Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 9 **(2010)**. Number 1. pp. 16 – 30

On commutative nilalgebras of low dimension

Juan C. Gutierrez Fernandez

Communicated by Efin Zelmanov

Abstract. We prove that every commutative non-associative nilalgebra of dimension ≤ 7, over a field of characteristic zero or sufficiently large is solvable.

Introduction

Throughout this paper the term *algebra* is understood to be a commutative not necessarily associative algebra. We will use the notations and terminology of [6] and [7]. Let $\mathfrak{A}$ be an (commutative nonassociative) algebra over a field F. We define inductively the following powers, $\mathfrak{A}^1 = \mathfrak{A}$ and $\mathfrak{A}^s = \sum_{i+j=s} \mathfrak{A}^i\mathfrak{A}^j$ for all positive integers $s \geq 2$. We shall say that $\mathfrak{A}$ is *nilpotent* if there is a positive integer s such that $\mathfrak{A}^s = (0)$. The least such number is called the index of nilpotency of the algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is called *nilalgebra* if given $a \in \mathfrak{A}$ we have that $\mathrm{alg}(a)$, the subalgebra of $\mathfrak{A}$ generated by a, is nilpontent. The *(principal) powers* of an element a in $\mathfrak{A}$ are defined recursively by $a^1 = a$ and $a^{i+1} = aa^i$ for all integers $i \geq 1$. The algebra $\mathfrak{A}$ is called *left-nilalgebra* if for every a in $\mathfrak{A}$ there exists an integer $k = k(a)$ such that $a^k = 0$. The smallest positive integer k which this property is the *index*. Obviously, every nilalgebra is left-nilalgebra. For any element a in $\mathfrak{A}$, the linear mapping L_a of $\mathfrak{A}$ defined by $x \to ax$ is called *multiplication operator* of $\mathfrak{A}$. An *Engel algebra* is an algebra in which every multiplication operator is nilpotent in the sense that for $a \in \mathfrak{A}$ there exists a positive integer j such that $L_a^j = 0$.

Partially Supported by FAPESP 05/01790-9 and FAPESP 2005/60337-2, Brazil
2000 Mathematics Subject Classification: *17A05,17A30.*
Key words and phrases: *solvable, commutative, nilalgebra.*

An important question is that of the existence of simple nilalgebras in the class of finite-dimensional algebras. We have the following *Shestakov's Conjecture:* there exists an example of commutative finite-dimensional simple nilalgebras. In [6] we proved that every nilagebra $\mathfrak{A}$ of dimension ≤ 6 over a field of characteristic $\neq 2, 3, 5$ is solvable and hence $\mathfrak{A}^2 \subsetneq \mathfrak{A}$. For power-associative nilalgebras of dimension ≤ 8 over a field of characteristic $\neq 2, 3, 5$, we have shown in [8] that they are solvable, and hence there is no simple algebra in this subclass. See also [4] and [6] for power-associative nilalgebras of dimension ≤ 7.

We show now the process of linearization of identities, which is an important tool in the theory of varieties of algebras. See [9], [12] and [13] for more information. Let P be the free commutative nonassociative polynomial ring in two generators x and y over a field F. For every $\alpha_1, \ldots, \alpha_r \in P$, the *operator linearization* $\delta[\alpha_1, \ldots, \alpha_r]$ can be defined as follows: if $p(x, y)$ is a monomial in P, then $\delta[\alpha_1, \ldots, \alpha_r]p(x, y)$ is obtained by making all the possible replacements of r of the k identical arguments x by $\alpha_1, \ldots, \alpha_r$ and summing the resulting terms if $x-$degree of $p(x, y)$ is $\geq r$, and is equal to zero in other cases. Some examples of this operator are $\delta[y](x^2(xy)) = 2(xy)^2 + x^2y^2$, $\delta[x^2, y](x^2) = 2x^2y$ and $\delta[y, xy^2, x](x^2) = 0$. For simplicity, $\delta[\alpha : r]$ will denote $\delta[\alpha_1, \ldots, \alpha_r]$, where $\alpha_1 = \cdots = \alpha_r = \alpha$. We observe that if $p(x)$ is a polynomial in P, then $p(x + y) = p(x) + \sum_{j=1}^{\infty} \delta[y : j]p(x)$, where $\delta[y : j]p(x)$ is the sum of all the terms of $p(x+y)$ which have degree j with respect to y.

The following known results are a basic tool in our investigation. See [2], [3] and [7].

Lemma 1. *Let $\mathfrak{A}$ be a commutative left-nilalgebra of index ≤ 4 over a field F of characteristic different form 2 or 3. Then $\mathfrak{A}$ satisfies the identities*

$$x^2x^3 = -x(x^2x^2), \quad x^3x^3 = (x^2)^3 = x(x(x^2x^2)), \tag{1}$$

$$x^3y = -x(x^2y) - 2x(x(xy)), \tag{2}$$

$\mathfrak{A}$ is a nilalgebra of index ≤ 7 and every monomial in P of x-degree ≥ 10 and y-degree 1 is an identity in $\mathfrak{A}$. Furthermore, for every $a \in \mathfrak{A}$ the associative algebra $\mathfrak{A}_a$ generated by all L_c with $c \in \mathrm{alg}(a)$ is in fact generated by L_a and L_{a^2}.

For simplicity, we will denote by L and U the multiplication operators, L_x and L_{x^2} respectively, where x is an element in $\mathfrak{A}$.

Lemma 2. *[7] Let $\mathfrak{A}$ be a commutative algebra over a field of characteristic $\neq 2$ or 3 satisfying the identities $x^4 = 0$ and $x(x^2x^2) = 0$. Then $\mathfrak{A}$*

satisfies the following multiplication identities:

$$L_{x^2x^2} = -4LUL, \qquad UU = -2ULL + 2LUL + 4L^4, \tag{3}$$

Table i, Multiplication identities of degree 5,

	ULU	LUL^2	L^3U	L^5
UUL	0	2	0	0
LUU	0	−2	−2	−4
L^2UL	0	0	−1	−4
UL^3	0	0	0	2

Table ii, Multiplication identities of degree 6,

	$ULLU$	L^4U	L^6
UUU	−2	4	8
$UULL$	0	0	4
$ULUL$	−1	2	4
$LUUL$	0	2	0
$LULU$	0	0	4

	$ULLU$	L^4U	L^6
$LLUU$	0	−4	−4
UL^4	0	0	2
LUL^3	0	0	2
L^2UL^2	0	1	0
L^3UL	0	−1	−4

Furthermore, every monomial in P of x-degree ≥ 7 and y-degree 1 is an identity in $\mathfrak{A}$ and the algebra generated by L_x and L_{x^2} is spanned, as vector space, by $L, U, L^2, UL, LU, L^3, UL^2, LUL, L^2U, L^4, ULU, LUL^2$, L^3U, L^5, UL^2U, L^4U, L^6.

