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Commutator subgroups of the power subgroups of generalized Hecke groups

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Abstract. Let p, $q \geqslant 2$ be relatively prime integers and let $H_{p,q}$ be the generalized Hecke group associated to p and q. The generalized Hecke group $H_{p,q}$ is generated by $X(z) = -(z-\lambda_p)^{-1}$ and $Y(z) = -(z+\lambda_q)^{-1}$ where $\lambda_p = 2\cos\frac{\pi}{p}$ and $\lambda_q = 2\cos\frac{\pi}{q}$. In this paper, for positive integer m, we study the commutator subgroups $(H^m_{p,q})'$ of the power subgroups $H^m_{p,q}$ of generalized Hecke groups $H_{p,q}$. We give an application related with the derived series for all triangle groups of the form $(0; p, q, n)$, for distinct primes p, q and for positive integer n.

1. Introduction

In [12], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda},$$

where $\lambda \in \mathbb{R}$. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geqslant 3$ integer, or $\lambda \geqslant 2$. We consider the former case $q \geqslant 3$ integer and we denote it by $H_q = H(\lambda_q)$. Hecke group H_q is isomorphic to the free product of two finite cyclic groups of orders 2 and q, i.e.,

$$H_q = \langle T, S : T^2 = S^q = I\rangle \cong C_2 * C_q.$$

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The first few Hecke groups H_q are $H_3 = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$, and $H_6 = H(\sqrt{3})$. It is clear from the above that $H_q \subset PSL(2,\mathbb{Z}[\lambda_q])$ unlike in the modular group case (the case $q = 3$), the inclusion is strict and the index $|PSL(2,\mathbb{Z}[\lambda_q]) : H_q|$ is infinite as H_q is discrete whereas $PSL(2,\mathbb{Z}[\lambda_q])$ is not for $q \geqslant 4$.

Lehner and Newman studied in [20] more general class $H_{p,q}$ of Hecke groups H_q, by taking

$$X = -\frac{1}{z-\lambda_p} \quad \text{and} \quad Y = -\frac{1}{z+\lambda_q},$$

where p and q are integers such that $2 \leqslant p \leqslant q$, $p+q > 4$. The groups $H_{p,q}$ have the presentation,

$$H_{p,q} = \langle X, Y : X^p = Y^q = I \rangle \cong C_p * C_q. \tag{1.1}$$

We call these groups *generalized Hecke groups* $H_{p,q}$. Generalized Hecke groups admit representations as triangle Fuchsian groups with one cusp at infinity. More precisely, a triangle group with signature $(0; p, q, \infty)$ is isomorphic to $H_{p,q}$. There is a relationship to Veech groups, see [13] and [37].

We know from [19] that $H_{2,q} = H_q$, $|H_q : H_{q,q}| = 2$, and there is no group $H_{2,2}$. Also, all Hecke groups H_q are included in generalized Hecke groups $H_{p,q}$. Generalized Hecke groups $H_{p,q}$ and $(p,q,\infty)-$triangle groups have been also studied by many authors, in [3], [7], [8], [10], [11], [14], [16], [21], [22], [26], [36] and [38].

On the other hand, if m is a positive integer, then the power subgroup $H_{p,q}^m$ is generated by the m^{th} powers of all elements of $H_{p,q}$. As $H_{p,q}^m$ are fully invariant subgroups, they are normal in $H_{p,q}$.

If m and n are positive integers, then from the definition one can easily deduce that

$$H_{p,q}^{mn} \leqslant H_{p,q}^m \tag{1.2}$$

and that

$$H_{p,q}^{mn} \leqslant (H_{p,q}^m)^n.$$

The last two inequalities imply that

$$H_{p,q}^m \cdot H_{p,q}^n = H_{p,q}^{(m,n)}.$$

We know from [16] and [34] that

a) If $(m, p) = d$ and $(m, q) = 1$, then

$$H_{p,q}^{m} = \langle Y \rangle * \langle XYX^{-1} \rangle * \cdots * \langle X^{d-1}YX^{-(d-1)} \rangle * \langle X^{d} \rangle,$$
$$H_{p,q}^{m} \cong \underbrace{\mathbb{Z}_q * \cdots * \mathbb{Z}_q}_{d \text{ times}} * \mathbb{Z}_{p/d},$$

and the signature of $H_{p,q}^{m}$ is $(0;\, q^{(d)},\, p/d,\, \infty)$.

