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Extended star graphs

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Abstract. Chordal graphs, which are intersection graph of subtrees of a tree, can be represented on trees. Some representation of a chordal graph often reduces the size of the data structure needed to store the graph, permitting the use of extremely efficient algorithms that take advantage of the compactness of the representation. An *extended star* graph is the intersection graph of a family of subtrees of a tree that has exactly one vertex of degree at least three. An *asteroidal triple* in a graph is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. Several subclasses of chordal graphs (interval graphs, directed path graphs) have been characterized by forbidden asteroids. In this paper, we define, a subclass of chordal graphs, called extended star graphs, prove a characterization of this class by forbidden asteroids and show open problems.

Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [6] states that a graph is chordal if and only if it is the (vertex) intersection graph of a family of subtrees of a tree. Families of subtrees of a tree together with the tree are called representation of a graph.

Some representation of a chordal graph often reduces the size of the data structure needed to store the graph, permitting the use of

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extremely efficient algorithms that take advantage of the compactness of the representation. Since some chordal graphs have many distinct representations, it is interesting to consider which one is most desirable under various circumstances, for example minimum diameter [1], minimum number of leaves [11], [4], and imposing conditions on trees, subtrees and intersection sizes [15].

The *leafage* of a chordal graph is the minimum integer ℓ such that the graph admits a representation whose tree has exactly ℓ leaves [14]. This number is related with the existence of asteroidal sets [14].

An *asteroidal set* A in a graph G is a set of non-adjacent vertices such that for any $v \in A$ the vertices of $A \setminus \{v\}$ appears in the same connected component of $G \setminus N[v]$. Note that this definition is compatible with the definition of asteroidal triple already given. The *asteroidal number* of a graph G is the maximum integer a such that G admits an asteroidal set of cardinality a. If G is a chordal graph containing an asteroidal set A of size k, then in any representation of G, its tree has at least k leaves. Thus the asteroidal number of a chordal graph is less or equal to its leafage, and this inequality can be strict [14].

Habib and Stacho [11] found a polynomial algorithm to compute the leafage of a chordal graph and built a representation of it.

Natural subclass of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and interval graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. A graph is a *rooted directed path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. A graph is an *interval graph* if it is the intersection graph of a family of subpaths of a path.

By definition we have the following inclusions between the different considered classes (and these inclusion are strict):

$$\text{interval} \subset \text{rooted directed path} \subset \text{directed path} \subset \text{path} \subset \text{chordal}.$$

Chaplick and Stacho [4] proved that for path graphs there is a representation, where the subtrees are paths, that reaches the leafage, and then it is also true for directed path graphs [5]. However, it is not true for rooted directed path graphs [9].

Lekkerkerler and Boland [12] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple. As byproduct, they found a characterization of interval graphs by forbidden induced subgraphs.

Panda [16] found the characterization of directed path graph by forbidden induced subgraphs and then Cameron, Hoáng and Lévêque [3] gave a characterization of this class in terms of forbidden asteroidal triples.

Lévêque, Maffray and Preissman [13], found the characterization of path graphs by forbidden induced subgraphs but there is still no nice characterization in terms of forbidden asteroids for this class.

Characterizing rooted directed path graph by forbidden induced subgraphs or forbidden asteroids are open problems. It is certainly too difficult to characterizing rooted directed path graphs by forbidden induced subgraphs as there are too many (families of) graphs to exclude but Cameron, Hoáng and Lévêque [2] suggest that directed path graphs could be characterized by forbidding some particular type of asteroidal quadruples (a set of four non-adjacent vertices such that any three of them is an asteroidal triple). Thus, several subclasses of rooted directed path graphs [10], [8] have been characterized by forbidden asteroids, and as byproduct it was found the characterization of them by forbidden induced subgraphs.

Other subclass of chordal graphs is extended star graphs. A graph G is *an extended star* if it is the intersection graph of families of subtrees of a tree which has exactly one vertex of degree at least tree. Clearly this class is a natural generalization of interval graphs.

By definition we have the following inclusions between the different considered classes (and these inclusion are strict):

$$\text{interval} \subset \text{extended star} \subset \text{chordal}$$

On the other hand, this class is hereditary, i.e is closed under vertex-induced subgraphs. It is known that hereditary classes admit a characterization by forbidden induced subgraphs. Characterize extended star graphs by forbidden induced subgraphs or by forbidden asteroids are open problems. Also it is an open problem answer if for extended star graph there is a representation that reaches the leafage.

In this paper we study properties of extended star graphs, and give a characterization of this class by forbidden asteroids.

The paper is organized as follows: in Section 2, we give some definitions and background. In Section 3, we prove a characterization of this class by forbidden asteroids. Finally, in Section 4, we show conclusions and open problems.

1. Definitions and background

A *clique* in a graph G is a set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all maximal cliques* of G. We denote by C_x the set of the maximal cliques that contain x.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$. A vertex s is *simplicial* if its closed neighborhood is a maximal clique.

A *clique tree* T of a graph G is a tree whose vertices are the elements of $\boldsymbol{C}(G)$ and such that for each vertex x of G, C_x induces a subtree of T, which we will denote by T_x.

