© Algebra and Discrete Mathematics
Volume **30** (2020). Number 2, pp. 172–178
DOI:10.12958/adm584 RESEARCH ARTICLE

Some results on the main supergraph of finite groups

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Communicated by A. Yu. Olshanskii

Abstract. Let G be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. In this paper, we will show that $G \cong \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$ if and only if $\mathcal{S}(G) \cong \mathcal{S}(\mathrm{PSL}(2,p))$ or $\mathcal{S}(\ \mathrm{PGL}(2,p))$, respectively. Also, we will show that if M is a sporadic simple group, then $G \cong M$ if only if $\mathcal{S}(G) \cong \mathcal{S}(M)$.

Introduction

Let G be a finite group and $x \in G$. The order of x is denoted by $o(x)$. The set of all element orders of G is denoted by $\pi_e(G)$ and the set of all prime divisors of $|G|$ is denoted by $\pi(G)$. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. Set $m_i = m_i(G) = |\{g \in G|\ o(g) = i\}|$. In this paper, we use p for denoting a prime number unless specifically stated otherwise.

We define the graph $\mathcal{S}(G)$ with vertex set G such that two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. This graph is called the *main supergraph* of *power graph* G and was introduced in [13]. Power graph $\mathcal{P}(G)$ is a graph with the vertex set G, in which two distinct elements are adjacent if one is a power of the other. The main properties

2010 MSC: 20D08; 05C25.
Key words and phrases: graph, main supergraph, finite groups, Thompson's problem.

of this graph were investigated by Cameron [9] and Chakrabarty et al. [10]. The proper main supergraph $\mathcal{S}^*(G)$ are defined as graphs constructed from $\mathcal{S}(G)$ by removing the identity element of G. We write $x \sim y$ when two vertices x and y are adjacent.

We say that groups G_1 and G_2 are of the same order type if and only if $m_t(G_1) = m_t(G_2)$ for all t. By the definition of the main supergraph, it is clear that if G_1 and G_2 are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. The converse of this result is not generally correct. To prove, we consider $G_1 = Z_4 \times Z_4$ and $G_2 = Z_4 \times Z_2 \times Z_2$. Since G_1 and G_2 are 2-groups, we have $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. But $m_4(G_1) = 12 > 8 = m_4(G_2)$ and $m_2(G_1) = 3 < 7 = m_2(G_2)$.

In 1987, J. G. Thompson [16, Problem 12.37] posed the following problem:

Thompson's Problem. Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Let $\mathrm{nse}(G)$ be the set of the number of elements of the same order in G. If G_1 and G_2 are the same order type, then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$. Therefore, if a group G has been uniquely determined by its order and $\mathrm{nse}(G)$, then Thompson's problem is true for G. In [1, 4, 6], it is proved that the group $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$, and the sporadic simple groups characterizable by their nse and order. Consequently, there is no solvable group that has the same order type as $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$, and the sporadic simple groups.

Clearly, for two groups G_1 and G_2 that are the same order type, $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. So, if a group G has been uniquely determined by $\mathcal{S}(G)$, then Thompson's problem is true for G. In [5, 7, 8], the authors of this paper proved that the alternating group of degree p, $p+1$, $p+2$, the symmetric group of degree p, the small Ree group ${}^2G_2(3^{2n+1})$, and the Suzuki group are uniquely determined by their main supergraph. So, there is no solvable group that has the same order type as these simple groups. Also, in this paper, by the main supergraph, we show that there is no solvable group that has the same order type as $\mathrm{PSL}(2, p)$, $\mathrm{PGL}(2, p)$, and the sporadic simple groups.

We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq $(p \neq q)$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose

$2 \in \pi_1$. If $r \in \pi(G)$ and $\{r\}$ is a connected component of $\Gamma(G)$, then we say r is an isolated vertex of the prime graph of G.

Throughout this paper we denote by ϕ the Euler's totient function. We denote by G_q a Sylow q-subgroup of G and $n_q(G)$ is the number of Sylow q-subgroup of G, that is, $n_q(G) = |\mathrm{Syl}_q(G)|$. The other notation and terminology in this paper are standard, and the reader is referred to [11] if necessary.