b) If $(m, p) = 1$ and $(m, q) = d$, then

$$H_{p,q}^{m} = \langle X \rangle * \langle YXY^{-1} \rangle * \cdots * \langle Y^{d-1}XY^{-(d-1)} \rangle * \langle Y^{d} \rangle,$$
$$H_{p,q}^{m} \cong \underbrace{\mathbb{Z}_p * \cdots * \mathbb{Z}_p}_{d \text{ times}} * \mathbb{Z}_{q/d},$$

and the signature of $H_{p,q}^{m}$ is $(0;\, p^{(d)},\, q/d,\, \infty)$.

c) $\left| H_{p,q} : H'_{p,q} \right| = pq$ and $H'_{p,q}$ is a free group of rank $(p-1)(q-1)$ with basis

$$\begin{array}{cccc} [X,Y], & [X,Y^2], & \ldots, & [X,Y^{q-1}], \\ [X^2,Y], & [X^2,Y^2], & \ldots, & [X^2,Y^{q-1}], \\ & & \ldots, & \\ [X^{p-1},Y], & [X^{p-1},Y^2], & \ldots, & [X^{p-1},Y^{q-1}] \end{array}$$

and of signature $\left(\frac{pq-p-q-(p,q)+2}{2}; \infty^{(p,q)}\right)$.

d) If $(p, q) = 1$, then

$$H'_{p,q} = H_{p,q}^{p} \cap H_{p,q}^{q}. \tag{1.3}$$

e) If $(p, q) = 1$ and if $m = pqn$, $n \in \mathbb{Z}^{+}$, then the subgroups $H_{p,q}^{m}$ are free groups.

The power subgroups H_q^m of the Hecke groups H_q and their commutator subgroups $(H_q^m)'$ have been studied by many authors in [1], [2], [5], [6], [9], [15], [17], [18], [24], [29], [30], [31], [32], [33] and [35].

Also, in [31], Sahin and Koruoğlu gave an application related with the derived series for all triangle groups of the form $(0; 2, q, n)$, where q is a odd prime and n is a positive integer. Using some results given in [39] and [31], they showed that there is a nice connection between the derived series for all triangle groups of the form $(0; 2, q, n)$ and the signatures of the power subgroups H_q^m of the Hecke groups H_q and their commutator subgroups.

In this paper, our aim is to generalize some results given in [31] and [32] for the Hecke groups H_q, to generalized Hecke groups $H_{p,q}$ where $p,\ q \geqslant 2$ are relatively prime integers. Firstly, we obtain the group structures and the signatures of commutator subgroups $(H_{p,q}^m)'$ of the power subgroups $H_{p,q}^m$ of the Hecke groups $H_{p,q}$. We achieve this by applying standard techniques of combinatorial group theory (the Reidemeister-Schreier method and the permutation method). Also, we make some numerical examples for the case $p < q$. Finally, for positive integer n, we give an application related with the derived series for all triangle groups of the form $(0; p, q, n)$, for distinct primes p and q.

2. Commutator subgroups of the power subgroups of generalized Hecke groups $H_{p,q}$

Theorem 1. *Let $p,\ q \geqslant 2$ be relatively prime integers and let m be a positive integer. If m is coprime to one of the them, say q, and let $d = (m, p)$, then*

i) $\left|H_{p,q}^m : (H_{p,q}^m)'\right| = q^d.\frac{p}{d}$,

ii) *The group $(H_{p,q}^m)'$ is a free group of rank $1 + q^{d-1}(pq - p - q)$,*

iii) *The group $(H_{p,q}^m)'$ is of index q^{d-1} in $H'_{p,q}$.*

Proof. i) From (1.1), let $k_1 = Y$, $k_2 = XYX^{-1}, \cdots, k_d = X^{d-1}YX^{-(d-1)}$, $k_{d+1} = X^d$. Then the quotient group $H_{p,q}^m/(H_{p,q}^m)'$ is obtained by adding the relation $k_ik_j = k_jk_i$ to the relations of $H_{p,q}^m$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, d+1\}$. Thus we have

$$H_{p,q}^m/(H_{p,q}^m)' \cong \underbrace{\mathbb{Z}_q \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q}_{d \text{ times}} \times \mathbb{Z}_{p/d}.$$

