were announced without proof at the AAA 76 – 76th International Workshop on General Algebra (Linz 2008).

### 1.1. Main notation and notions

The paper is closely connected with [5] and [6], so similar terminology and notation is used. For convenience of the reader we repeat some material from [5] and [6] without proofs, making our presentation self-contained. All undefined notions and notations are standard and can be found in [3] or [8]. Some basic notions specific to quasigroup theory are analogous to those introduced in [1].

We also use the following notations and terminology:

- $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}_k \overset{\text{def}}{=} \mathbb{N} - \{0, \ldots, k-1\}$.
- If $n, m \in \mathbb{N}_1$, then $n \mid m$ means that $n$ divides $m$.
- If $A \subseteq \mathbf{Cn}$ is a set of cardinal numbers, then $\mathrm{lcm}(A)$ and $\gcd(A)$ denote the least common multiple and the greatest common divisor of the set $A$ respectively.
- If $A$ and $B$ are sets, then $A^B \overset{\text{def}}{=} \{f \mid f\colon B \longrightarrow A\}$ denotes the set of all maps from $B$ to $A$.

By a groupoid is meant a pair $\mathfrak{G} = (G, \cdot\,)$ with universe (base set) $G$ and binary operation

$$\cdot : G \times G \longrightarrow G \quad ((x, y) \mapsto xy).$$

In the following, the symbol $\mathfrak{G}$ stands for groupoids only.

Recall that in groupoid theory, by a *right* (resp. *left*) *quasigroup* we mean a groupoid $\mathfrak{G}$ such that for all $a, b \in G$ the equation $xa = b$ (resp. $ax = b$) has a unique solution. By a *quasigroup* we mean a groupoid which is a right and left quasigroup simultaneously. For a groupoid $\mathfrak{G} = (G, \cdot\,)$ we have the *dual groupoid* $\mathfrak{G}^{\leftarrow} = (G, \circ\,)$ where $x \circ y \overset{\text{def}}{=} yx$. Clearly

$(\mathfrak{G}^{\leftarrow})^{\leftarrow} = \mathfrak{G}$. Let $t$ be a term over a language appropriate for groupoid theory. Let $\mathfrak{G}$ be a groupoid. Then the interpretation $t^{\mathfrak{G}^{\leftarrow}}$ is named the *dual sentence* to the interpretation $t^{\mathfrak{G}}$. Thus, if a groupoid $\mathfrak{G}$ is a right quasigroup then its dual groupoid is a left quasigroup and vice versa. This duality establishes a symmetrical correspondence between 'right' and 'left' versions of statements (to every theorem in the right version corresponds its dual left version and vice versa). Therefore, for conciseness we formulate almost all statements below in one (right) version only. The term '*one-sided finite exponent* groupoid' will mean a finite right exponent groupoid or a finite left exponent groupoid (the definitions are recalled in Section 1.2). The term '*one-sided torsion* groupoid' is understood analogously.

### 1.2. Classes of groupoids with right (left) quasigroup properties

Let us recall some idea presented in [6].

The family of 'power' terms $\{xy^n \mid n \in \mathbb{N}_1\}$ is inductively defined as follows:

$$xy^1 = xy, \quad xy^n = (xy^{n-1})y,$$

and similarly for $\{{}^n yx \mid n \in \mathbb{N}_1\}$. With these families of terms there are naturally associated some families of identities. For $n, m \in \mathbb{N}_1$ we have the following 'power' identities:

$$xy^n = x, \tag{$*^n$}$$

$$^m yx = x. \tag{$^m*$}$$

Let $\mathcal{Q}_n$ (resp. ${}_m\mathcal{Q}$) denote the variety of groupoids defined by the identity $(*^n)$ (resp. $({}^m*)$). Moreover ${}_m\mathcal{Q}_n \overset{\text{def}}{=} {}_m\mathcal{Q} \cap \mathcal{Q}_n$. The elements of $\mathcal{Q}_n$ will be called *groupoids of right exponent* $n$. Let $\mathcal{Q} \overset{\text{def}}{=} \{\mathcal{Q}_n \mid n \in \mathbb{N}_1\}$. Set $\mathcal{Q}^* \overset{\text{def}}{=} \bigcup \mathcal{Q}$. A groupoid $\mathfrak{G}$ is said to be of *finite right exponent* if $\mathfrak{G} \in \mathcal{Q}^*$. The varieties $\mathcal{Q}_n$ ($n \in \mathbb{N}_1$) and the class $\mathcal{Q}^*$ were studied in [6].