1. Preliminary results

In this section, we present some preliminary results which will turn out to be useful in what follows. We start with a classical theorem of Frobenius.

Lemma 1 ([12]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2 ([2, Theorem 2.3]). *Let p be a prime that is not a Mersenne prime. If G is a group satisfying*
(1) $|G| = |\mathrm{PSL}(2,p)|$,
(2) $|\mathrm{Syl}_p(G)| = |\mathrm{Syl}_p(\mathrm{PSL}(2,p))|$,
then $G \cong \mathrm{PSL}(2,p)$.

Lemma 3 ([2, Theorem 2.5]). *Let p be a Mersenne prime. If G is a group satisfying*
(1) $|G| = |\mathrm{PSL}(2,p)|$,
(2) $|\mathrm{Syl}_p(G)| = |\mathrm{Syl}_p(\mathrm{PSL}(2,p))|$, *and*
(3) *p is an isolated vertex of the prime graph of G,*
then $G \cong \mathrm{PSL}(2,p)$ *unless it is the case that* $p = 7$ *and* $G \cong \mathrm{P\Gamma L}_2(8)$.

Lemma 4 ([3, Theorem 1.1]). *Let p be a prime and G be a group such that $|G| = |\mathrm{PGL}(2,p)|$ and $|N_G(R)| = |N_{\mathrm{PGL}(2,p)}(S)|$, where $R \in \mathrm{Syl}_p(G)$ and $S \in \mathrm{Syl}_p(\mathrm{PGL}(2,p))$. Then the following assertions are true.*
(1) *If p is not a Mersenne prime, then G is isomorphic to* $\mathrm{PSL}(2,p) \times C_2$, $\mathrm{SL}(2,p)$ *or* $\mathrm{PGL}(2,p)$.
(2) *If $p > 3$ is a Mersenne prime and p is an isolated vertex of the prime graph of G, then* $G \cong \mathrm{PGL}(2,p)$.

Lemma 5 ([15]). *Let S be a sporadic simple group and p be the greatest element of $\pi(S)$. Then S is uniquely determined by $|S|$ and $n_p(S)$.*

Remark 1. Let $m_n = m_n(G)$ be the number of elements of order n of a finite group G. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. If $n \mid |G|$, then by Lemma 1 we have $\phi(n) \mid m_n$ and $n \mid \sum_{d|n} m_d$.

2. Main results

In this section, we prove that the groups $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, and the sporadic simple groups uniquely determined by their main supergraph.

Theorem 1. *Let G be a finite group and $p > 2$ a prime number. If $\mathcal{S}(\mathrm{PSL}(2,p)) \cong \mathcal{S}(G)$, then $G \cong \mathrm{PSL}(2,p)$.*

Proof. By the definition of the main supergraph and our assumption, we have $|G| = |\mathrm{PSL}(2,p)|$. Also, by $\mathcal{S}(\mathrm{PSL}(2,p)) \cong \mathcal{S}(G)$ and the definition of the proper main supergraph, we have $\mathcal{S}^*(\mathrm{PSL}(2,p)) \cong \mathcal{S}^*(G)$.

By [14, p. 213], $\mu(\mathrm{PSL}(2,p)) = \{(p-1)/2, p, (p+1)/2\}$. Thus $\mathrm{PSL}(2,p)$ has not any element of order rp, where $r \in \pi(G)$. By [1, Lemmas 2.3 and 2.4], $\mathrm{PSL}(2,p)$ contains exactly p^2-1 elements of order p. It follows that $\mathcal{S}^*(G)$ is a disconnected graph and at least one of the connected components of this graph is a complete graph of order p^2-1. Denote by K_{p^2-1} a complete subgraph of order p^2-1 in the graph $\mathcal{S}^*(G)$. We prove that the vertices of K_{p^2-1} are elements of order p.