Therefore, we find the index $\left|H_{p,q}^m : (H_{p,q}^m)'\right| = q^d.\frac{p}{d}$.

ii) Now we choose a Schreier transversal Σ for $(H_{p,q}^m)'$. Here, Σ consists of identity element I; $\binom{d}{1} \times (q-1)$ elements of the form k_i^a where $1 \leqslant i \leqslant d$ and $1 \leqslant a \leqslant q-1$; $(\frac{p}{d} - 1)$ elements of the form k_{d+1}^t, where $1 \leqslant t \leqslant \frac{p}{d} - 1$; $\binom{d}{2} \times (q-1)^2$ elements of the form $k_i^a k_j^b$ where $1 \leqslant i < j \leqslant d$ and $1 \leqslant a, b \leqslant q-1$; $\binom{d}{1} \times (q-1) \times (\frac{p}{d} - 1)$ elements of the form $k_i^a k_{d+1}^t$ where $1 \leqslant i \leqslant d$, $1 \leqslant a \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d} - 1$; $\binom{d}{3} \times (q-1)^3$ elements of the form $k_i^a k_j^b k_s^c$ where $1 \leqslant i < j < s \leqslant d$ and $1 \leqslant a, b, c \leqslant q-1$; $\binom{d}{2} \times (q-1)^2 \times (\frac{p}{d} - 1)$ elements of the form $k_i^a k_j^b k_{d+1}^t$ where $1 \leqslant i < j \leqslant d$, $1 \leqslant a, b \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d} - 1$; $\cdots$; $(q-1)^d \times (\frac{p}{d} - 1)$ elements of the form $k_1^{a_1} k_2^{a_2} \cdots k_d^{a_d} k_{d+1}^t$ where $1 \leqslant i \leqslant d$, $1 \leqslant a_i \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d} - 1$.

Using the Reidemeister-Schreier method and after some calculations, we have the generators of $(H^m_{p,q})'$ as the followings.

There are $1 \times \binom{d}{2} \times (q-1)^2$ generators of the form $[k_i^a,\, k_j^b]$ where $1 \leqslant i < j \leqslant d$, $1 \leqslant a, b \leqslant q-1$; and $1 \times \binom{d}{1} \times (q-1) \times (\frac{p}{d}-1)$ generators of the form $[k_i^a,\, k_{d+1}^t]$ where $1 \leqslant i \leqslant d$, $1 \leqslant a \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d}-1$.

There are $2 \times \binom{d}{3} \times (q-1)^3$ generators of the form $[k_i^a, k_j^b k_s^c]$ or $[k_i^a k_j^b, k_s^c]$ (for the difference, please see the place of the comma) where $1 \leqslant i < j < s \leqslant d$ and $1 \leqslant a, b, c \leqslant q-1$; and $2 \times \binom{d}{2} \times (q-1)^2(\frac{p}{d}-1)$ generators of the form $[k_i^a, k_j^b k_{d+1}^t]$ or $[k_i^a k_j^b, k_{d+1}^t]$ where $1 \leqslant i < j \leqslant d$, $1 \leqslant a, b \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d}-1$.