**Lemma 1.** ([6, Lemma 1]) *Let $\mathfrak{G}$ be a groupoid, and $a, b \in G$.*

(i) *If $ab^n = a$ and $k \in \mathbb{N}_1$, then $ab^{kn} = a$.*
(ii) *If $ab^n = a$ and $ab^m = a$, then $ab^{\gcd(m,n)} = a$.*
(iii) *If $ab^n = a$ and ${}^k ba = a$, then $ab^{\mathrm{lcm}(k,n)} = a$ and ${}^{\mathrm{lcm}(k,n)} ba = a$.*

**Proposition 2.** ([6, Proposition 2]) *For $n, k \in \mathbb{N}_1$ the following statements hold:*

(i) *$\mathcal{Q}_n \subseteq \mathcal{Q}_{kn}$;*

(ii) $\mathcal{Q}_k \cup \mathcal{Q}_n \subseteq \mathcal{Q}_m$, *where* $m = \mathrm{lcm}(k, n)$;
(iii) $\mathcal{Q}_k \cap \mathcal{Q}_n = \mathcal{Q}_d$, *where* $d = \gcd(k, n)$;
(iv) ${}_k\mathcal{Q}_n \subseteq {}_m\mathcal{Q}_m$, *where* $m = \mathrm{lcm}(k, n)$.

**Proposition 3.** ([6, Proposition 9])

(i) *If* $\mathfrak{G}$ *satisfies* $(*^n)$ *for some* $n \in \mathbb{N}_1$, *then* $\mathfrak{G}$ *is a right quasigroup.*
(ii) *If* $\mathfrak{G}$ *satisfies both* $(*^n)$ *and* $({}^m*)$ *for some* $n, m \in \mathbb{N}_1$, *then* $\mathfrak{G}$ *is a quasigroup.*

**Theorem 4.** ([6, Theorem 13]) *The lattice* $\mathfrak{Q}^* = (\mathcal{Q}, \wedge, \vee)$ *is isomorphic to* $\mathfrak{N}_1 = (\mathbb{N}_1, \gcd, \mathrm{lcm})$.

The following formulas can be seen as a natural generalization of the identities $(*^n)$ (cf. [6]):

$$\forall\, x, y\ \exists\, n \in \mathbb{N}_1\ \ xy^n = x. \tag{$*^t$}$$

We call $\mathfrak{G}$ *right-torsion* if the formula $(*^t)$ is satisfied in $\mathfrak{G}$. The class of right-torsion groupoids is denoted by $\mathcal{T}^*$.

Denote by $\mathcal{QG}^*$ (resp. ${}^*\mathcal{QG}$) the class of right (resp. left) quasigroups.

**Proposition 5.** ([6, Proposition 14]) $\mathcal{Q}^* \subsetneq \mathcal{T}^* \subsetneq \mathcal{QG}^*$.

Clearly neither $\mathcal{Q}^*$ nor $\mathcal{T}^*$ is a variety.

### 1.3. Steinitz numbers

We recall briefly an idea of extension of positive integers introduced by E. Steinitz. The construction and notation we propose are presented in the form suitable for our further considerations.