Lemma 3. *[7] Let $\mathfrak{A}$ be a commutative algebra over a field of characteristic $\neq 2, 3$ or 5, satisfying the identities $x^4 = 0$ and $x(x(x^2x^2)) = 0$. Then $\mathfrak{A}$ satisfies the following multiplication identities:*

$$LUU = -2LUL^2 - 2L^3U - 4L^5, \tag{4}$$
$$LUL^3 = -\frac{1}{2}(L^2UL^2 + L^3UL), \tag{5}$$
$$L^4UL = -3L^5U - 16L^7, \tag{6}$$
$$L^2ULU = -L^3UL^2 + 5L^5U + 28L^7, \tag{7}$$
$$UL^4U = -\frac{1}{2}L^2UL^2U + 24L^6U + 62L^8, \tag{8}$$
$$L^2UL^2U = 48L^6U + 156L^8, \tag{9}$$
$$L^6U = -2L^8. \tag{10}$$

Furthermore, every monomial in P of x-degree ≥ 9 and y-degree 1 is an identity in $\mathfrak{A}$.

We now study (commutative nonassociative) nilalgebras of dimension ≤ 7, over a field F of characteristic zero or sufficiently large. We will show that nilalgebras over F with dimension ≤ 7, are solvable. An algebra $\mathfrak{A}$ is called *solvable* if there exists a positive integer t such that $\mathfrak{A}^{[t]} = (0)$, where we define inductively $\mathfrak{A}^{[1]} = \mathfrak{A}$ and $\mathfrak{A}^{[j+1]} = \mathfrak{A}^{[j]}\mathfrak{A}^{[j]}$ for all positive integers j.

Let $\mathfrak{A}$ be a finite-dimensional nilgalgebra over F. We will denote by $\deg(\mathfrak{A})$, the *degree* of $\mathfrak{A}$, the smallest number m such that for every $a \in \mathfrak{A}$, the subalgebra $\mathrm{alg}(a)$ of $\mathfrak{A}$ generated by a has $\dim(\mathrm{alg}(a)) \leq m$. If $\deg(\mathfrak{A}) \leq 2$, then $\mathfrak{A}$ satisfies the identity $x^3 = 0$ and hence this algebra is Jordan. It is well-known that any finite-dimensional Jordan nilalgebra is nilpotent. Therefore $\mathfrak{A}$ is nilpotent if $\deg(\mathfrak{A}) \leq 2$. Because any nilpotent algebra is solvable, we have that $\mathfrak{A}$ is solvable if $\deg(\mathfrak{A}) \leq 2$.

The following lemma, proved in [6], is an immediate consequence of a result of [10] and [11] for linear spaces of nilpotent matrices.

Lemma 4. *Let $\mathfrak{A}$ be a nilalgebra over the field F. Then $\mathfrak{A}^2\mathfrak{A}^2 \subset \mathfrak{B}$ for every subalgebra $\mathfrak{B}$ of codimension ≤ 2.*

By above lemma, if $\deg(\mathfrak{A}) \geq \dim(\mathfrak{A}) - 2$, then $\mathfrak{A}^2\mathfrak{A}^2$ is nilpotent and hence $\mathfrak{A}$ is solvable. Summarizing, $\mathfrak{A}$ is solvable in the following cases: (i) $\dim(\mathfrak{A}) \leq 5$; (ii) $\dim(\mathfrak{A}) = 6$ and $\deg(\mathfrak{A}) \neq 3$; (iii) $\dim(\mathfrak{A}) = 7$ and $\deg(\mathfrak{A}) \neq 3$ or 4. Thus, for $\dim(\mathfrak{A}) \leq 7$, it remains to be shows that $\mathfrak{A}$ is solvable if $\deg(\mathfrak{A}) = 3$ or 4.

The following lemma is clear from Lemma 1. For any subset S of $\mathfrak{A}$ we denote by $\langle S \rangle$ the vector space spanned by S.

Lemma 5. *Let $\mathfrak{A}$ be an algebra over F satisfying the identity $x^4 = 0$. Consider an element a in $\mathfrak{A}$.* (i) *If $a(a(a^2a^2)) \neq 0$, then* $\dim(\mathrm{alg}(a)) = 6$*;* (ii) *If $a(a(a^2a^2)) = 0$ and $a(a^2a^2) \neq 0$, then* $\dim(\mathrm{alg}(a)) = 5$.

Proof. By Lemma 1 we observe that $\mathfrak{A}$ is a nilalgebra of nilindex ≤ 7 and $\mathrm{alg}(a) = \langle a, a^2, a^3, a^2a^2, a(a^2a^2), a(a(a^2a^2)) \rangle$. Assume $a(a(a^2a^2)) \neq 0$. We will prove that $a, a^2, a^3, a^2a^2, a(a^2a^2), a(a(a^2a^2))$ are linearly independent. Let $\lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2a^2 + \lambda_5 a(a^2a^2) + \lambda_6 a(a(a^2a^2)) = 0$. Then $0 = L_a^2 L_{a^2} L_a(0) = L_a^2 L_{a^2} L_a(\lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2a^2 + \lambda_5 a(a^2a^2) + \lambda_6 a(a(a^2a^2))) = \lambda_1 a(a(a^2a^2))$ and hence $\lambda_1 = 0$. Analogously, $0 = L_a^2 L_{a^2}(0) = L_a^2 L_{a^2}(\lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2a^2 + \lambda_5 a(a^2a^2) + \lambda_6 a(a(a^2a^2))) = \lambda_2 a(a(a^2a^2))$ so that $\lambda_2 = 0$. Next, $0 = L_a L_{a^2}(0) = L_a L_{a^2}(\lambda_3 a^3 + \lambda_4 a^2a^2 + \lambda_5 a(a^2a^2) + \lambda_6 a(a(a^2a^2))) = -\lambda_3 a(a(a^2a^2))$ so that $\lambda_3 = 0$. And analogously we can prove that $\lambda_4 = \lambda_5 = \lambda_6 = 0$. The case (ii) is similar. $\square$

Corollary 1. *Let $\mathfrak{A}$ be an algebra over F satisfying the identity $x^4 = 0$. Assume $\deg(\mathfrak{A}) = 3$ or 4 and let a be an element in $\mathfrak{A}$. Then $\mathrm{alg}(a) = \langle a, a^2, a^3, a^2a^2 \rangle$ and $\langle a^3, a^2a^2 \rangle \cdot \mathrm{alg}(a) = 0$.*

1. The case degree(A)=3

Now we will study nilalgebras of degree 3. In this section $\mathfrak{A}$ will be a nilalgebra of degree 3 and dimension ≤ 7 over the field F. Consider a an element in $\mathfrak{A}$. Because $\mathfrak{A}$ is nilalgebra, there exists a positive integer t such that $a^t = 0$. We can assume that $a^t = 0$ and $a^{t-1} \neq 0$. Clearly, the elements $a, a^2, \ldots, a^{t-1}$ are linearly independent, and hence $t \leq 4$, since $\deg(\mathfrak{A}) = 3$. Consequently, the algebra $\mathfrak{A}$ satisfies the identity $x^4 = 0$. By Corollary 1, the sequence a^3, a^2a^2 is linearly dependent and $\mathfrak{A}$ satisfies the identities $x(x^2x^2) = 0, \quad x^2x^3 = 0$. Consequently, $\mathfrak{A}$ satisfies multiplication identities (3), Tables i and ii and Lemma 2.

