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A morphic ring of neat range one

O. Pihura, B. Zabavsky

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ABSTRACT. We show that a commutative ring R has neat range one if and only if every unit modulo principal ideal of a ring lifts to a neat element. We also show that a commutative morphic ring R has a neat range one if and only if for any elements $a, b \in R$ such that $aR = bR$ there exist neat elements $s, t \in R$ such that $bs = c$, $ct = b$. Examples of morphic rings of neat range one are given.

The notion of principal ideals being uniquely generated first appeared in Kaplansky's classic paper [4]. He had raised the question of when a ring R satisfies the property of being uniquely generated. He remarked that for commutative rings, the property holds for principal ideal rings and artinian rings. In the case of a left quasi morphic ring the property of being uniquely generated is equivalent to that a ring has stable range one. The concept of a neat range one ring is introduced by the first named author in [9]. In this paper we show that for a commutative morphic ring the condition of a neat range one is equivalent to the a uniquely generated weak condition relation with a neat elements.

Throughout this paper we assume that R is a commutative ring with an identity element. To make the paper almost self-contained, we recall basic definitions and some results used later. We recall that:

(i) R is a *Bezout ring*, if each finitely generated ideal of R is principal, see [10].

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(ii) Two rectangular matrices A and B are *equivalent* if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$, see [10].

(iii) The ring R is *Hermite* if every rectangular matrix A over R is equivalent to an upper or a lower triangular matrix, see [10].

(iv) R is *an elementary divisor ring* if every square n by n matrix A with coefficients in R can be converted to a diagonal matrix $diag(a_{11}, \ldots, a_{nn})$ such that every a_{ii} divides $a_{i+1,i+1}$, see [4].

(v) a ring R is *a ring of stable range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a unit of R, see Bass [1].

(vi) An element $a \in R$ is defined to be a *clean* element of R, if a can be written as the sum of a unit and an idempotent. The ring R is defined to be a *clean ring*, if every element of R is clean, see [10].

(vii) An element $a \in R$ is defined to be a *neat* element of R, if R/aR is a clean ring. The ring R is defined to be a *neat ring*, if every elements in a ring R are neat, see [6].

(viii) R is defined to be of *neat range one*, if for any $a, b \in R$ such that $aR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element of R, see[9].

(ix) An element $a \in R$ is defined to be *morphic*, if $\mathrm{Ann}(a) \cong R/aR$, where $\mathrm{Ann}(a)$ denotes the annihilator of a in R. The ring R is defined to be morphic, if every its element is morphic, see [7].

We recall from [4] that every elementary divisior ring R is both a Bezout ring and a Hermite ring. Note also that unity elements of R are neat elements and, hence, every ring of stable range one is a ring of neat range one.

In our next result we need the following definition.

Definition 1. (a) An element $a \in R$ is a unit modulo a principal ideal cR if $ax - 1 \in cR$ for some $x \in R$.

(b) A unit $a \in R$ modulo a principal ideal cR lifts to a neat element, if $a - t \in bR$ for a neat element $t \in R$.

Proposition 1. *Let R be a commutative ring. Then the following are equivalent:*

1) R has a neat rang one;

2) Every unit lifts to a neat element modulo every principal ideal.

Proof. We assume that R has neat range one. Let $a, b, c \in R$ be such that $ab - 1 \in cR$, i.e. b is a unit modulo the principal ideal cR. We show that there exists a neat element $t \in R$ such that $b - t \in R$.

Let $x \in R$ be such that $ab - 1 = cx$. Then $ab - cx = 1$. Since R has neat range one, there exists an element $s \in R$ and a neat element $t \in R$ such that $b - cs = t$. Therefore $b - t \in cR$ where t is a neat element in R.

To prove the implication (2) $\Rightarrow$ (1), assume that every unity of R lifts to a neat element modulo every principal ideal. We show that R has a neat range one. Let $a, b, c \in R$ such that $ab + cd = 1$. Then $ab - 1 \in cR$. Therefore, by our hypothesis there exists a neat element $t \in R$ such that $b - t \in cR$. Thus $b - t = cx$ for some $x \in R$ i.e. $b + c(-x) = t$ is a neat element i.e. R has neat range one. □

Proposition 2. *A morphic ring is a ring of neat range one if and only if for any pair of elements $a, b \in R$ such that $aR = bR$ there are neat elements $s, t \in R$ such that $as = b$ and $a = bt$.*

Proof. In view of Proposition 1 it suffices to show that every unit lifts to a neat element modulo every principal ideal in R.

Let x be a unit that lifts to a neat element modulo the principal ideal yR, i.e there exists $z \in R$ such that $zx - 1 \in yR$. We would like to show that there exists a neat elements $t \in R$ such that $x - t \in yR$. Since R is a morphic, there exists a, b such that $yR = \mathrm{Ann}(a)$ and $xaR = \mathrm{Ann}(b)$.