Let $\mathbb{P}$ be the set of prime numbers. There exists a 1-1 map

$$\alpha\colon \mathbb{N}_1 \longrightarrow \mathbb{S}^\circ \quad (n \mapsto \alpha(n)), \tag{1}$$

where

$$\mathbb{S}^\circ = \{x \mid x \in \mathbb{N}^{\mathbb{P}} \text{ and } |\{p \mid x_p \neq 0\}| < \aleph_0\},$$

i.e. $\mathbb{S}^\circ$ is the set of maps from $\mathbb{P}$ to $\mathbb{N}$ which are zero almost everywhere. The mapping $\alpha(n)$ is defined as the infinite sequence

$$\alpha(n) = 2^{\alpha(n)_2} 3^{\alpha(n)_3} 5^{\alpha(n)_5} \ldots p^{\alpha(n)_p} \ldots$$

which corresponds to the unique prime factorization of $n$. Thus we can write slightly informal equalities

$$n = \prod_{p\in\mathbb{P},\, \alpha(n)_p\neq 0} p^{\alpha(n)_p} = \alpha(n) = 2^{\alpha(n)_2}3^{\alpha(n)_3}5^{\alpha(n)_5}\dots p^{\alpha(n)_p}\dots \tag{2}$$

identifying these formally different objects.

Let us consider the following two lattices:

- $\mathfrak{N}_1 = (\mathbb{N}_1, |)$, the lattice of positive integers with the divisibility relation,
- $\mathfrak{S}^\circ = (\mathbb{S}^\circ, \leq)$ where $\leq$ denotes the *product relation*, i.e. for $a, b \in \mathbb{S}^\circ$ we have $a \leq b$ if and only if $a_p \leq b_p$ for every $p \in \mathbb{P}$.

It is clear that

$$n \mid m \iff \alpha(n) \leq \alpha(m), \tag{3}$$

so that

$$\alpha\colon \mathfrak{N}_1 \xrightarrow{\text{iso}} \mathfrak{S}^\circ, \tag{4}$$

i.e. $\alpha$ is an isomorphism of lattices.

Using the traditional intuitive and slightly informal notation and applying the identification (2) we write

$$n = \alpha(n) = \prod_{p\in\mathbb{P}} p^{\alpha(n)_p} \tag{5}$$

where in the formal symbol $\prod_{p\in\mathbb{P}} p^{\alpha(n)_p}$ almost all factors are equal to 1. Using this notation we can write

$$\gcd(n,m) = \min\{\alpha(n), \alpha(m)\} = \prod_{p\in\mathbb{N}} p^{\min\{\alpha(n)_p, \alpha(m)_p\}}, \tag{6}$$

$$\operatorname{lcm}(n,m) = \max\{\alpha(n), \alpha(m)\} = \prod_{p\in\mathbb{N}} p^{\max\{\alpha(n)_p, \alpha(m)_p\}}. \tag{7}$$

Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ be an extension of $\mathbb{N}$ by adding a new element $\omega$. We extend the standard relation $\leq$ by defining $n \leq \omega$ for all $n \in \bar{\mathbb{N}}$. Thus $\bar{\mathfrak{N}} = (\bar{\mathbb{N}}, \leq)$ is a bounded chain. Define $\mathbb{S} = \bar{\mathbb{N}}^{\mathbb{P}}$. Then $\mathbb{S}$ is called the *set of Steinitz numbers*. Thus $\mathfrak{S} = (\mathbb{S}, \leq)$ with the product relation $\leq$ is a complete (and evidently bounded) lattice called the *lattice of Steinitz numbers*. It is clear that $\mathfrak{S}^\circ$ is a sublattice of $\mathfrak{S}$. For Steinitz numbers we introduce the notation analogous to (5). If $s \in \mathbb{S}$ then we write

$$s = \prod_{p\in\mathbb{P}} p^{s_p} \tag{8}$$

where $s_p \in \bar{\mathbb{N}}$ and it is not assumed that almost all factors are equal to 1. On $\mathbb{S}$ one can define the divisibility relation $|$ as follows. For $r, s \in \mathbb{S}$,