Lemma 6. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then $L^6 = 0$ is a multiplication identity in $\mathfrak{A}$.*

Proof. Assume that there exist $a, b \in \mathfrak{A}$ such that $L_a^6(b) \neq 0$. Then the sequence $\Psi = \{L_a^i(b) : i = 0, 1, \ldots, 6\}$ is a basis of $\mathfrak{A}$. On the other hand, we note that from Table ii and (3) we have

$$L_a^6(b) = \frac{1}{2}a(a^2(a(a(ab)))) = -\frac{1}{8}(a^2a^2)(a(ab)),$$

so that $a^2a^2 \neq 0$. Because Ψ is a basis and $a(a^2a^2) = 0$, we get that

$$a^2a^2 = \lambda L_a^6(b),$$

for any $0 \neq \lambda \in F$. Combining above relations we get that

$$a^2a^2 = (a^2a^2)[(-\lambda/8)a(ab)],$$

but this is impossible because $\mathfrak{A}$ is an Engel algebra. Therefore $L_a^6 = 0$ for all $a \in \mathfrak{A}$. □

We may use (2) combined with (3) to yield

$$L_{x^2x^2}L - 4L_{x^3}L^2 = 8L^5. \tag{11}$$

We shall use this formula now.

Lemma 7. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then $L^5 = 0$ is a multiplication identity in $\mathfrak{A}$.*

Proof. Assume that there exist $a, b \in \mathfrak{A}$ such that $L_a^5(b) \neq 0$. By identity (11) we have that either $a^3 \neq 0$ or $a^2a^2 \neq 0$. The proof now splits into two cases.

Case 1. If $a^2a^2 \neq 0$, then $a^3 = \beta a^2a^2$ for any $\beta \in F$ and using (11) we obtain $8L_a^5 = L_{a^2a^2}L_a - 4\beta L_{a^2a^2}L_a^2$. Multiplying this relation from the right side with L_a yields $L_{a^2a^2}L_a^2 = 0$, so that $L_{a^3}L_a^2 = 0$ and $L_{a^2a^2}L_a = 8L_a^5$. Now, it is easy to prove that $\Psi = \{a^2a^2, L_a^i(b) : i = 0, 1, \ldots, 5\}$ is linearly independent and hence a basis of $\mathfrak{A}$. Let $a^2 = \lambda a^2a^2 + \sum_{i=0}^{5} \mu_i L_a^i(b)$. Multiplying by a, 2 times, we get $0 = \mu_0 L_a^2(b) + \mu_1 L_a^3(b) + \mu_2 L_a^4(b) + \mu_3 L_a^5(b)$, so that $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$. Now, multiplying with a^2, we get $a^2a^2 = \lambda(a^2)^3 + \mu_4 L_{a^2}L_a^4(b) + \mu_5 L_{a^2}L_a^5(b) = 0$, but this is impossible.

Case 2. If $a^2a^2 = 0$, then $L_{a^3}L_a^2 = -2L_a^5$. Now, it is easy to prove that $\Phi = \{a^3, L_a^i(b) : i = 0, 1, \ldots, 5\}$ is linearly independent and hence a basis of $\mathfrak{A}$. Let $a = \lambda a^3 + \sum_{i=0}^{5} \mu_i L_a^i(b)$. Multiplying by a, 3 times, we get $0 = \mu_0 L_a^3(b) + \mu_1 L_a^4(b) + \mu_2 L_a^5(b)$, so that $\mu_0 = \mu_1 = \mu_2 = 0$. Next, multiplying by a two time, we have $a^3 = \mu_3 L_a^5(b)$, but this is impossible because Φ is a basis. □

Lemma 8. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then every monomial in P of x-degree ≥ 6 and y-degree 1 is an identity in $\mathfrak{A}$.*

Proof. By Lemma 2 and Lemma 6 we only need to prove that $L^4U = 0$ and $UL^2U = 0$ are multiplication identities in $\mathfrak{A}$. Using identity (3), Table ii and relation $0 = \delta[x^2]\{x(x(x(x(xy))))\}$ we have that $0 = UL^4 + LUL^3 + L^2UL^2 + L^3UL + L^4U = L^2UL^2 + L^3UL + L^4U = L^4U$. Now, from Lemma 2, multiplication identities $L^6 = 0$ and $L^4U = 0$, and identity (11), we see that

$$ULLU = -ULUL = \frac{1}{4}UL_{x^2x^2} = UL_{x^3}L.$$

Let $a \in \mathfrak{A}$. If $a^2a^2 = 0$, if follows immediately that $L_{a^2}L_a^2L_{a^2} = 0$. If $a^2a^2 \neq 0$, then there exists $\lambda \in F$ such that $a^3 = \lambda a^2a^2$. Therefore, $L_{a^2}L_a^2L_{a^2} = L_{a^2}L_{a^3}L_a = \lambda L_{a^2}L_{a^2a^2}L_a = 0$. This proves the lemma. □

Using identity (2), Lemma 2 and Lemma 7, we can prove easily the following multiplication identities

$$L^3U = -L^2UL = -L^2L_{x^3} = LL_{x^3}L = \frac{1}{4}LL_{x^2x^2},$$

$$LUL^2 = -L_{x^3}L^2 = -\frac{1}{4}L_{x^2x^2}L,$$

for nilalgebras of dimension ≤ 7 and degree 3 over the field F. We shall use these formulas now.

