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RESEARCH ARTICLE

A criterion of elementary divisor domain for distributive domains

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Abstract. In this paper we introduce the notion of the neat range one for Bezout duo-domains. We show that a distributive Bezout domain is an elementary divisor domain if and only if it is a duo-domain of neat range one.

A problem of describing elementary divisor rings is classical and far from its completion. The most full history of this problem and close to it problems can be found in [4]. In the case of commutative rings there are many developments on this problem in the case of noncommutative rings it is little investigated and fragmented. A general picture is far from its full description.

Among these results are should especially note a result of [5] which shows that a distributive elementary divisor domain is a duo-domain. Tuganbaev extended this result in case of a distributive ring [3].

In this paper we give a criterion when a distributive domain is an elementary divisor domain.

We start with necessary definitions and facts. Under a ring R we understand an associative ring with 1, and $1 \neq 0$. We say that matrices A and B over a ring R are equivalent if exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. The fact that matrices A and B are equivalent is denoted by $A \sim B$. If for a matrix A there exists a diagonal matrix $D = (d_i)$ such that $A \sim D$ and $Rd_{i+1}R \subseteq d_iR \cap Rd_i$ for every i

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then we say that the matrix A has a canonical diagonal reduction. A ring R is an elementary divisor ring if every matrix over R has a canonical diagonal reduction. If over a ring R every 1×2 (2×1) matrix has a canonical diagonal reduction then R called a right (left) Hermite ring.

A ring which is both a right and left Hermite ring is called an Hermite ring. We note that a right Hermite ring is a right Bezout ring that is a ring in which every finitely generated right ideal is principal [1], [4].

A ring R is called clean if every element of R is the sum of an idempotent and a unit. A ring R is called an exchange ring if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$, $1 - e \in (1 - a)R$. [2].

A ring R is called a ring of stable range one if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that a $(a + bt)R = R$.

A ring R is called right (left) distributive if every lattice right (left) ideal of ring R is distributive. A distributive ring is a ring which is both right and left distributive ring [3].

A right (left) quasi-duo ring is a ring in which every a right (left) maximal ideal is ideal. In the case of distributive right (left) Bezout rings a connection with right (left) quasi-duo rings is established by the following theorem.

Theorem 1. [3] *The following properties are equivalent for a Bezout ring R.*

1) *R is a distributive ring.*
2) *R is a quasi-duo ring.*
3) *From the condition $aR + bR = R$ it follows that $Ra + Rb = R$ for every elements $a, b \in R$.*
4) *From the condition $Ra + Rb = R$ it follows that $aR + bR = R$ for every elements $a, b \in R$.*

Theorem 2. [5] *Any distributive elementary divisor domain is a duo-domain.*

Definition 1. We say that a duo-ring R has neat range one if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that a $R/(a + bt)R$ is a clean ring.

We note that every duo-ring of stable range one is a ring of neat range one.

The following two theorems are the main result of this paper.

Theorem 3. *Any Bezout duo-domain is an elementary divisor domain if and only if it is a domain of neat range one.*

Theorem 4. *Any distributive Bezout domain is an elementary divisor domain if and only if it is a duo-domain of neat range one.*

Theorem 3 is a consequence of Theorem 5 and Proposition 4.

Theorem 4 is a consequence of Theorems 2 and 3.

We prove the following result which will be useful in the forthcoming research. Recall that a row $(a_1, \ldots, a_n)$ of elements of a ring R is called unimodular if $a_1R + \ldots + a_nR = R$.

Proposition 1. *Let R be a right Hermite ring, then every unimodular row $(a_1, \ldots, a_n)$ with elements of the ring R can be completed to an invertible matrix.*

Proof. Since R is a right Hermite ring and $a_1R + \ldots + a_nR = R$, then

$$(a_1, \ldots, a_n)P = (1, 0 \ldots 0) \tag{1}$$

for some matrix P of order n over the ring R. Note that

$$P^{-1} = (p_{ij}).$$

From equality (1) we have

$$(a_1, \ldots, a_n) = (1, 0 \ldots 0)P^{-1},$$

then $a_1 = p_{11}, \ldots, a_n = p_{1n}$ and hence the row $(a_1, \ldots, a_n)$ is the first row invertible matrix P^{-1}. The proposition is proved. □

Proposition 2. *A Hermite duo-ring R is an elementary divisor ring if for such any elements $a, b, c \in R$ such that $aR + bR + cR = R$ there exist elements $p, q \in R$ such that $(pa)R + (pb + qc)R = R$.*

Proof. Let R be an elementary divisor ring. Let $aR + bR + cR = R$. The matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ has canonical diagonal reduction, i.e., there exists invertible matrices $P = \begin{pmatrix} p & q \\ * & * \end{pmatrix} \in GL_2(R)$, $Q \in GL_2(R)$ such that

$$PAQ = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}.$$

Hence we get that $paR + (pb + qc)R = R$. The necessity is proved.