If we continue similarly, then we find that there are $(d-1) \times \binom{d}{d} \times (q-1)^d$ generators of the form $[k_1^{a_1}, k_2^{a_2} \cdots k_d^{a_d}]$ or $[k_1^{a_1} k_2^{a_2}, \cdots k_d^{a_d}]$ or $\cdots$ or $[k_1^{a_1} k_2^{a_2} \cdots, k_d^{a_d}]$ where $1 \leqslant i \leqslant d$, $1 \leqslant a_i \leqslant q-1$; and $(d-1) \times \binom{d}{d-1} \times (q-1)^{d-1}(\frac{p}{d}-1)$ generators of the form $[k_1^{a_1}, k_2^{a_2} \cdots k_{d-1}^{a_{d-1}} k_{d+1}^t]$ or $[k_1^{a_1} k_2^{a_2}, \cdots k_{d-1}^{a_{d-1}} k_{d+1}^t]$ or $\cdots$ or $[k_1^{a_1} k_2^{a_2} \cdots k_{d-1}^{a_d-1}, k_{d+1}^t]$ where $1 \leqslant i \leqslant d-1$, $1 \leqslant a_i \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d}-1$. Finally, there are $d \times \binom{d}{d} \times (q-1)^d \times (\frac{p}{d}-1)$ generators of the form $[k_1^{a_1}, k_2^{a_2} \cdots k_d^{a_d} k_{d+1}^t]$ or $[k_1^{a_1} k_2^{a_2}, \cdots k_d^{a_d} k_{d+1}^t]$ or $\ldots$ or $[k_1^{a_1} k_2^{a_2} \cdots k_d^{a_d}, k_{d+1}^t]$ where $1 \leqslant i \leqslant d$, $1 \leqslant a_i \leqslant q-1$ and $1 \leqslant t \leqslant \frac{p}{d}-1$.

In fact, there are totally generators

$$\sum_{i=2}^{d}(i-1)\binom{d}{i}(q-1)^i + \sum_{i=1}^{d} i\binom{d}{i}(q-1)^i(\frac{p}{d}-1) = 1 + q^{d-1}(pq-p-q).$$

Notice that the number of the generators can be also seen from [25].

iii) We know that

$$\left|H_{p,q} : H^m_{p,q}\right| = d, \quad \left|H_{p,q} : H'_{p,q}\right| = pq \quad \text{and} \quad \left|H_{p,q} : (H^m_{p,q})'\right| = q^d p.$$

Then we have

$$\left|H'_{p,q} : (H^m_{p,q})'\right| = q^{d-1}.$$

Finally, we find the signature of $(H^m_{p,q})'$ as $(1+\frac{(pq-p-q-1)q^{d-1}}{2};\infty^{(q^{d-1})})$. $\square$

Here we give some examples.

Example 1. 1) Let $p = 2$, $q = 3$ and $m = 2$. As $d = 2$, $k_1 = Y$, $k_2 = XYX^{-1}$. Then we get $\left|H^2_{2,3} : (H^2_{2,3})'\right| = 9$. Here, a Schreier transversal Σ consist of 9 elements. These are identity element I; 4 elements of the form

k_1, k_2, k_1^2, k_2^2; 4 elements of the form k_1k_2, $k_1k_2^2$, $k_1^2k_2$, $k_1^2k_2^2$ for $(H_{2,3}^2)'$. As $(\frac{p}{d}-1)=0$, the number of the generators of $(H_{2,3}^2)'$ is

$$\sum_{i=2}^{2}(i-1)\binom{2}{i}2^i = 1\cdot 1\cdot 2^2 = 4.$$

According to the Reidemeister-Schreier method, we have the generators of $(H_{2,3}^2)'$ as the following. There are 4 generators of the form

$$[k_1,k_2],\quad [k_1^2,k_2],\quad [k_1,k_2^2]\quad \text{and}\quad [k_1^2,k_2^2].$$

Therefore, the group $(H_{2,3}^2)'$ is a free group of rank 4 and of signature $(1;\infty^{(3)})$.

Notice that these results coincide with the results given in [24] for the modular group.

2) Let $p=6$, $q=7$ and $m=10$. As $d=2$, we have $k_1=Y$, $k_2=XYX^{-1}$, $k_3=X^2$ and $\left|H_{6,7}^{10}:(H_{6,7}^{10})'\right|=147$. We choose a Schreier transversal $\Sigma=\{\ I$, 12 elements of the form k_i^a where $1\leqslant i\leqslant 2$ and $1\leqslant a\leqslant 6$; 2 elements of the form k_3^t, where $1\leqslant t\leqslant 2$; 36 elements of the form $k_1^ak_2^b$ where $1\leqslant a,b\leqslant 6$; 24 elements of the form $k_i^ak_3^t$ where $1\leqslant i\leqslant 2$, $1\leqslant a\leqslant 6$ and $1\leqslant t\leqslant 2$; 72 elements of the form $k_1^ak_2^bk_{d+1}^t$ where $1\leqslant a,b\leqslant 6$ and $1\leqslant t\leqslant 2\}$ for $(H_{6,7}^{10})'$. The number of the generators of $(H_{6,7}^{10})'$ is