Lemma 9. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then $L^3U = 0$ and $LUL^2 = 0$ are multiplications identities in $\mathfrak{A}$.*

Proof. Let a be an element in $\mathfrak{A}$. If $a^2a^2 = 0$ then, from above identities we obtain immediately that $L_a^3 L_{a^2} = (1/4)L_a L_{a^2a^2} = 0$ and $L_a L_{a^2} L_a^2 = -(1/4)L_{a^2a^2}L_a = 0$. If $a^2a^2 \neq 0$ then there exists $\lambda \in F$ such that $a^3 = \lambda a^2a^2$. This means that $L_{a^3} = \lambda L_{a^2a^2}$. Then we have $L_a^3 L_{a^2} = L_a^2 L_{a^3} = \lambda L_a^2 L_{a^2a^2} = 0$ and $L_a L_{a^2} L_a^2 = -L_{a^3} L_a^2 = \lambda L_{a^2a^2} L_a^2 = 0$ by Lemma 8. This proves the lemma. □

Lemma 10. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then $LUL = 0$ is a multiplication identity in $\mathfrak{A}$.*

Proof. We will assume the contrary, there exist two elements $a, b \in \mathfrak{A}$ such that $a(a^2(ab)) \neq 0$. We know by (3) that

$$a(a^2(ab)) = -(1/4)(a^2a^2)b.$$

Therefore, $a^2a^2 \neq 0$ and also the sequence $\{a^2a^2, a(a^2(ab))\}$ is linearly independent, because L_b is nilpotent. For any $\lambda \in F$ we have that $a^3 = \lambda a^2a^2$. Obviously, this forces $L_{a^3} = \lambda L_{a^2a^2}$. From identity (2) and above lemma, we have immediately that $L_a L_{a^2} L_a = -L_{a^3} L_a - 2L_a^4 = -\lambda L_{a^2a^2} L_a - 2L_a^4 = 4\lambda L_a L_{a^2} L_a^2 - 2L_a^4 = -2L_a^4$, that is

$$L_a L_{a^2} L_a = -2L_a^4. \tag{12}$$

We will now prove that $\Psi = \{b, ab, a^2(ab), a(a^2(ab)), a, a^2, a^2a^2\}$ is a basis of $\mathfrak{A}$. Let $\lambda_1 b + \lambda_2 ab + \lambda_3 a^2(ab) + \lambda_4 a(a^2(ab)) + \mu_1 a + \mu_2 a^2 + \mu_3 a^2a^2 = 0$, with $\lambda_i, \mu_j \in F$. Multiplying with a, a^2 and a successively, we get $\lambda_1 = 0$. Multiplying with a^2 and a successively, we have $\lambda_2 = 0$. Multiplying with a and a^2 successively, we obtain $\mu_1 = 0$, so that

$$\lambda_3 a^2(ab) + \lambda_4 a(a^2(ab)) + \mu_2 a^2 + \mu_3 a^2a^2 = 0.$$

Multiplying with a it follows that $\lambda_3 = 0$ since $a^2a^2, a(a^2(ab))$ are linearly independent and $a^3 \in \langle a^2a^2 \rangle$. Multiplying with a^2 we have $\mu_2 = 0$. Now, relation $\lambda_4 a(a^2(ab)) + \mu_3 a^2a^2 = 0$ forces $\lambda_4 = \mu_3 = 0$. Therefore, we have proved that the sequence Ψ is linearly independent. Since $\dim(\mathfrak{A}) \leq 7$, it follows that Ψ is a basis of $\mathfrak{A}$.

On the other hand, because Ψ is a basis of $\mathfrak{A}$, we have a representation $a(ab) = \alpha_1 b + \alpha_2 ab + \alpha_3 a^2(ab) + \alpha_4 a(a^2(ab)) + \alpha_5 a + \alpha_6 a^2 + \alpha_7 a^2a^2$, with $\alpha_i \in F$. Using the operators $L_a L_{a^2} L_a$, $L_a L_{a^2}$, $L_{a^2} L_a$ and $L_a L_a$, we prove that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_5 = 0$ and $a(a(a(ab))) = 0$ respectively, but this is impossible since by identity (12) we have that $2a(a(a(ab))) = -a(a^2(ab))$ and by hypothesis this element is diferent form zero. This proves the lemma. □

It was proved in [8] the following result for power-associative nilalgebras.

Lemma 11. *Every commutative power-associative nilalgebra of dimension ≤ 8 over a field of characteristic $\neq 2, 3$ or 5 is solvable.*

Theorem 1. *Let $\mathfrak{A}$ be a nilalgebra over the field F with dimension ≤ 7 and degree 3. Then, the algebra $\mathfrak{A}$ is solvable.*

Proof. By Lemma 10 we have that a^2a^2 belong to the annihilator of the algebra $\mathfrak{A}$, for every $a \in \mathfrak{A}$. This means that the linear subspace $J = \langle a^2a^2 : a \in \mathfrak{A}\rangle$ is an ideal of $\mathfrak{A}$ and $\mathfrak{A}J = 0$. Thus, $\mathfrak{A}/J$ is a commutative power-associative nilalgebra of dimension ≤ 7, and hence solvable. This implies that $\mathfrak{A}$ is solvable. □

2. The case degree(A) = 4 and x(x(xx))=0

For any subalgebra $\mathfrak{B}$ of an algebra $\mathfrak{A}$, the set $\mathrm{st}(\mathfrak{B}) = \{x \in \mathfrak{A} : x\mathfrak{B} \subset \mathfrak{B}\}$ is called *stabilizer* of $\mathfrak{B}$ in $\mathfrak{A}$. For every element $a \in \mathrm{st}(\mathfrak{B})$, we can define a linear transformation $\overline{L_a}$ on the quotient vector space $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{B}$ as follows,

$$\overline{L_a}(x + \mathfrak{B}) = ax + \mathfrak{B},$$

for all $x \in \mathfrak{A}$. We will now denote by $M_{\mathfrak{B}}$ the linear space $\{\overline{L_a} : a \in \mathrm{st}(\mathfrak{B})\}$ and by $N_{\mathfrak{B}}$ the linear subspace $\{\overline{L_b} : b \in \mathfrak{B}\}$. Evidently, we have that $N_{\mathfrak{B}} \subset M_{\mathfrak{B}}$.

The following result will be useful. Items (iv), (v), (vi) and (vii) follow immediately from (i)-(iii) proved in [6].

Lemma 12. *Let V be a vector space of dimension 3 over a field F of characteristic $\neq 2$ and let $\mathfrak{M}$ be a vector space of nilpotent linear endomorphisms in $End_F(V)$. Then $\dim \mathfrak{M} \leq 3$ and either $\mathfrak{M}^3 = 0$ or: (i) $\dim \mathfrak{M} = 2$; (ii) for every nonzero $f \in \mathfrak{M}$ we have that $rank(f) = 2$; (iii) if $\mathfrak{M} = \langle f_1, f_2\rangle$, then there exists a basis ϕ of V and $0 \neq \lambda \in F$ such that the matrices (using columns) of f_1 and λf_2 with respect to ϕ are*

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively; (iv) if f, g and fg are all in $\mathfrak{M}$, then $f = 0$ or $g = 0$; (v) if f, g, h and $f(g + fh)$ are all in $\mathfrak{M}$ and $f \neq 0$, then $g = 0$ and $h \in \langle f\rangle$; (vi) if $f, g, h \in \mathfrak{M}$ and $f(f + gh) = 0$, then $f = 0$; (vii) if $f, g \in \mathfrak{M}$ and $f^2g^2 = 0$, then the sequence $\{f, g\}$ is linearly dependent.