In order to prove sufficiency according to [1] it is enough to prove that every matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $aR + bR + cR = R$ has canonical diagonal reduction. We see that $(pa)R + (pb + qc)R = R$ for some elements $p, q \in R$. Hence $pR + qR = R$, as R is an Hermite ring and the row (p, q), by Proposition 1, is adding to an invertible matrix $P \in GL_2(R)$.

Obviously, the matrix PA has canonical diagonal reduction. The proposition is proved. □

Proposition 3. *Let R be a Bezout duo-domain. For every elements $a, b, c \in R$ such that $aR + bR + cR = R$ the following conditions are equivalent:*

1) *There exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;*
2) *There exist elements $\lambda, u, v \in R$ such that $b + \lambda c = v \cdot u$, where $uR + aR = R$, $vR + cR = R$.*

Proof. 1) ⇒ 2) Let condition 1) be true. Then it follows that $pR + qcR = R$ and hence $pR + cR = R$. Since R is a duo-ring, $Rp + Rc = R$. Hence $vp + jc = 1$ for some elements $v, j \in R$. Then $vpb - b = jcb = ct$ for $t \in R$. Note that since R is a duo-ring, then $t = jc$, where $jc = cj'$.

Then $v(pb + qc) = vpb + vqc = b + ct + vqc = b + ct + ck$, that is $v(pb + qc) - b \in cR$, that is $v(pb + qc) - b = c\lambda$ for some $\lambda \in R$. We note that such an element k exists, since R is a duo-ring. Namely, $vqc = ck$. Hence $vR + cR = R$ and $uR + aR = R$ where $u = pb + qc$. We note that the condition $uR + aR = R$ follows obviously from the condition $paR + (pb + qc)R = R$. Condition 2) is proved.

2) ⇒ 1) We assume that exists an element $\lambda \in R$ such that $b + c\lambda = vu$, where $vR + cR = R$ and $uR + aR = R$. Since $vR + cR = R$ then $Rv + Rc = R$ and $pv + jc = 1$ for some elements $p, j \in R$.

We note that $pR + cR = R$. Then $pb = p(vu - c\lambda) = (pv)u - pc\lambda = (1 - jc)u - pc\lambda = u - qc$ for an element $q \in R$. Hence $u = pb + qc$. Therefore, $(pb + qc)R + aR = R$ and $pR + cR = R$. Since R is a Bezout duo-domain, let $pR + qR = dR$, where $p = dp_1$, $q = dq_1$ and $p_1R + q_1R = R$ such that $p_1R + (p_1b + q_1c)R = p_1R + q_1cR$ since $pR + cR = R$ and $p_1R + q_1R = R$ then $p_1R + (p_1b + q_1c)R = R$.

Hence $(p_1b + q_1c)R + aR = R$ and $(p_1b + q_1c)R + p_1R = R$ and hence $p_1aR + (p_1b + q_1c)R = R$. Condition 1) is true.

The proposition is proved. □

Remark 1. In Proposition 3 we can choose the elements u and v such that $uR + vR = R$.

Theorem 5. *Let R be a Bezout duo-domain. Then the following conditions are equivalent.*

1) *R is an elementary divisor duo-domain;*
2) *For every elements $x, y, z \in R$ such that $xR + yR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = vu$, where $uR + zR = R$, $vR + (1 - z)R = R$.*

Proof. 1) $\Rightarrow$ 2) Let R be an elementary divisor domain. By Proposition 2, then for every elements $a, b, c \in R$ such that $aR + bR + cR = R$ there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$.

We obtain Condition 2 of Proposition 3 to the elements $a = z, b = x, c = y(1 - z)$.

It is complicated to prove the fact that from Condition 2) of our theorem we obtain the condition that for every $a, b, c \in R$ such that $aR + bR + cR = R$ there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$. Let $bR + cR = dR$ and $b = db_1$, $c = dc_1$ where $b_1R + c_1R = R$. Since $aR + dR = R = aR + bR + cR = R$ then $dR + aR = R$ hence $1 - d_1d \in aR$ for an element $d_1 \in R$.