$$\begin{aligned}\sum_{i=2}^{2}(i-1)\binom{2}{i}6^i+\sum_{i=1}^{2}i\binom{2}{i}6^i\cdot 2\\ = 1\cdot 1\cdot 6^2+1\cdot 2\cdot 6\cdot 2+2\cdot 1\cdot 6^2\cdot 2=204.\end{aligned}$$

Using the Reidemeister-Schreier method, we get 204 generators of $(H_{6,7}^{10})'$ as the following. There are 36 generators of the form,

$$[k_1^a,k_2^b],\quad \text{where } 1\leqslant a,b\leqslant 6,$$

24 generators of the form

$$[k_i^a,k_3^t],\quad \text{where } 1\leqslant i\leqslant 2, 1\leqslant a\leqslant 6 \text{ and } 1\leqslant t\leqslant 2,$$

144 generators of the form,

$$[k_1^a,k_2^bk_3^t]\quad \text{or}\quad [k_1^ak_2^b,k_3^t],\quad \text{where } 1\leqslant a,b\leqslant 6 \text{ and } 1\leqslant t\leqslant 2.$$

Thus, the group $(H^{10}_{6,7})'$ is a free g roup of rank 204 and of signature $(99; \infty^{(7)})$.

3) Let $p = 2$, $q = 3$ and $m = 3$. Since $d = 3$, we get $k_1 = X$, $k_2 = YXY^{-1}$, $k_3 = Y^2XY^{-2}$. Then we have $\left|H^3_{2,3} : (H^3_{2,3})'\right| = 8$. We choose a Schreier transversal $\Sigma = \{\, I, k_1, k_2, k_3, k_1k_2, k_1k_3, k_2k_3, k_1k_2k_3\}$. As $(\frac{q}{d} - 1) = 0$, the number of the generators of $(H^3_{2,3})'$ is

$$\sum_{i=2}^{3}(i-1)\binom{3}{i}1^i = 1\cdot 3\cdot 1^2 + 2\cdot 1\cdot 1^2 = 5.$$

According to the Reidemeister-Schreier method and after some calculations, we have the generators of $(H^3_{2,3})'$ as the followings

$$[k_1, k_2], \quad [k_1, k_3], \quad [k_2, k_3], \quad [k_1, k_2k_3] \quad \text{and} \quad [k_1k_2, k_3].$$

Also the signature of $(H^3_{2,3})'$ is $(1;\ \infty^{(4)})$.

Notice that these results coincide with the result given in [24] for the modular group.

Now we can give the following result.

Corollary 1. *Let p, $q \geqslant 2$ be relatively prime integers. We have*

$$H'_{p,q} = (H^p_{p,q})'(H^q_{p,q})'.$$

Proof. For the proof, we will use the results in [24] and [31]. We know that

$$(H^p_{p,q})' \trianglelefteq H'_{p,q} \quad \text{and} \quad (H^q_{p,q})' \trianglelefteq H'_{p,q}.$$

Then we have the chains

$$(H^p_{p,q})' \subseteq (H^p_{p,q})'(H^q_{p,q})' \subseteq H'_{p,q} \quad \text{and} \quad (H^p_{p,q})' \subseteq (H^q_{p,q})'(H^q_{p,q})' \subseteq H'_{p,q}.$$