First, let x and y be two vertices of K_{p^2-1} such that $o(x) = r$, $o(y) = s$, r, $s \in \pi(G)$ and $r \neq s$. Since K_{p^2-1} is a complete graph, we have $x \sim y$, which is a contradiction. Let r be a prime and the vertices of K_{p^2-1} be all of $x \in G$ such that $o(x) = r$, r^2,..., or r^k (note that $\exp(G_r) = r^k$). Then with considering $n = |G_r|$ in Remark 2.1, $|G_r| \mid (1+m_r+m_{r^2}+...+m_{r^k}) = 1+p^2-1 = p^2$. It follows that $r = p$. Hence, the vertices of K_{p^2-1} are $x \in G$ such that $o(x) = p^k$, where $k \geqslant 1$ is an integer.

Since $|G| = p(p^2-1)/2$, we have $p^2 \nmid |G|$ and so $p^2 \notin \pi_e(G)$. Therefore, the vertices of K_{p^2-1} is all of elements of order p in G. Thus $m_p(G) = p^2-1$.

Let $pr \in \pi_e(G)$, where $r \in \pi(G)\backslash\{p\}$. Then there exists an element $x \in G$ such that $o(x) = r$ and x is one of the vertices of K_{p^2-1}, which is a contradiction. Thus p is an isolated vertex of the prime graph of G.

By Remark 1, $m_p(G) = \phi(p)n_p(G) = (p-1)n_p(G) = p^2-1$, so $n_p(G) = p+1$. By Lemmas 2 and 3, $G \cong \mathrm{PSL}(2,p)$ unless it is the case that $p = 7$. If $p = 7$, then $G \cong \mathrm{PSL}(2,7)$ or $\mathrm{P\Gamma L}_2(8)$. Let $G \cong \mathrm{P\Gamma L}_2(8)$. We have $\pi_e(\mathrm{P\Gamma L}_2(8)) = \{1, 2, 3, 6, 7\}$. So, $\mathcal{S}^*(\mathrm{P\Gamma L}_2(8))$ has two connected components, but $\mathcal{S}^*(\mathrm{PSL}(2,7))$ has three connected components, we get a contradiction by $\mathcal{S}^*(\mathrm{PSL}(2,7)) \cong \mathcal{S}^*(G)$. Therefore, $G \cong \mathrm{PSL}(2,7)$ for the case $p = 7$. The proof is completed. □

Theorem 2. *Let G be a finite group and $p > 2$ a prime number. If $\mathcal{S}(\mathrm{PGL}(2,p)) \cong \mathcal{S}(G)$, then $G \cong \mathrm{PGL}(2,p)$.*

Proof. The proof of this theorem is similar to the proof of the Theorem 1. Since some part of the proof is different from Theorem 1, we have repeated the proof in the following.

By the definition of the main supergraph and our assumption, we have $|G| = |\mathrm{PGL}(2,p)|$ and $\mathcal{S}^*(\mathrm{PGL}(2,p)) \cong \mathcal{S}^*(G)$.

By [17, Lemma 2.1], $\mu(\mathrm{PGL}(2,p)) = \{p-1, p, p+1\}$. Thus $\mathrm{PGL}(2,p)$ has not any element rp, where $r \in \pi(G)$. By [4, Lemma 2.4], G contains exactly $p^2 - 1$ elements of order p. It follows that $\mathcal{S}^*(G)$ is a disconnected graph and at least one of the connected components of this graph is a complete graph of order $p^2 - 1$. Denote by K_{p^2-1} a complete subgraph of order $p^2 - 1$ in the graph $\mathcal{S}^*(G)$. We prove that the vertices of K_{p^2-1} are elements of order p.

First, let x and y be two vertices of K_{p^2-1} such that $o(x) = r$, $o(y) = s$, r, $s \in \pi(G)$ and $r \neq s$. Since K_{p^2-1} is a complete graph, we have $x \sim y$, which is a contradiction. Let r be a prime and the vertices of K_{p^2-1} be $x \in G$ such that $o(x) = r$, r^2,..., r^k (note that $\exp(G_r) = r^k$). Then with considering $n = |G_r|$ in Remark 1, $|G_r| \mid (1 + m_r + m_{r^2} + ... + m_{r^k}) = 1 + p^2 - 1 = p^2$. It follows that $r = p$. Hence, the vertices of K_{p^2-1} are $x \in G$ such that $o(x) = p^k$, where $k \geqslant 1$ is an integer.