Let $\mathfrak{A}$ be a nilalgebra over the field F with degree 4 and dimension ≤ 7 satisfying the identity $x^4 = 0$. From Lemma 4, $\mathfrak{A}$ is solvable if $\dim(\mathfrak{A}) \leq 6$, so that throughout this section we will assume that $\mathfrak{A}$ has dimension 7. By Corollary 1, the algebra $\mathfrak{A}$ satisfies the identities $x(x^2x^2) = 0$ and $x^2x^3 = 0$. Now we may take an element b in $\mathfrak{A}$ such that $\mathfrak{B}$, the subalgebra of $\mathfrak{A}$ generated by b, has dimension 4. By Corollary 1, we have

$$\mathfrak{B} = \langle b, b^2, b^3, b^2b^2 \rangle,$$

and

$$\langle b^3, b^2b^2 \rangle \mathfrak{B} = (0). \tag{13}$$

If $\dim N_{\mathfrak{B}} = 0$, then $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ and hence $\mathfrak{A}$ is solvable because $\mathfrak{A}/\mathfrak{B}$ is solvable. If $M_{\mathfrak{B}}$ is nilpotent, then there exists $a \in \mathfrak{A}$ but not in $\mathfrak{B}$ such that

$$f(a + \mathfrak{B}) = 0 + \mathfrak{B}, \tag{14}$$

for all $f \in M_{\mathfrak{B}}$. There exists a smallest integer m, $1 \leq m \leq 3$, such that $M_{\mathfrak{B}}^m = (0)$. If $m = 1$ take $a \in \mathfrak{A}$ but not in $\mathfrak{B}$; if $m > 1$, take $0 \neq g \in M_{\mathfrak{B}}^{m-1}$ and $a + \mathfrak{B}$ in $g(\mathfrak{A}/\mathfrak{B})$ with $a + \mathfrak{B} \neq 0 + \mathfrak{B}$. Then (14) is satisfied. Since $a \in \mathrm{st}(\mathfrak{B})$ we have that $\overline{L_a} \in M_{\mathfrak{B}}$. Then relation (14) implies that $0 + \mathfrak{B} = \overline{L_a}(a + \mathfrak{B})$ and hence $a^2 \subset \mathfrak{B}$. Let $\mathfrak{B}' = \langle b, b^2, b^3, b^2b^2, a \rangle$. We have that $\mathfrak{B}'$ is a subalgebra of $\mathfrak{A}$ with codimension 2. Using Lemma 4 we get that $\mathfrak{A}^2\mathfrak{A}^2 \subset \mathfrak{B}'$ so that $\mathfrak{A}$ is solvable.

We now consider the case $N_{\mathfrak{B}} \neq (0)$ and $M_{\mathfrak{B}}^3 \neq (0)$. Then $M_{\mathfrak{B}}$ satisfies (i)-(vii) of Lemma 12. By Lemma 1 we have that $N_{\mathfrak{B}}$ is nilpotent, so that Lemma 12 forces $\dim(N_{\mathfrak{B}}) = 1$ since $N_{\mathfrak{B}} \subset M_{\mathfrak{B}}$ and $M_{\mathfrak{B}}$ is not nilpotent. Let $0 \neq h \in N_{\mathfrak{B}}$. Then $\overline{L_{b^i}} = \alpha_i h$ for any $\alpha_i \in F$ and for $i = 1, 2, 3$. From identities (3) and (2) we have $\overline{L_{b^2b^2}} = -4\alpha_1^2\alpha_2 h^3 = 0$ and $\alpha_3 h = \overline{L_{b^3}} = -\alpha_1\alpha_2 h^2 - 2\alpha_1^3 h^3 = -\alpha_1\alpha_2 h^2$ so that $\overline{L_{b^3}} = 0$ since $h^3 = 0$. Next (3) forces $\overline{L_{b^2}}^2 = -2\alpha_1^2\alpha_2 h^3 + 2\alpha_1^2\alpha_2 h^3 + 4\alpha_1^4 h^4 = 0$. Therefore $\overline{L_{b^2}} = 0$ since $\overline{L_{b^2}} \in M_{\mathfrak{B}}$ and from Lemma 12 every nonzero element in $M_{\mathfrak{B}}$ is nilpotent of index 3. Thus, we have proved that

$$\mathfrak{B}^2\mathfrak{A} = \langle b^2, b^3, b^2b^2 \rangle \mathfrak{A} \subset \mathfrak{B}.$$

This yields $N_{\mathfrak{B}} = \langle \overline{L_b} \rangle$. Because $N_{\mathfrak{B}} \subsetneq M_{\mathfrak{B}}$ and $\dim(M_{\mathfrak{B}}) = 2$, we can take $a \in \mathrm{st}(\mathfrak{B})$, but not in $\mathfrak{B}$ such that $M_{\mathfrak{B}} = \langle \overline{L_b}, \overline{L_a} \rangle$. By Lemma 12 there exists a basis $\Phi = \{v_1 + \mathfrak{B}, v_2 + \mathfrak{B}, v_3 + \mathfrak{B}\}$ of $\mathfrak{A}/\mathfrak{B}$ such that the matrices of $\overline{L_b}$ and $\overline{L_a}$ with respect to Φ are respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\lambda}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

for any $0 \neq \lambda \in F$. This means that $v_3 + \mathfrak{B} = \alpha a + \mathfrak{B}$, $v_2 + \mathfrak{B} = \alpha a^2 + \mathfrak{B}$ and $v_1 + \mathfrak{B} = \alpha a^3 + \mathfrak{B}$ for any $\alpha \in F$, $\alpha \neq 0$. We can assume, without lost of generality, that $\lambda = 1$ and $\alpha = 1$. Now, by equation (3) and (13) we have $(b^2 b^2)a = -4b(b^2(ba)) \subset b(b^2\mathfrak{B}) = (0)$, so that

