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RESEARCH ARTICLE

Amply (weakly) Goldie-Rad-supplemented modules

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Abstract. Let R be a ring and M be a right R-module. We say a submodule S of M is a *(weak) Goldie-Rad-supplement* of a submodule N in M, if $M = N + S$, ($N \cap S \leqslant Rad(M)$) $N \cap S \leqslant Rad(S)$ and $N\beta^{**}S$, and M is called *amply (weakly) Goldie-Rad-supplemented* if every submodule of M has ample (weak) Goldie-Rad-supplements in M. In this paper we study various properties of such modules. We show that every distributive projective weakly Goldie-Rad-Supplemented module is amply weakly Goldie-Rad-Supplemented. We also show that if M is amply (weakly) Goldie-Rad-supplemented and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian.

Introduction

Throughout this article, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules unless indicated otherwise. Let M be an R-module. $N \leqslant M$ will mean N is a submodule of M. $End(M)$ and $Rad(M)$ will denote the ring of endomorphisms of M and the Jacobson radical of M, respectively. The notions which are not explained here will be found in [6].

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Recall that a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. A module H is called *hollow* if every proper submodule of H is small in H. Let N and L be submodules of M. Then N is called a *supplement* of L in M if $N + L = M$ and N is minimal with respect to this property, or equivalently, N is a supplement of L in M if $M = N + L$ and $N \cap L \ll N$. N is said to be a *supplement submodule* of M if N is a supplement of some submodule of M. Recall from [3] that M is called a *supplemented module* if any submodule of M has a supplement in M. M is called an *amply supplemented module* if for any two submodule A and B of M with $A + B = M$, B contains a supplement of A. M is called a *weakly supplemented module* if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. Let $K, N \leqslant M$. K is a *(weak) Rad-supplement* of N in M, if $M = N + K$ and $(N \cap K \leqslant Rad(M))$ $N \cap K \leqslant Rad(K)$ (in this case K is a *(weak) generalized supplement* of N (see, [5])). K is said to be a *(weak) Rad-supplement submodule* of M if K is a (weak) Rad-supplement of some submodule of M (in this case K is a generalized (weakly) supplement submodule (see, [5])). A module M is called *(weakly) Rad-supplemented* if every submodule of M has a (weak) Rad-supplement (in this case M is a generalized (weakly) supplemented module (see, [5])).

In [2], the authors introduced a new class of modules namely Goldie*-Supplemented by defining and studying the β^* relation as the following: Let $X, Y \leqslant M$. X and Y are β^* equivalent, $X\beta^*Y$, provided $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$. After this work, Talebi et. al. [4] defined and studied the β^{**} relation and investigated some properties of this relation. In [4], this β^{**} relation was defined as the following:
Let $X, Y \leqslant M$. X and Y are β^{**} equivalent, $X\beta^{**}Y$, provided $\frac{X+Y}{X} \leqslant \frac{Rad(M)+X}{X}$ and $\frac{X+Y}{Y} \leqslant \frac{Rad(M)+Y}{Y}$.
Based on definition of β^{**} relation they introduced a new class of modules namely Goldie-Rad-supplemented. A module M is called *Goldie-Rad-supplemented* if for any submodule N of M, there exists a Rad-supplement submodule D of M such that $N\beta^{**}D$.

Let M be an R-module. We say a submodule S is a *(weak) Goldie-Rad-supplement* of a submodule N in M, if $M = N + S$, $(N \cap S \leqslant Rad(M))$ $N \cap S \leqslant Rad(S)$ and $N\beta^{**}S$. We say that M is *weakly Goldie-Rad-supplemented* if every submodule of M has a weak Goldie-Rad-supplement in M. We say that a submodule N of M has *ample (weak) Goldie-Rad-supplements* in M if, for every $L \leqslant M$ with $N + L = M$, there exists a (weak) Goldie-Rad-supplement S of N with $S \leqslant L$. We say that M is *amply (weakly) Goldie-Rad-supplemented* if every submodule of M has ample (weak) Goldie-Rad-supplements in M.