$$s \mid r \overset{\text{def}}{\iff} s \leq r.$$

Thus the map $\alpha$ is an embedding of the lattice $\mathfrak{N}_1$ into $\mathfrak{S} = (\mathbb{S}, |)$ (see (4)). Statements (9) and (10) below can be seen as the analogues of (6) and (7) respectively for Steinitz numbers. Let $A \subseteq \mathbb{S}$ and $A \neq \varnothing$. Then

$$\gcd(A) = \min\{s \mid s \in A\} = \prod_{p\in\mathbb{N}} p^{\min\{s_p \mid s\in A\}}, \tag{9}$$

$$\operatorname{lcm}(A) = \max\{s \mid s \in A\} = \prod_{p\in\mathbb{N}} p^{\max\{s_p \mid s\in A\}}. \tag{10}$$

$\mathfrak{S}$ as a relational system determines the algebra $\mathfrak{S}_a = (\mathbb{S}, \gcd, \operatorname{lcm}, 1, \omega)$ of type $(2, 2, 0, 0)$ which is a bounded lattice in algebraic interpretation.

**Some examples**

The positive integer 30 can be interpreted as the Steinitz number $2^1 3^1 5^1$ (factors equal to 1, i.e. of the form $p^0$, are omitted). Numbers like $2^\omega 3^1 5^1$ or $\prod_{p\in\mathbb{N}} p$ have no interpretation as positive integers. For every $n \in \mathbb{N}$ we have $2^n 3^1 5^1 \mid 2^\omega 3^1 5^1$. Moreover, $\omega = \prod_{p\in\mathbb{N}} p^\omega$ and $1 = \prod_{p\in\mathbb{N}} p^0$ are the maximum and minimum in $\mathfrak{S}$ respectively.

## 2. Classes of groupoids with one-sided quasigroup conditions

The notion of Steinitz numbers allows us to present the classes of finite one-sided exponent groupoids and of one-sided torsion groupoids as collections of subclasses satisfying so-called bounded conditions.

Let $s \in \mathbb{S}$. The following formulas can be seen as a generalization of the identities $(*^n)$ for Steinitz numbers:

$$\exists\, n \in \mathbb{N}_1\ \forall\, x, y \quad n \mid s,\ xy^n = x. \tag{$*^s$}$$

Thus $\mathfrak{G}$ satisfies $(*^s)$ if and only if there exists $n \in \mathbb{N}_1$ such that $n \mid s$ and $\mathfrak{G}$ satisfies $(*^n)$. Let $\mathcal{Q}_{\bar{s}}$ be the class of groupoids satisfying $(*^s)$. The members of $\mathcal{Q}_{\bar{s}}$ will be called *s-bounded right exponent groupoids.* Clearly (for example by Proposition 2(i)), if $m \in \mathbb{N}_1$ then $\mathcal{Q}_m = \mathcal{Q}_{\bar{m}}$. This identity allows us to simplify notation by omitting the bar over $s$ in $\mathcal{Q}_{\bar{s}}$. In this terminology (clearly $\mathcal{Q}^* = \mathcal{Q}_\omega$), the class of finite right exponent groupoids is the class of $\omega$-bounded right exponent groupoids, and the variety $\mathcal{Q}_1$ (i.e. the variety $\mathbf{0}^*$ of left-zero groupoids) is the class of 1-bounded right exponent groupoids. Clearly if $s \in \mathbb{S}$ then

$$\mathcal{Q}_s = \bigcup_{n \in \mathbb{N}_1,\, n|s} \mathcal{Q}_n. \tag{11}$$

As a generalization of the condition $(*^t)$ we have:

$$\forall\, x, y\ \exists\, n \in \mathbb{N}_1 \quad n \mid s,\ xy^n = x. \tag{$*^t_s$}$$

Let $\mathcal{T}_s$ be the class of all groupoids satisfying $(*^t_s)$. The elements of $\mathcal{T}_s$ will be called *s-bounded right torsion groupoids.*