$$(b^2 b^2)a = 0.$$

On the other hand, ab can be expressed as a linear combination of b, b^2, b^3, b^2b^2. Let $ab = \mu_1 b + \mu_2 b^2 + \mu_3 b^3 + \mu_4 b^4 b^2$. Then $cb = \mu_1 b + \mu_4 b^2 b^2$, where $c = a - \mu_2 b - \mu_3 b^2$. Therefore $c(cb) = \mu_1 cb + \mu_4 c(b^2 b^2) = \mu_1 cb$. Since $\mathfrak{A}$ is an Engel algebra, L_c is nilpotent and hence either $\mu_1 = 0$ or $cb = 0$. This implies

$$ab \in \mathfrak{B}^2.$$

Using relation (3) we have that $b^2(b^2 a) = -2b^2(b(ba)) + 2b(b^2(ba)) + 4b(b(b(ba))) = 0$. This forces

$$b^2 a \in \mathfrak{B}^3.$$

Finally, using (2) and (3) we have

$$\begin{aligned}(b^2b^2)a^2 &= -4b(b^2(ba^2)) \in \langle b(b^2(-a+\mathfrak{B}))\rangle = \langle b(b^2a)\rangle = 0,\\ (b^2b^2)a^3 &= -a(a^2(b^2b^2)) - 2a(a(a(b^2b^2))) = 0,\end{aligned}$$

and hence $b^2b^2 \in \operatorname{ann}(\mathfrak{A})$. Let $J = \langle b^2b^2 : b \in \mathfrak{A}, \dim(\operatorname{alg}(b)) = 4\rangle$. Then $\mathfrak{A}/J$ is a commutative nilalgebra of dimension ≤ 6 and degree ≤ 3, so that solvable. This implies that $\mathfrak{A}$ is solvable.

3. The case degree(A) = 4

Let $\mathfrak{A}$ be a nilalgebra with degree 4. If $a \in \mathfrak{A}$, then there exists an integer t such that $a^t \neq 0$ and $a^{t+1} = 0$ so that the elements $a, a^2, \ldots, a^t$ are linearly independent. Since $\deg(\mathfrak{A}) = 4$, we have that $t \leq 4$ and hence $\mathfrak{A}$ satisfies the identity

$$x^5 = 0.$$

Now we will see that $\mathfrak{A}$ is a nilalgebra of index ≤ 9. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ generated by a single element and let k_1 be the index of $\mathfrak{B}$ as left-nilalgebra, that is k_1 is the smallest integer such that $x^{k_1} = 0$ for all $x \in \mathfrak{B}$. Evidently, $\dim \mathfrak{B} \leq 4$ and $k_1 \leq 5$. If $k_1 \leq 3$, then $\mathfrak{B}$ is a Jordan algebra and hence nilpotent with $\mathfrak{B}^{k_1} = 0$. If $k_1 = 4$, then by Lemma 1 and Lemma 5, we have that $p(x) = 0$ is an identity in $\mathfrak{B}$ for every monomial $p(x)$ of degree ≥ 5. Finally, if $k_1 = 5$, then there exists $b \in \mathfrak{B}$ such that $\mathfrak{B} = \langle b, b^2, b^3, b^4\rangle$. Now, because $\mathfrak{B}$ is nilpotent, we have that

$$b^2b^2 \in \langle b^3, b^4\rangle, \quad b^2b^3, b^3b^3 \in \langle b^4\rangle, \quad b^4\mathfrak{B} = (0).$$

Thus, $\mathfrak{B}^3, \mathfrak{B}^4 \subset \langle b^3, b^4\rangle$, $\mathfrak{B}^5 \subset \langle b^4\rangle$ and $\mathfrak{B}^t = 0$ for all $t \geq 9$. It has the following consequences.

Lemma 13. *The algebra $\mathfrak{A}$ satisfies the identities*

$$x^i(x^j(x^t x^2)) = 0, \qquad i, j, t \geq 1,$$

and $p(x) = 0$ for every monomial $p(x) \in P$ of degree ≥ 9.

Linearizing the above identities we have the following multiplication identities (in order to simplify, we will write L_3 instead of L_{x^3} and L_4 instead of L_{x^4}):

$$L_4 + LL_3 + L^2U + 2L^4 = 0, \tag{15}$$
$$L_{x^2x^3} + 2LL_3L + LUU + 2LUL^2 = 0, \tag{16}$$
$$2L_4L^2 + L_4U + L_3L_3 + L_3LU + 2L_3L^3 = 0, \tag{17}$$
$$L_{x^3x^3} + 2LL_3U + 4LL_3L^2 = 0, \tag{18}$$
$$U^3 + 2UUL^2 + 2UL_3L + 2L_{x^2x^3}L = 0, \tag{19}$$
$$L_4(L_3 + LU + 2L^3) = 0. \tag{20}$$

Lemma 14. *[9] Every nilalgebra of bounded index over F is an Engel algebra.*

We have proved that the index of a nilalgebra of degree 4 is ≤ 9. We then apply Lemma 14 to obtain

Corollary 2. *Every nilalgebra of degree 4 over F is an Engel algebra.*

Theorem 2. *Let $\mathfrak{A}$ be a nilalgebra over the field F. If $\dim(\mathfrak{A}) \leq 7$, then $\mathfrak{A}$ is solvable.*

Proof. We already prove that $\mathfrak{A}$ is solvable if either $\deg(\mathfrak{A}) \neq 4$ or $x^4 = 0$ is an identity. Thus, it remains to prove that $\mathfrak{A}$ is solvable if $\dim(\mathfrak{A}) = 7$, $\deg(\mathfrak{A}) = 4$ and $x^4 = 0$ is not an identity in $\mathfrak{A}$.

Let $\mathfrak{A}$ be a nilalgebra of dimension 7 and degree 4 such that there exists $b \in \mathfrak{A}$ with $b^4 \neq 0$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by b. Because $\mathfrak{A}$ has degree 4, we have $\mathfrak{B} = \langle b, b^2, b^3, b^4\rangle$ and $b^5 = 0$. As in Section 3, if $M_{\mathfrak{B}}$ is nilpotent, then the algebra $\mathfrak{A}$ is solvable. Also, the algebra is solvable if $N_{\mathfrak{B}} = 0$. Thus, we can assume that $M_{\mathfrak{B}}$ is not nilpotent and $\dim N_{\mathfrak{B}} \geq 1$. By Lemma 12, we have that $\dim(M_{\mathfrak{B}}) = 2$. From (15) we have

$$\overline{L_{b^4}} = -\overline{L_bL_{b^3}} - \overline{L_b}^2\,\overline{L_{b^2}} = -\overline{L_b}(\overline{L_{b^3}} + \overline{L_bL_{b^2}}) \in N_{\mathfrak{B}}.$$