We prove that every distributive projective weakly Goldie-Rad-supplemented module is amply weakly Goldie-Rad-supplemented. We show that if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian. In addition, let M be a radical module ($Rad(M) = M$). Then M is Artinian if and only if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Moreover, we also show that the class of amply (weakly) Goldie-Rad-supplemented modules is closed under supplement submodules and homomorphic images.

Lemma 1. ([6, 41.1]) *Let M be a module and K be a supplement submodule of M. Then $K \cap Rad(M) = Rad(K)$.*

Theorem 1. ([1, Theorem 5]) *Let R be any ring and M be a module. Then $Rad(M)$ is Artinian if and only if M satisfies DCC on small submodules.*

1. Amply (weakly) Goldie-Rad-supplemented modules

In this section, we discuss the concept of amply (weakly) Goldie-Rad-supplemented modules and we give some properties of such modules.

Proposition 1. *Every amply (weakly) Goldie-Rad-supplemented module is a (weakly) Goldie-Rad-supplemented module.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and N be a submodule of M. Then $N + M = M$. Since M is amply (weakly) Goldie-Rad-supplemented, M contains a (weak) Goldie-Rad-supplement S of N. So S is a (weak) Goldie-Rad-supplement of N in M. Hence M is (weakly) Goldie-Rad-supplemented. □

Example 1. An hollow radical module M ($Rad(M) = M$) is amply Goldie-Rad-supplemented.

Lemma 2. *Let M be an R-module and $L \leqslant N \leqslant M$. If S is a (weak) Goldie-Rad-supplement of N in M, then $(S + L)/L$ is a (weak) Goldie-Rad-supplement of N/L in M/L.*

Proof. By the proof of [5, Proposition 2.6 (1)], $(S+L)/L$ is a (weak) Rad-supplement of N/L in M/L. By [4, Proposition 2.3 (1)], $\frac{N}{L}\beta^{**}\left(\frac{S+L}{L}\right)$. Hence $(S+L)/L$ is a (weak) Goldie-Rad-supplement of N/L in M/L. □

Proposition 2. *Every factor module of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and M/K be any factor module of M. Let $N/K \leqslant M/K$. For $L/K \leqslant M/K$, let $N/K + L/K = M/K$. Then $N + L = M$. Since M is an amply (weakly) Goldie-Rad-supplemented module, there exists a (weak) Goldie-Rad-supplement S of N with $S \leqslant L$. By Lemma 2, $(S+K)/K$ is a (weak) Goldie-Rad-supplement of N/K in M/K. Since $(S+K)/K \leqslant L/K$, N/K has ample (weak) Goldie-Rad-supplements in M/K. Thus M/K is amply (weakly) Goldie-Rad-supplemented. □

Corollary 1. *Every direct summand of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module. Since every direct summand of M is isomorphic to a factor module of M, then by Proposition 2, every direct summand of M is amply (weakly) Goldie-Rad-supplemented. □

Corollary 2. *Every homomorphic image of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module. Since every homomorphic image of M is isomorphic to a factor module of M, every homomorphic image of M is amply (weakly) Goldie-Rad-supplemented by Proposition 2. □

Let M be a module. Then M is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules K, L, N of M, $N+(K\cap L) = (N+K)\cap(N+L)$ or $N\cap(K+L) = (N\cap K)+(N\cap L)$.

Proposition 3. *Every supplement submodule of a distributive amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module and S be any supplement submodule of M. Then there exists a submodule N of M such that S is a supplement of N. Let $L \leqslant S$ and $L + S' = S$

for $S' \leqslant S$. Then $N + L + S' = M$. Since M is amply (weakly) Goldie-Rad-supplemented, $N + L$ has a (weak) Goldie-Rad-supplement S'' in M with $S'' \leqslant S'$.