Obviously, $xR \subset \mathrm{Ann}(ab)$ and $yR \subseteq \mathrm{Ann}(ab)$.

Since $zx - 1 \in yR$, we have $xR + yR = R$ and $xR + yR = \mathrm{Ann}(ab)$. Then $ab = 0$ and $a \in \mathrm{Ann}(b)$. Also we have $\mathrm{Ann}(b) = xaR \subseteq aR$. Therefore $\mathrm{Ann}(b) = xaR = aR$. Under the assumption on the ring there exists a neat element $t \in R$ such that $xa = ta$. This implies that $(x - t)a = 0$. We have $x - t \in \mathrm{Ann}(a) = yR$. Thus from Proposition 1, the R has neat range one.

Let $aR = bR$. Then there exist $x, y \in R$ such that $a = bx$, $b = ay$. Therefore $b = bxy$, $b(1 - xy) = 0$. This shows that $1 - xy \in \mathrm{Ann}(b)$.

Now $xy + (1 - xy) = 1$ where $xy \in xR$ and $1 - xy \in (1 - xy)R$. Therefore $xR + (1 - xy)R = R$. Since R is assumed to have neat range one, there exists $s \in R$ such that $x + (1 - xt)s = t$ is a neat element in R. Since $1 - xy \in \mathrm{Ann}(b)$, we have $(x + (1 - xy)s)b = tb$, $xb = tb$ where $xb = a$. Thus $a = tb$ for some neat element $t \in R$. Similarly we have $b = sa$, for some neat element $s \in R$, which completes the proof. □

Theorem 1. *If R is an elementary divisor ring, then R is a ring of neat range one.*

Proof. By [8] for any elements $a, b, c \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $s = a + bt = uv$, where $uR + cR = R$,

$vR + (1 - c)R$, $uR + vR = R$. Let $\overline{u} = u + sR$, $\overline{v} = v + sR$. Since $uR + vR = R$, one has $ux + vy = 1$ and $\overline{u}^2\overline{x} = \overline{u}$, $\overline{v}^2\overline{y} = \overline{v}$, where $\overline{x} = x + sR$, $\overline{y} = y + sR$. Let $\overline{vy} = \overline{e}$, obviously $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{ux}$. Since $uR + cR = R$, we obtain $\overline{ce}\overline{\beta} = \overline{e}$, for some element $\overline{\beta} \in R/sR$. Similarly, $(\overline{1} - \overline{c})\overline{\alpha}(\overline{1} - \overline{e}) = \overline{1} - \overline{e}$ for some element $\overline{\alpha} \in R/sR$. We proved that for any element $\overline{c} = c + sR$ there exists an idempotent $\overline{e}$ such that $\overline{e} \in \overline{c}\overline{R}$ and $\overline{1} - \overline{e} \in (\overline{1} - \overline{c}\overline{R})$. We have proved that R/sR is a clean ring [6] which completes the proof. $\square$

As a consequence we obtain the following result.

Theorem 2. *If R is an elementary divisor domain and $a \in R \setminus \{0\}$, then the factor-ring R/aR is a morphic ring of neat range one.*

Proof. Since every elementary divisor domain is a Bezout ring [4], by [9] R/aR is a morphic ring. Since every homomorphic image of an elementary divisor ring is an elementary divisor ring, by Theorem 3, R/aR is a morphic ring of neat range one, which completes the proof. $\square$

We say that R has almost stable range one if every finite proper homomorphic image R has stable range one. By [5] a Bezout ring of almost stable range one is an elementary divisor ring.

A well-known Henriksen example of a Bezout domain, namely $R = \mathbb{Z} + x\mathbb{Q}[x]$ (see [2]; for a general theorem on pullbacks of Bezout domains [3]), R is an elementary divisor that does not have almost stable range one [8].

Let R be an elementary divisor domain which is not of almost stable range one. Then there exists an element $a \in R$ such that in the factor-ring $\overline{R} = R/aR$ there exist elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{bR} = \overline{c}\overline{R}$. There exist noninvertible neat elements $\overline{s}, \overline{t} \in R$ such that $\overline{bs} = \overline{c}$, $\overline{ct} = \overline{b}$.

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Contact information

O. Pihura Department of Mechanics and Mathematics, Ivan Franko National Univ., Lviv, Ukraine
E-Mail(s): `pihuraoksana@mail.ru`

B. V. Zabavsky Department of Mechanics and Mathematics, Ivan Franko National Univ., Lviv, Ukraine
E-Mail(s): `zabavskii@gmail.com`

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