From the first chain that

$$\left|H'_{p,q} : (H^p_{p,q})'(H^q_{p,q})'\right| \,|q^{p-1},$$

and the second chain that

$$\left|H'_{p,q} : (H^p_{p,q})'(H^q_{p,q})'\right| \,|p^{q-1}.$$

As p and q are relatively primes, we find that $(q^{p-1}, p^{q-1}) = 1$ and

$$\left|H'_{p,q} : (H^p_{p,q})'(H^q_{p,q})'\right| = 1$$

Therefore we have

$$H'_{p,q} = (H^p_{p,q})'(H^q_{p,q})'. \qquad \square$$

As a consequence, we have

Corollary 2. *Let p and q be distinct primes and let m be a positive integer. Then*

i) *If $(m,p)=1$ and $(m,q)=1$, then $H^m_{p,q}\cong H_{p,q}$ and so $(H^m_{p,q})'=H'_{p,q}$. In this case, the series of the signatures of $H_{p,q}$, $H^m_{p,q}$ and $(H^m_{p,q})'$, respectively, is*

$$H_{p,q}(0;p,q,\infty)\supseteq (H^m_{p,q})'(\frac{pq-p-q+1}{2};\infty). \tag{2.1}$$

ii) *If m is coprime to one of p and q, say q, and if $p=(m,p)$, then $H^m_{p,q}\cong H^p_{p,q}$ and so $(H^m_{p,q})'\cong (H^p_{p,q})'$. In this case, the series of the signatures of $H_{p,q}$, $H^m_{p,q}$ and $(H^m_{p,q})'$, respectively, is*

$$\begin{aligned} H_{p,q}(0;p,q,\infty) &\supseteq H^m_{p,q}(0;q^{(p)},\infty)\\ &\supset (H^m_{p,q})'(1+\frac{(pq-p-q-1)q^{p-1}}{2};\infty^{(q^{p-1})}). \end{aligned}$$

If $(m,p)=p$ and $(m,q)=q$, then the factor group $H_{p,q}/H^m_{p,q}$ is an infinite group. In this case, we can not say much about $H^m_{p,q}$ apart from the fact that they are all free normal subgroups.

Notice that some cases of the results are previously known, due partly to the relation between generalized Hecke groups and torus knot groups (e.g. the Theorem 1.1 in the case $(m,p)=(m,q)=1$, as well as Corollary 1.3. i) in a more general case, where p,q are not necessarily assumed both prime as here, but only assumed coprime). The statements, for m coprime to both p,q (i.e. case $d=1$ in the Theorem 1.1), there is a coincidence $(H^m_{p,q})'=H'_{p,q}$, free group of rank $(p-1)(q-1)$, and an isomorphism $H^m_{p,q}\cong H_{p,q}$, including some of generating systems, are known, and can be found in [36]. Also, see [23] for the freeness of $H'_{p,q}$ and [4] for the commutator subgroups of knot groups.

3. An application to the triangle groups $(0;p,q,n)$

Now we give an application to triangle groups of the form $(0;p,q,n)$ where p and q are distinct primes, and n is a positive integer.

We use a Fuchsian group Γ with signature $(0;p,q,n)$. It is well known that Γ is isomorphic to one relator quotient group $H_{p,q}/R(X,Y)$ of the generalized Hecke group $H_{p,q}$. , Here, $H_{p,q}/R(X,Y)$ is obtained by adding one extra relator $R(X,Y) = (XY)^n = I$ to the standard presentation of $H_{p,q}$. Thus Γ has the following presentation

$$\Gamma \cong \langle X, Y \mid X^p = Y^q = (XY)^n = I \rangle.$$

Then, the quotient group Γ/Γ' is the group obtained by adding the relation $XY = YX$ to the relations of Γ. Then Γ/Γ' has a presentation

$$\Gamma/\Gamma' \cong \langle X, Y \mid X^p = Y^q = (XY)^n = I,\ XY = YX \rangle$$

Let $\ell(\Gamma)$ denote the drive length of Γ and $\Gamma \rhd \Gamma' \rhd \Gamma'' \rhd \cdots \rhd \Gamma^{(k)} \rhd \cdots$ is its derived series. From [39], we know that if Γ is any non-perfect co-compact Fuchsian group, then the drived length $\ell(\Gamma)$ of Γ is bounded by 4. Now we give the following example:

Example 2. i) If $(n,p) = 1$ and $(n,q) = 1$, then we get $X = Y = I$ from the relations $(XY)^n = (XY)^{pq} = I$. Then $\Gamma = \Gamma'$ and therefore $\Gamma = \Gamma' = \Gamma'' = \cdots = \Gamma^{(k)} = \cdots$. Consequently, we have $\ell(\Gamma)=\infty$.