In this case $(N+L)+S'' = M$, $((N+L)\cap S^{''} \leqslant Rad(M))$ $(N+L)\cap S^{''} \leqslant Rad(S^{''})$ and $(N+L)\beta^{**}S^{''}$. Since $L+S'' \leqslant S$ and S is a supplement of N in M, $L+S'' = S$. On the other hand, $L\cap S'' \leqslant (N+L)\cap S'' \leqslant Rad(S^{''})$. Now, we show that $L\beta^{**}S^{''}$ in S. By Lemma 1, $S \cap Rad(M) = Rad(S)$. Therefore, since $(N+L)\beta^{**}S^{''}$,

$$\begin{aligned} \frac{L+S^{''}}{S^{''}} = \frac{S\cap(L+S^{''})}{S^{''}} \leqslant \frac{S\cap(N+L+S^{''})}{S^{''}} \leqslant \frac{S\cap(Rad(M)+S^{''})}{S^{''}} &= \frac{S^{''}+(S\cap Rad(M))}{S^{''}} \\ &= \frac{S^{''}+Rad(S)}{S^{''}}, \end{aligned}$$

and since $N \cap S \ll S$, $N + L + S^{''} \leqslant Rad(M) + N + L$,

$$\begin{aligned} \frac{L+S^{''}}{L} = \frac{S\cap(L+S^{''})}{L} \leqslant \frac{S\cap(L+S^{''}+N))}{L} &\leqslant \frac{S\cap(Rad(M)+N+L)}{L} \\ &= \frac{L+(S\cap(Rad(M)+N))}{L} \leqslant \frac{L+Rad(S)}{L}. \end{aligned}$$

Hence $S^{''}$ is a (weak) Goldie-Rad-supplement of L in S. Since $S^{''} \leqslant S^{'}$, L has ample (weak) Goldie-Rad-supplements in S. Thus S is amply (weakly) Goldie-Rad-supplemented. □

A module M is said to be *π-projective* if, for every two submodules N, L of M with $L + N = M$, there exists $f \in End(M)$ with $Imf \leqslant L$ and $Im(1-f) \leqslant N$ (see, [6]).

Theorem 2. *Let M be a distributive weakly Goldie-Rad-supplemented and π-projective module. Then M is an amply weakly Goldie-Rad-supplemented module.*

Proof. Let $N \leqslant M$ and $L + N = M$ for $L \leqslant M$. Since M is weakly Goldie-Rad-supplemented, there exists a weak Goldie-Rad-supplement S of N in M. Then $S + N = M$, $S \cap N \leqslant Rad(M)$ and $S\beta^{**}N$. Since M is π-projective, there exists $f \in End(M)$ such that $f(M) \leqslant L$ and $(1-f)(M) \leqslant N$. Note that $f(N) \leqslant N$ and $(1-f)(L) \leqslant L$. Then

$$M = f(M) + (1-f)(M) \leqslant f(S+N) + N = f(S) + N.$$

Let $n \in N \cap f(S)$. Then there exists $s \in S$ with $n = f(s)$. In this case $s - n = s - f(s) = (1-f)(s) \in N$ and then $s \in N$. Hence $s \in N \cap S$ and

$N \cap f(S) \leqslant f(N \cap S)$. Since $N \cap S \leqslant Rad(M)$, $f(N \cap S) \leqslant f(Rad(M))$. Then

$$N \cap f(S) \leqslant f(N \cap S) \leqslant f(Rad(M)) \leqslant Rad(f(M)) \leqslant Rad(M)$$

Next we show that $f(S)\beta^{**}N$. Since $S\beta^{**}N$, $S + N \leqslant Rad(M) + N$ and $S + N \leqslant Rad(M) + S$. Hence

$$f(S) + N = M = S + N \leqslant Rad(M) + N,$$

and since $S \cap N \leqslant Rad(M)$,

$$\begin{aligned} f(S) + N = f(S) + (N \cap M) &= f(S) + (N \cap (Rad(M) + S)) \\ &\leqslant f(S) + Rad(M). \end{aligned}$$

Hence $f(S)$ is a weak Goldie-Rad-supplement of N in M. Since $f(S) \leqslant L$, N has ample weak Goldie-Rad-supplements in M. Thus M is amply weakly Goldie-Rad-supplemented. □

Corollary 3. *Every projective distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module.*

Proof. Since every projective module is π-projective, every projective and distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module by Theorem 2. □

Corollary 4. *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a distributive module and $M_1, M_2, \cdots$, M_n be projective modules. Then $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ is amply weakly Goldie-Rad-supplemented if and only if for every $1 \leqslant i \leqslant n$, M_i is amply weakly Goldie-Rad-supplemented.*