Combining above relation and (v) of Lemma 12 we get that $\overline{L_{b^4}}=0$. Now (17) implies $\overline{L_{b^3}}(\overline{L_{b^3}}+\overline{L_bL_{b^2}})=0$ and by (vi) of Lemma 12 we get that $\overline{L_{b^3}}=0$. This means that

$$\mathfrak{B}^3\mathfrak{A}\subset\mathfrak{B}, \tag{21}$$

and $N_{\mathfrak{B}}=\langle\overline{L_b},\overline{L_{b^2}}\rangle$. Now relation (19) for $x=b$ forces $0=\overline{L_{b^2}}^3+2\overline{L_{b^2}}^2\overline{L_b}^2+2\overline{L_{b^2}L_{b^3}L_b}+2\overline{L_{b^2b^3}L_b}=2\overline{L_{b^2}}^2\overline{L_b}^2$ and hence using (vii) of Lemma 12 we have that

$$\dim(N_B)=1. \tag{22}$$

We can assume, without loss of generality, that

$$\overline{L_b}\neq 0, \tag{23}$$

since if $\overline{L_b}=0$, then $\overline{L_{b^2}}\neq 0$ and we can take $0\neq\lambda\in F$ such that $(b+\lambda b^2)^4=b^4+\lambda[b(b^2b^2)+b^2b^3]+\lambda^2(b^2)^3\neq 0$. Because $\dim(N_{\mathfrak{B}})=1$, there exists $\alpha\in F$ such that $\overline{L_{b^2}}=\alpha\overline{L_b}$. As in Section 3 there exists $a\in\mathfrak{A}$ such that $M_{\mathfrak{B}}=\langle\overline{L_b},\overline{L_a}\rangle$, $\Phi=\{a^3+\mathfrak{B},a^2+\mathfrak{B},a+\mathfrak{B}\}$ is a basis of $\mathfrak{A}/\mathfrak{B}$ and the matrices of $\overline{L_b}$ and $\overline{L_a}$ with respect to Φ are respectively $\begin{bmatrix}0&0&0\\1&0&0\\0&-1&0\end{bmatrix}$ and $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$. This means that

$$ba^3-a^2,b^2a^3-\alpha a^2,ba^2+a,b^2a^2+\alpha a,ba,b^2a,a^4\in\mathfrak{B}. \tag{24}$$

By (24) we have that $ba\in\mathfrak{B}$ so that $ba=\lambda_1b+\lambda_2b^2+\lambda_3b^3+\lambda_4b^4$, with $\lambda_i\in F$. Therefore $[a-\lambda_2b-\lambda_3b^3-\lambda_4b^3]b=\lambda_1b$. This implies that $\lambda_1=0$ and hence

$$ba\in\mathfrak{B}^2, \tag{25}$$

since every multiplication operator on $\mathfrak{A}$ is nilpotent. Let j be a positive integer. From (15) and (21) we get

$$b^4a^j=-b(b^3a^j)-b(b(b^2a^j))-2b(b(b(ba^j)))\in\mathfrak{B}^2, \tag{26}$$

so that

$$b^4\mathfrak{A}\subset\mathfrak{B}^2. \tag{27}$$

By (20) we see that $0=b^4(b^3a^3+b(b^2a^3)+2b(b(ba^3)))=b^4(b(b^2a^3))=-\alpha b^4a$ and now using identity (17) we have $2b^4(b(ba^3))+b^4(b^2a^3)+b^3(b^3a^3)+b^3(b(b^2a^3))+2b^3(b(b(ba^3)))=0$ and hence $-2b^4a+\alpha(b^4a^2-b^3a)=-b^3(b^3a^3)\in\langle b^4\rangle$. Therefore

$$b^4a=0,\qquad \alpha(b^4a^2-b^3a)\in\langle b^4\rangle, \tag{28}$$

since $\mathfrak{A}$ is an Engel algebra and $\alpha b^4a=0$.

The proof now splits into two cases:

Case 1. The relation $x^3x^3 = 0$ is not an identity in $\mathfrak{A}$. In this case, we can assume without loss of generality that

$$b^4 \neq 0, \quad \text{and} \quad b^3b^3 \neq 0. \tag{29}$$

Let b_1 be an element in $\mathfrak{A}$ such that $b_1^3b_1^3 \neq 0$ and $\mathfrak{B}$ the subalgebra of $\mathfrak{A}$ generated by b_1. Then $\{b_1, b_1^2, b_1^3, b_1^3b_1^3\}$ is a basis of $\mathfrak{B}$ and products satisfy the following properties, $b_1^2b_1^2 \in \langle b_1^3, b_1^3b_1^3\rangle$, $b_1^3\mathfrak{B} \subset \langle b_1^3b_1^3\rangle$ and $(b_1^3b_1^3)\mathfrak{B} = (0)$, because $\mathfrak{B}$ is nilpotent. By Corollary 1 we have that $x^4 = 0$ is not an identity in $\mathfrak{B}$. Thus, there exists an element b in $\mathfrak{B}$ of the form $\lambda_1b_1 + \lambda_2b_1^2 + \lambda_3b_1^3 + \lambda_4b_1^3b_1^3$ such that $b^4 \neq 0$ and also we can assume that $\overline{L_b} \neq 0$. Evidently, we have $\lambda_1 \neq 0$. Now $b^2 \in \lambda_1^2b_1^2 + \langle b_1^3, b_1^3b_1^3\rangle$, $b^3 \in \lambda_1^3b_1^3 + \langle b_1^3b_1^3\rangle$ and $b^3b^3 = \lambda_1^6b_1^3b_1^3 \neq 0$. Then (29) is satisfied. Evidently, we have $b^3b^3 = \gamma b^4$ for any $0 \neq \gamma \in F$. Combining (18) and (28) it follows $0 = (b^3b^3)a^2 + 2b(b^3(b^2a^2)) + 4b(b^3(b(ba^2))) = (b^3b^3)a^2 - 2\alpha b(b^3a) = \gamma b^4a^2 - 2\alpha b(b^4a^2)$. Thus $b^4b^2 = (2\alpha/\gamma)b(b^4a^2)$. Since $\mathfrak{A}$ is an Engel algebra it follows that

$$b^4a^2 = 0.$$

Combining this identity with (28) we have that

$$\alpha b^3a \in \langle b^4\rangle. \tag{30}$$

Now, relation (16) with $x = b$ for the element a^2 implies $2b(b^3a) + \alpha b(b^2a) \in \langle b^4\rangle$. Combining this relation with (30) we see that $b(b^3a) \in \langle b^4\rangle$, so that $b^3a \in \langle b^3, b^4\rangle$. Therefore

$$b^3a \in \langle b^4\rangle,$$

since $\mathfrak{A}$ is an Engel algebra. Next, we put $x = b$ in (15) to obtain $0 = b^4a + bb^3a + b(b(b^2a)) + 2b(b(b(ba))) = b(b(b^2a))$, and hence