**Proposition 6.**

(i) *If $m \in \mathbb{N}_1$, then $\mathcal{Q}_m = \mathcal{T}_m$.*

(ii) *Let $s_1, s_2 \in \mathbb{S}$ be such that $s_1 \mid s_2$. Then $\mathcal{Q}_{s_1} \subseteq \mathcal{Q}_{s_2}$ and $\mathcal{T}_{s_1} \subseteq \mathcal{T}_{s_2}$.*

(iii) *If $s \in \mathbb{S}$, then $\mathcal{Q}_s \subseteq \mathcal{T}_s$. If $s \in \mathbb{S}$ and $s$ is not a positive integer, then $\mathcal{Q}_s \subsetneq \mathcal{T}_s$.*

*Proof.* (i) By the definitions of $\mathcal{Q}_m$ and $\mathcal{T}_m$, clearly $\mathcal{Q}_m \subseteq \mathcal{T}_m$. Let $\mathfrak{G} \in \mathcal{T}_m$. Then for every $a, b \in G$ there exists $n \in \mathbb{N}$ such that $n \mid m$ and $ab^n = a$. By Lemma 1 we see that for every $a, b \in G$ the equality $ab^m = a$ is satisfied. Hence the identity $xy^m = x$ is satisfied in $\mathfrak{G}$ and $\mathfrak{G} \in \mathcal{Q}_m$.

(ii) This is an easy consequence of the definition of $\mathcal{Q}_s$ and $\mathcal{T}_s$ for $s \in \mathbb{S}$.

(iii) It is clear that $\mathcal{Q}_s \subseteq \mathcal{T}_s$. Assume that $s$ is not a positive integer. To prove that $\mathcal{Q}_s \neq \mathcal{T}_s$ we use a construction similar to the one in [6, Example 8]. By assumption, $s$ is a Steinitz number that is not a positive integer. Thus $s = \prod_{p \in \mathbb{P}} p^{s_p}$ and there are two mutually not necessarily exclusive possibilities:

(a) There exists $p \in \mathbb{P}$ such that $s_p = \omega$,

(b) The set $\{s_p \mid s_p \neq 0\}$ is not finite.

In case (a), fix $p \in \mathbb{P}$ such that $s_p = \omega$. Let $f = (f_n)_{n \in \mathbb{N}_1}$ be a family of permutations of $\mathbb{N}_1$ such that

$$f_n \stackrel{\text{def}}{=} (1\ \dots\ np) \cup id_{\mathbb{N}_{np+1}}$$

i.e. $f_n$ is a cyclic permutation of $\{1, \dots, np\}$ and the identity elsewhere. Let $\mathfrak{N}_f = (\mathbb{N}_1, \cdot_f)$ with $a \cdot_f b = f_b(a)$. In the following we omit the symbol $\cdot_f$ in products. Evidently the groupoid $\mathfrak{N}_f$ satisfies condition $(*^t_{p^\omega})$. Thus $\mathfrak{N}_f \in \mathcal{T}_{p^\omega}$. It is clear that $p^\omega \mid s$. Therefore $\mathcal{T}_{p^\omega} \subseteq \mathcal{T}_s$ by (ii). Hence $\mathfrak{N}_f \in \mathcal{T}_s$. Suppose that $\mathfrak{N}_f$ satisfies condition $(*^s)$. Then $\mathfrak{N}_f$ satisfies the identity $xy^m = x$ for some $m \in \mathbb{N}_1$ such that $m \mid s$. Evidently $\alpha(m)_p < \omega$. Let $b > \alpha(m)_p$. Then $bp > \alpha(m)_p$. Therefore $1b^{bp} = 1$ (by the construction of $\mathfrak{N}_f$) and $1b^m = 1$ (by the assumption that $\mathfrak{N}_f$ satisfies the identity $xy^m = x$). From Lemma 1 (ii) we have $1 = 1b^{\gcd(bp,m)} = 1b^{\alpha(m)_p}$, a contradiction. Hence $\mathfrak{N}_f \in \mathcal{T}_s - \mathcal{Q}_{\bar{s}}$.