Since $|G| = p(p^2-1)$, we have $p^2 \nmid |G|$ and so $p^2 \notin \pi_e(G)$. Therefore, the vertices of K_{p^2-1} is all of elements of order p in G. Thus $m_p(G) = p^2 - 1$.

Let $pr \in \pi_e(G)$, where $r \in \pi(G)\backslash\{p\}$. Then there exists an element x in G such that $o(x) = r$ and x is one of the vertices of K_{p^2-1}, a contradiction.

By Remark 1, $m_p(G) = \phi(p)n_p(G) = (p-1)n_p(G) = p^2 - 1$, so $n_p(G) = p + 1$.

If p is not a Mersenne prime, then by Lemma 4 G is isomorphic to $\mathrm{PSL}(2,p) \times C_2$, $\mathrm{SL}(2,p)$ or $\mathrm{PGL}(2,p)$. Let G be isomorphic to $\mathrm{PSL}(2,p) \times C_2$ or $\mathrm{SL}(2,p)$. Then G has an element of order $2p$, which is a contradiction. Therefore, $G \cong \mathrm{PGL}(2,p)$.

If p is a Mersenne prime, then by Lemma 4, $G \cong \mathrm{PGL}(2,p)$. □

Theorem 3. *Let G be a finite group. If $\mathcal{S}(M) \cong \mathcal{S}(G)$ where M is a sporadic simple group, then $G \cong M$.*

Proof. Since $\mathcal{S}(M) \cong \mathcal{S}(G)$, we have $|G| = |M|$ and $\mathcal{S}^*(M) \cong \mathcal{S}^*(G)$. Let p be the greatest element of $\pi(M)$. By [11], $p^2 \nmid |G|$ and $pr \notin \pi_e(M)$ for every $r \in \pi(M)$. Thus $\mathcal{S}^*(M) \cong \mathcal{S}^*(G)$ is a disconnected graph and at least one of the connected components of this graph is complete graph of order $t = m_p(M)$. Denote by K_t a complete subgraph of order t in the graph $\mathcal{S}^*(G)$. We will show that the vertices of K_t are elements of order p.

First, let x and y be two vertices of K_t such that $o(x) = r$, $o(y) = s$, r, $s \in \pi(G)$ and $r \neq s$. Since K_t is a complete graph, we have $x \sim y$, a contradiction. Let r be a prime and the vertices of K_t be all of $x \in G$ such that $o(x) = r$, r^2,..., or r^k (note that $\exp(G_r) = r^k$). Then by Remark 1, $|G_r| \mid (1 + m_r + m_{r^2} + ... + m_{r^k}) = 1 + t$. Now, it is easy to check (case by case for all of sporadic simple group M) that $r = p$. Hence, the vertices of K_t are $x \in G$ such that $o(x) = p^k$, where $k \geqslant 1$ is an integer.

Since $p^2 \nmid |G|$, we have $p^2 \notin \pi_e(G)$. Therefore, the vertices of K_t is all of elements of order p in G. Hence, $m_p(G) = m_p(M)$. Remark 1 follows that G and M have equal numbers of Sylow p-subgroups and by Lemma 5, G is isomorphic to M and now the proof is completed. □

By our example in the introduction, we have seen if $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$, then it is not necessary that $G_1 \cong G_2$. But by Theorems 1 and 3, we have just seen that sporadic simple groups and projective special linear simple group $\mathrm{PSL}(2,p)$ uniquely determined by their main supergraph. So, we can pose the following question:

Question. Let G be a finite group and M a non-abelian finite simple group. Is it true that $G \cong M$ if only if $\mathcal{S}(G) \cong \mathcal{S}(M)$?

Acknowledgment

The authors would like to thank the referee for his/her careful reading and useful suggestions which led us to improve the paper.

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Received by the editors: 01.12.2017
and in final form 18.03.2018.