2) $\Rightarrow$ 1) Put $x = b_1, y = c_1, z = d_1d$. By Condition 2) of our theorem, there exists an element $\lambda_1 \in R$ such that $b_1 + c_1\lambda_1 = vu_1$ where $u_1R + (1 - d_1d)R = R$, $vR + d_1dR = R$. Since $(1 - d_1d) \in aR$ and also the fact that $u_1R + (1 - d_1d)R = R$, then $u_1R + aR = R$. We show that $u = u_1d$ hence $uR + aR = R$. Let $\lambda \in R$ be such that $c_1\lambda_1 = \lambda c_1$.

We have that $b + \lambda c = (b_1 + \lambda c_1)d = vu_1d = vu$. As $vR + d_1R = R$ then $vR + dR = R$. Remark that $vR + cR = vR + dc_1R = vR + c_1R$ as $b_1 + \lambda c_1 = vu_1$, $vR + c_1R = R$ therefore $vR + cR = R$ and this means that Condition 2) of Proposition 3 is true. Therefore according to Proposition 3 we conclude that for every $a, b, c \in R$ with $aR + bR + cR = R$ there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$, that is according to Proposition 2, R is an elementary divisor ring.

The theorem is proved. □

Proposition 4. *Let R be a Bezout duo-domain and $c \in R \setminus \{0\}$. Then $\overline{R} = R/cR$ is a clean ring if and only if for every element $a \in R$ there exist elements v, u such that $c = vu$ where $uR + aR = R$ $vR + (1 - a)R = R$, $uR + vR = R$.*

Proof. Let R be a clean ring. According to [2], R is an exchange ring. Let $\bar{a} = a + cR$. Then there exists an idempotent $\bar{e} \in \bar{R}$ such that $\bar{e} \in \bar{a}\bar{R}$, $\bar{1} - \bar{e} \in (\bar{1} - \bar{a})\bar{R}$. Since $\bar{e} \in \bar{a}\bar{R}$, $e - ap = cs$ for elements $p, s \in R$. Similarly, $1 - e - (1 - a)\alpha = c\beta$ for elements $\alpha, \beta \in R$. Since $\bar{e}^2 = \bar{e}$, then $e(1 - e) = ct$

for an element $t \in R$. Let $eR + cR = dR$. Hence $e = de_0, c = dc_0$ for elements $e_0, c_0 \in R$ such that $e_0R + c_0R = R$, hence $e_0(1-e) = c_0t$ and $e + c_0j \equiv 1$ for every element $j \in R$.

Denote that $v = d, u = c_0$ we have $c = vu$. Since $e = 1 - c_0j$, then $uR + eR = R$. Since $e = ap + cs$, then $uR + aR = R$. We show that $vR + (1-a)R = R$. As $1 - e + (1-a)\alpha = c\beta$ and $e = de_0, c = dc_0$ hence $1 - de_0 + (1-a)\alpha = dc_0\beta$ and this means that $d(e_0 + c_0\beta) + (1-a)\alpha = 1$, thus $dR + (1-a)R = R$ that is $vR + (1-a)R = R$. The necessity is proved.

Let $c = vu$, where $uR + aR = R$, $vR + (1-a)R = R$. Let $\bar{u} = u + cR$, $\bar{v} = v + cR$. From the equality $uR + vR = R$ we have $ur + vs = 1$ for some elements $r, s \in R$. Hence $vur + v^2s = v$ and $u^2r + uvs = u$ and this means that $\bar{v}^2\bar{s} = \bar{v}$, $\bar{u}^2\bar{r} = \bar{u}$.

Let $\bar{v}\bar{s} = \bar{e}$, it is obvious that $\bar{e}^2 = \bar{e}$ and $\bar{1} - \bar{e} = \bar{u}\bar{r}$. Since $uR + aR = R$, we have $ux + ay = 1$ for elements $x, y \in R$. Hence $vux + vay = v$, $vuxs + vays = vs$.

Let $va = av'$ for some element v'. Hence $vuxs + av'ys = vs$ and this means that $\bar{a}\bar{v}'\bar{y} \cdot \bar{s} = \bar{v} \cdot \bar{s}$ that is $\bar{a}\bar{j} = \bar{e}$ for $\bar{j} \in R$ that is $\bar{e} \in \bar{a}\bar{R}$. Similarly, from the equality $vR + (1-a)R = R$ it follows that $\bar{1} - \bar{e} \in (\bar{1} - \bar{a})R$. According to [2], $\bar{R}$ is a clean ring. The proposition is proved. $\square$

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