In case (b), there exists an infinite sequence $\bar{p} = (p_1, p_2 \ldots)$ of prime numbers such that $s_{p_n} \geq 1$ for every $n \in \mathbb{N}_1$. Let $f = (f_n)_{n \in \mathbb{N}_1}$ be a family of permutations of $\mathbb{N}_1$ such that

$$f_n \overset{\text{def}}{=} \begin{cases} (1\ \ldots\ p_i) \cup id_{\mathbb{N}_{p_i+1}} & \text{if } n = p_i \text{ for some } i \in \mathbb{N}_1, \\ id_{\mathbb{N}_1} & \text{otherwise.} \end{cases}$$

As in the previous item, let $\mathfrak{N}_f = (\mathbb{N}_1, \cdot_f)$ with $a \cdot_f b = f_b(a)$. Thus $\mathfrak{N}_f \in \mathcal{T}_{s'}$ where $s' = \prod_{p \in \mathbb{P}} p^{\alpha(s')_p}$ is such that

$$\alpha(s')_p = \begin{cases} 1 & \text{if } p = p_i \text{ for some } i \in \mathbb{N}_1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathfrak{N}_f \in \mathcal{T}_s$, because $\mathfrak{N}_f \in \mathcal{T}_{s'}$ and $s' \mid s$ (see (ii)). Suppose that $\mathfrak{N}_f$ satisfies condition $(*^s)$. Then $\mathfrak{N}_f$ satisfies the identity $xy^m = x$ for some $m \in \mathbb{N}_1$ such that $m \mid s$. Let $p_i$ be a prime number appearing in the sequence $\bar{p}$ and not appearing in the prime decomposition of $m$. We have $1p_i^{p_i} = 1$ and $1p_i^m = 1$. Therefore $1 = 1p_i^{\gcd(p_i,m)} = 1p_i$ by Lemma 1 (ii), a contradiction. Thus $\mathfrak{N}_f \in \mathcal{T}_s - \mathcal{Q}_{\bar{s}}$. □

Let $\bar{\mathcal{Q}} \overset{\text{def}}{=} \{\mathcal{Q}_s \mid s \in \mathbb{S}\}$ and $\bar{\mathcal{T}} \overset{\text{def}}{=} \{\mathcal{T}_s \mid s \in \mathbb{S}\}$. We have $\mathcal{Q}^* = \mathcal{Q}_\omega = \bigcup \bar{\mathcal{Q}}$ and $\mathcal{T}^* = \mathcal{T}_\omega = \bigcup \bar{\mathcal{T}}$.

**Theorem 7.** *$\bar{\mathfrak{Q}}^* = (\bar{\mathcal{Q}}, \subseteq)$ and $\bar{\mathfrak{T}}^* = (\bar{\mathcal{T}}, \subseteq)$ are complete lattices, both isomorphic to the lattice of Steinitz numbers $\mathfrak{S} = (\mathbb{S}, |)$.*

*Proof.* Let us consider the relational system $\bar{\mathfrak{Q}}^* = (\bar{\mathcal{Q}}, \subseteq)$ and the complete lattice $\mathfrak{S} = (\mathbb{S}, |)$. Let us take the map

$$\varphi_{\mathcal{Q}} \colon \mathbb{S} \longrightarrow \bar{\mathcal{Q}} \quad (x \mapsto \varphi_{\mathcal{Q}}(x) \overset{\text{def}}{=} \mathcal{Q}_x).$$

We will prove that $\varphi_{\mathcal{Q}} \colon \mathfrak{S} \xrightarrow{\text{iso}} \bar{\mathfrak{Q}}^*$ (i.e. $\varphi_{\mathcal{Q}}$ is an isomorphism of relational systems). The proof of this fact is divided into two steps:

(a) $\varphi_{\mathcal{Q}}$ is a bijection,

(b) $\varphi_{\mathcal{Q}}$ and $\varphi_{\mathcal{Q}}^{-1}$ are homomorphisms of relational systems.