ii) If n is coprime to one of p and q, say q, and if $(n,p) = p$, then we have $Y = I$, since $Y^n = Y^q = I$. Hence we get $\Gamma/\Gamma' \cong \mathbb{Z}_p$. Using the Reidemeister-Schreier method, the permutation method and the Riemann-Hurwitz formula, we have the derived series of Γ as

$$\begin{aligned}\Gamma(0;p,q,pr) &\supseteq \Gamma'(0;\underbrace{q,q,\cdots,q}_{p \text{ times}},r) \\ &\supseteq \Gamma''\Big(\frac{(p-2)q^{(p-1)} - pq^{(p-2)} + 2}{2};\underbrace{r,r,\cdots,r}_{q^{(p-1)} \text{ times}}\Big) \\ &\supseteq \Gamma'''\Big(\frac{(pq-p-q)q^{(p-2)}r^{q^{(p-1)}-1} - q^{(p-1)}r^{q^{(p-1)}-2}) + 2}{2};-\Big).\end{aligned}$$

where $r \in \mathbb{Z}^+$. Here, the quotient groups Γ/Γ', Γ'/Γ'' and Γ''/Γ''' are isomorphic to $\mathbb{Z}_p$, $\underbrace{\mathbb{Z}_q \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q}_{(p-1) \text{ times}}$ and $\underbrace{\mathbb{Z}_r \times \mathbb{Z}_r \times \cdots \times \mathbb{Z}_r}_{q^{(p-1)}-1 \text{ times}}$, respectively. Indeed, there are infinitely many automorphism groups covered by Γ which are residually soluble (Γ''' and all the terms following Γ''' in the series). Therefore we find $\ell(\Gamma)=4$.

iii) If $(n,p) = p$ and $(n,q) = q$, then we get $X^p = Y^q = I$. Since p and q are distinct primes, we have $\Gamma/\Gamma' \cong \mathbb{Z}_{pq}$. Using the Reidemeister-Schreier method and the permutation method, Γ' is a Fuchsian group generated by $z = (XY)^{pq}$, $[X,Y]$, $[X,Y^2]$, $\ldots$, $[X,Y^{q-1}]$, $[X^2,Y]$, $[X^2,Y^2]$, $\ldots$,

$[X^2, Y^{q-1}]$, ..., $[X^{p-1}, Y]$, $[X^{p-1}, Y^2]$, ..., $[X^{p-1}, Y^{q-1}]$. Here the only element of finite order is $z = (XY)^{pq}$ and its order is $n/(pq)$. Using the permutation method and the Riemann-Hurwitz formula, Γ' has signature $(\frac{pq-p-q+1}{2}; r)$ for $r \in \mathbb{Z}^+$. Then the second derived group Γ'' is of infinite index in Γ. Therefore we can find the following series:

$$\Gamma(0; p, q, pqr) \supset \Gamma' = \Gamma(\frac{pq-p-q+1}{2}; r) \supset \Gamma'' \supset \cdots . \tag{3.1}$$

Here Γ' is a free product of a finite cyclic group and $(p-1)(q-1)$ infinite cyclic groups and Γ'' is a free group. Also the corresponding quotient groups Γ/Γ' and Γ'/Γ'' are $\mathbb{Z}_{pq}$ and $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{(p-1)(q-1) \text{ times}}$, respectively. Therefore, we find $\ell(\Gamma)$=3.

Remark 1. 1) There are similar results between the derived series for all triangle groups Γ of the form $(0; p, q, n)$ and the series of the signatures of the power subgroups of the generalized Hecke groups $H_{p,q}$ and their commutator subgroups. There are similarities between (1.1), (1.2) and (2.3), (2.1), respectively. Of course, there are some differences in these signatures, since $(XY)^n = I$ in Γ and $(XY)^\infty = I$ in $H_{p,q}$.

2) In the previous example, if we take $p = 2$, $q \geqslant 3$ prime, then our results coincide with the results given in [31] and if we take $p = 3$ and $q \geqslant 5$ prime, then we obtain the derived series of the triangle group $(0; 3, q, n)$. These triangle groups are studied by many authors (for example, please see, [38] and [8]).

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