$$b^2a \in \langle b^3, b^4\rangle.$$

Now, by Lemma 13 we know that $x(x^2x^3) = 0$ is an identity in $\mathfrak{A}$ and hence $0 = (1/48)\delta[b : 4, a : 2]\{x(x^2x^3)\} = b(b^2(ba^2)) + 2b(b^2(a(ba))) + 4b((ba)(b(ba))) + 2a(b^2(b(ba))) + 2b((ba)(b^2a)) + a(b^2(b^2a)) + b(a^2b^3) + 2a((ba)b^3)$, forces $b(b^3a^2) \in \langle b^4\rangle$ so that

$$b^3a^2 \in \langle b^4\rangle.$$

Finally, (18) implies $0 = (b^3b^3)a^3 + 2b(b^3(b^2a^3)) + 4b(b^3(b(ba^3))) = (b^3b^3)a^3 + 2b(b^3(\alpha a^2 - 2a)) = (b^3b^3)a^3$. Therefore we must have $b^4a^3 = 0$. Consequently, we have proved, in this case, that $b^4 \in \mathrm{ann}(\mathfrak{A})$. Let $J =$

$\langle c^4 \colon c^3c^3 \neq 0,\, c \in \mathfrak{A}\rangle$. Then $\overline{\mathfrak{A}} = \mathfrak{A}/J$ is a nilalgebra of dimension ≤ 6 and hence solvable. This forces the solvability of $\mathfrak{A}$.

Case 2. The relation $x^3x^3 = 0$ is an identity in $\mathfrak{A}$. Linearizing this identity we have that $\mathfrak{A}$ satisfies the identity

$$x^3(x^2y) + 2x^3(x(xy)) = 0.$$

Taking $x = b$ and $y = a^2$ it follows immediately that $\alpha b^3 a \in \langle b^4\rangle$ and for $x = b$ and $y = a^3$ this identity forces $\alpha b^3a^2 - 2b^3a \in \langle b^4\rangle$. Therefore

$$b^3a \in \langle b^4\rangle.$$

Next, (15) forces $0 = b^4a + b(b^3a) + b(b(b^2a)) + 2b(b(b(ba))) = b(b(b^2a))$, so that

$$b^2a \in \langle b^3, b^4\rangle.$$

Now, taking the identity $\delta[b:4, a:3]\{x^4x^3\} = 0$ we have

$$\begin{aligned} -b^4a^3 &= \big[b^3a + b(b^2a) + 2b(b(ba))\big]\cdot\big[ba^2 + 2a(ab)\big] + \\ &\quad \big[b(ba^2) + 2b(a(ba)) + 2a(b(ba)) + a(ab^2)\big]\cdot\big[b^2a + 2b(ba)\big] + \\ &\quad \big[ba^3 + a(a^2b) + 2a(a(ab))\big]\cdot\big[b^3\big] \\ &\in \langle b^4\rangle\cdot\big\langle a, b, b^2, b^3, b^4\big\rangle + \mathfrak{B}\cdot\langle b^3, b^4\rangle + \mathfrak{B}\cdot\langle b^3\rangle \subset \langle b^4\rangle \end{aligned}$$

since by (24) we have that $ba^3 + a(a^2b) \in \mathfrak{B}$. This means that

$$b^4a^3 = 0,$$

because L_{a^3} is nilpotent. From $\delta[b:3, a:2]x^5 = 0$ we get $-b(b(ba^2)) = a(ab^3) + a(b(ab^2)) + 2a(b(b(ba))) + b(a(ab^2)) + 2b(a(b(ba))) + 2b(b(a(ba))) = 0$. This means that

$$b(ba^2) \in \langle b^4\rangle.$$

Finally, from $\delta[b:4, a:2]\{x^4x^2\} = 0$ it follows that

$$\begin{aligned} -b^4a^2 &= \big[b^3a + b(b^2a) + 2b(b(ba))\big]\cdot\big[2ba\big] + \\ &\quad \big[b(ba^2) + 2b(a(ba)) + 2a(b(ba)) + a(ab^2)\big]\cdot\big[b^2\big] \\ &\in \langle b^4\rangle\cdot\langle b^2, b^3, b^4\rangle + \langle b^4\rangle\cdot\langle b^2\rangle = (0). \end{aligned}$$

Therefore $b^4 \in \operatorname{ann}(\mathfrak{A})$ and as in the case 1, this implies the solvability of $\mathfrak{A}$. □

References

[1] A.A. Albert, *Power-associative rings*, Trans. Amer. Math. Soc., N.**64**, 1948, pp.552-593.

[2] I. Correa, I.R. Hentzel, A. Labra, *On the nilpotence of the multiplication operator in commutative right nil algebras*, Comm. in Algebra, N.**30**, 2002, pp.3473-3488.

[3] A. Elduque, A. Labra, *On the Classification of commutative right-nilalgebras of dimension at most four*, Comm. in Algebra, N.**35**, 2007, pp.577-588.

[4] L. Elgueta, A. Suazo, (2004). *Solvability of commutative power-associative nilalgebras of nilindex 4 and dimension* ≤ 8, Proyecciones, N.**23**, 2004, pp.123-129.

[5] L. Elgueta, J.C.G. Fernandez, A. Suazo, *Nilpotence of a class of commutative power-associative nilalgebras*, Journal of Algebra, N.**291**, 2005, pp.492-504.

[6] J.C.G. Fernandez, *On commutative power-associative nilalgebras*, Comm. in Algebra, N.**32**, 2004, pp.2243-2250.

[7] J.C.G. Fernandez, *On commutative left-nilalgebras of index 4*, Proyecciones, N.**27**(1), 2008, pp. 103-112.

[8] J.C.G. Fernandez, A. Suazo, *Commutative power-associative nilalgebras of nilindex 5* Result. Math., N.**47**, 2005, pp.296-304.

[9] M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices II*, Duke Math. J., N.**27**, 1960, pp.21-31.

[10] M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices III*, Ann. of Math., N.**70**, 1959, pp.167-205.

[11] B. Mathes, M. Omladic, H. Radjavi, *Linear spaces of nilpotent matrices*, Lin. Alg. Appl., N.**149**, 1991, pp.215-225.

[12] J.M. Osborn, *Varieties of algebras*, Adv. in Math., N.**8**, 1972, pp.163-396.

[13] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, A.I. (1982). *Rings that are nearly associative*, Academic Press, 1982.

Contact information

J. C. G. Fernandez Departamento de Matemática-IME, Universidade de São Paulo Caixa Postal 66281, CEP 05315-970, São Paulo, SP, Brazil
E-Mail: `jcgf@ime.usp.br`

Received by the editors: 19.09.2008
and in final form 19.09.2008.