To prove (a), let $s_1, s_2 \in \mathbb{S}$, $s_1 \neq s_2$. Without loss of generality we can assume that there exists a prime number $p$ such that $s_{1,p} < s_{2,p}$. Thus there exists a positive integer $s'_{2,p}$ such that $s_{1,p} < s'_{2,p} \leq s_{2,p}$. Let $\mathfrak{N}_f = (\mathbb{N}_1, \cdot_f)$ with

$$f_n \stackrel{\text{def}}{=} \begin{cases} (1 \ \dots \ s'_{2,p}) \cup id_{\mathbb{N}_{s'_{2,p}+1}} & \text{if } n = p, \\ id_{\mathbb{N}_1} & \text{otherwise.} \end{cases}$$

Thus $\mathfrak{N}_f \in \mathcal{Q}_{s_2} - \mathcal{Q}_{s_1}$. Therefore $\varphi_{\mathcal{Q}}(s_1) = \mathcal{Q}_{s_1} \neq \mathcal{Q}_{s_2} = \varphi_{\mathcal{Q}}(s_2)$. It is obvious that $\varphi_{\mathcal{Q}}$ is a surjection.

To prove (b), let $s_1, s_2 \in \mathbb{S}$ with $s_1 \mid s_2$. By Proposition 6 (ii), $\varphi_{\mathcal{Q}}(s_1) = \mathcal{Q}_{s_1} \subseteq \mathcal{Q}_{s_2} = \varphi_{\mathcal{Q}}(s_2)$. Now assume that $\mathcal{Q}_{s_1} \subseteq \mathcal{Q}_{s_2}$ and suppose that $s_1 \nmid s_2$. Then $s_{2,p} < s_{1,p}$ for some prime number $p$. Thus there exists a positive integer $s'_{1,p}$ such that $s_{2,p} < s'_{1,p} \leq s_{1,p}$. Let $\mathfrak{N}_f = (\mathbb{N}_1, \cdot_f)$ with

$$f_n \stackrel{\text{def}}{=} \begin{cases} (1 \ \dots \ s'_{1,p}) \cup id_{\mathbb{N}_{s'_{1,p}+1}} & \text{if } n = p, \\ id_{\mathbb{N}_1} & \text{otherwise.} \end{cases}$$

Thus $\mathfrak{N}_f \in \mathcal{Q}_{s_1} - \mathcal{Q}_{s_2}$, a contradiction.

Using similar arguments one can prove that

$$\varphi_{\mathcal{T}} \colon \mathbb{S} \longrightarrow \bar{\mathcal{T}} \quad (x \mapsto \varphi_{\mathcal{T}}(x) \stackrel{\text{def}}{=} \mathcal{T}_x)$$

is an isomorphism of the relational systems $\mathfrak{S}$ and $\bar{\mathfrak{T}}^*$. □

The complete lattices $\bar{\mathfrak{Q}}^*$, $\bar{\mathfrak{T}}^*$ as relational systems determine the algebras $\bar{\mathfrak{Q}}^*_a = (\bar{\mathcal{Q}}, \wedge, \vee, \mathbf{0}^*, \mathcal{Q}^*)$ and $\bar{\mathfrak{T}}^*_a = (\bar{\mathcal{T}}, \wedge, \vee, \mathbf{0}^*, \mathcal{T}^*)$ respectively, both of type $(2, 2, 0, 0)$; these are bounded lattices in algebraic interpretation. As an immediate consequence of Theorem 7 we obtain the following corollary:

**Corollary 8.** $\bar{\mathfrak{Q}}^*_a \simeq \bar{\mathfrak{T}}^*_a \simeq \mathfrak{S}_a$.

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## CONTACT INFORMATION

**J. Gałuszka**

Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

*E-Mail:* jan.galuszka@polsl.pl

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