Proof. "$\Longrightarrow$" is clear from Corollary 1.
"$\Longleftarrow$" Since M_i is amply weakly Goldie-Rad-supplemented, M_i is weakly Goldie-Rad-supplemented. Let $U \leqslant M$ and $U_i = M_i \cap U$. There exists $S_i \leqslant M_i$ such that $S_i\beta^{**}U_i$, $S_i + U_i = M_i$, $S_i \cap U_i \leqslant Rad(M_i)$ for $i = 1, \cdots n$. By [4, Proposition 2.5], $U\beta^{**}(\Sigma_{i=1}^{n} S_i)$. Moreover, $U + (\Sigma_{i=1}^{n} S_i) = M$ and

$$U \cap (\Sigma_{i=1}^{n} S_i) = \Sigma_{i=1}^{n}(S_i \cap U_i) \leqslant \Sigma_{i=1}^{n} Rad(M_i) \leqslant Rad(\Sigma_{i=1}^{n} M_i) = Rad(M).$$

This means that, $(\Sigma_{i=1}^{n} S_i)$ is a weak Goldie-Rad-supplement of U in M. Hence M is weakly Goldie-Rad-supplemented. Since, for every $1 \leqslant i \leqslant n$, M_i is projective, $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ is also projective. Then M is amply weakly Goldie-Rad-supplemented by Corollary 3.

□

Proposition 4. *Let M be an amply (weakly) Goldie-Rad-supplemented module. If M satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then M is Artinian.*

Proof. Let M be an amply (weakly) Goldie-Rad-supplemented module which satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Then $Rad(M)$ is Artinian by Theorem 1. It suffices to show that $M/Rad(M)$ is Artinian. Let N be any submodule of M containing $Rad(M)$. Then there exists a (weak) Goldie-Rad-supplement S of N in M, i.e, $M = N+S$, $N \cap S \leqslant Rad(S) \leqslant Rad(M)$ and $N\beta^{**}S$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((S + Rad(M))/Rad(M))$ and so every submodule of $M/Rad(M)$ is a direct summand. Therefore $M/Rad(M)$ is semisimple.

Now suppose that $Rad(M) \leqslant N_1 \leqslant N_2 \leqslant N_3 \leqslant \cdots$ is an ascending chain of submodules of M. Because M is amply (weakly) Goldie-Rad-supplemented, there exists a descending chain of submodules $S_1 \geqslant S_2 \geqslant S_3 \geqslant \cdots$ such that S_i is a (weak) Goldie-Rad-supplement of N_i in M for each $i \geqslant 1$. By hypothesis, there exists a positive integer t such that $S_t = S_{t+1} = S_{t+2} = \cdots$. Because $M/Rad(M) = N_i/Rad(M) \oplus (S_i + Rad(M))/Rad(M)$ for all $i \geqslant t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/Rad(M)$ is Noetherian and since $M/Rad(M)$ is semisimple, by [6, 31.3] $M/Rad(M)$ is Artinian, as desired. □

Corollary 5. *Let M be a finitely generated amply (weakly) Goldie-Rad-supplemented module. If M satisfies DCC on small submodules, then M is Artinian.*

Proof. Since $M/Rad(M)$ is semisimple and M is finitely generated, then by [6, 31.3] $M/Rad(M)$ is Artinian. Now that M satisfies DCC on small submodules, $Rad(M)$ is Artinian by Theorem 1. Thus M is Artinian. □

Corollary 6. *Let M be a radical module (Rad(M)=M). Then M is Artinian if and only if M is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules.*

Proof. "$\Longleftarrow$" is clear by Proposition 4.
"$\Longrightarrow$" It suffices to prove that M is amply (weakly) Goldie-Rad-supplemented. It is well known that a module M is Artinian if and only if M is an amply supplemented module and satisfies DCC on supplement submodules and on small submodules. Since an amply supplemented

module is amply Rad-supplemented and for every submodules N, S of M, $N\beta^{**}S$, M is amply (weakly) Goldie-Rad-supplemented, as desired. □

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