Note that G is the intersection graph the vertex sets of subtrees $(T_x)_{x \in V(G)}$. Gavril [6] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

It is clear that a graph is an *interval graph* if it admits a clique tree T that is a path such that T_x is a subpath of T for every $x \in V(G)$. A natural generalization of interval graphs are extended star graphs. A graph G is an *extended star* if there is a model of G that has at most exactly one vertex of degree at least three, such models are called *extended star models*. Clearly, interval graphs is a subclass of extended star graphs. Split graphs, minimal forbidden induced subgraphs for interval graphs, and path graphs minimal forbidden induced subgraphs for directed path graphs are examples of extended star graphs.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of G. In order to simplify the notation, we often write $Q \in T$ instead of $Q \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\cup_{Q \in V(T')} Q$.

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and the edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| = 0$, then $T + T'$ is a forest.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimal subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X, Y]$ is the subpath of T between X and Y. Let $T[X, Y) = T[X, Y] \setminus Y$, $T(X, Y] = T[X, Y] \setminus X$ and $T(X, Y) = T[X, Y] \setminus \{X, Y\}$. Note that some of these paths may be empty or reduced to a single vertex when X and Y are equal or adjacent.

We say that $T[X,Y]$ is a *branch* of T if X is a leaf of T and Y is its most next vertex of degree at least three of T.

For $X,Y,Z \in V(T)$ that are not on the same path in T, $T[X,Y,Z]$ is the subtree of T that has X,Y,Z, as its leaves. Let $T[X,Y,Z) = T[X,Y,Z]\backslash Z$ and $T(X,Y,Z) = T[X,Y,Z]\backslash\{X,Z\}$.

In a clique tree T, the *label* of an edge QQ' of T is defined as $lab(QQ') = Q \cap Q'$. Observe that the label of an edge of T is a minimal separator of G.

Let T be a tree, we denote by $ln(T)$ the number of leaves of T. The *leafage* of a chordal graph G is a minimum integer ℓ such that G admits a model T with $ln(T) = \ell$ [14].

In some cases the leafage of a graph decides if a graph is an extended star as shows the following Lemma.

Lemma 1. *Let G be a chordal graph. If $l(G) \leqslant 3$ then G is an extended star graph.*

Proof. Let T be a model of G that reaches the leafage, i.e $ln(T) = l(G)$. Clearly, $ln(T) \leqslant 3$. Thus T has at most exactly one vertex of degree three. Therefore, G is an extended star graph. □

An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. An *asteroidal n-tupla* in a graph G is a set of n non-adjacent vertices such that for any $(n-1)$ of them is an asteroidal $(n-1)$-tupla.

If G is a chordal graph containing an asteroidal n-tupla, then in any model T of G, T has at least n leaves. Thus the leafage of G is greater or equal to n.

In [7] has been proved that for any clique tree that reaches the leafage, every vertex of degree at least three, and every choice of three branches incident to it there is an asteroidal triple on these branches. Thus for extended star graphs we have the same result.

Lemma 2. *Let G be an extended star graph and T be an extended star model of G with minimum number of leaves equal $n > 2$. Then G has $\frac{n(n-1)(n-2)}{6}$ asteroidal triples.*

Proof. Let $H_1, H_2, \ldots, H_n$ be the leaves of T and Q be the vertex of degree at least three in T. Suppose that $G_{T[H_1,H_2,H_3]}$ does not have an asteroidal triple. Then there is an interval model T' of $G_{T[H_1,H_2,H_3]}$. Clearly $T - (T[H_1,Q) + T[H_2,Q) + T[H_3,Q)) + T'$ is an extended star model of

G which has less leaves than T, a contradiction. Hence $G_{T[H_i,H_j,H_k]}$ has an asteroidal triple for any three different $i, j, k \in \{1, 2 \ldots, n\}$. Therefore G has $\frac{n(n-1)(n-2)}{6}$ asteroidal triples. □

Lemma 3. *Let G be an extended star chordal graph and T be an extended star model of G with minimum number of leaves equal $n > 2$. If T has exactly k leaves whose distance to the vertex of degree at least three is greater than one then G has at least an asteroidal $(n-k)-$tuple.*

Proof. Let Q be the vertex of degree n of T, $H_1, \ldots, H_k$ be the leaves of T at distance greater than one to Q in T, and $H_{k+1}, \ldots, H_n$ be the other that are incident to the vertex Q. Let $a_{k+1}, \ldots, a_n$ be simplicial vertices of $H_{k+1}, \ldots, H_n$ respectively. Since a_i is a simplicial vertex of G, $N[a_i] = H_i$ for $i \in \{k+1, \ldots, n\}$. Let $T' = T[H_{k+1}, \ldots, H_n]$. Suppose that $G_{T'} \setminus N[a_n]$ is not a connected graph. So there is at least an edge H_iQ in T' for some $i \in \{k+1, \ldots n-1\}$ such that $lab(H_iQ) \subset H_n$. Then $T_1 = T - H_iQ + H_iH_n$ is an extended star model of G that has less leaves than T, a contradiction. Hence $G_{T'} \setminus N[a_i]$ is a connected graph for all $i \in \{k+1, \ldots, n\}$. Therefore $a_{k+1}, \ldots, a_n$ is an asteroidal $(n-k)$-tupla. □

Lemma 4. *Let s be a simplicial vertex of G, a minimally non extended star graph. Then*

1) *s is a vertex of some asteroidal triple;*
2) *there is a model T of G which has exactly two vertices of degree at least three Q and Q'. Moreover, there is at least two branches $T[Q', H_i']$ for $i = 1, 2$ such that $G_{T[H_1',H_2',Q]}$ is not an interval graph;*
3) *there is a model T of G which has exactly two vertices Q, Q' of degree at least three, it has at least two branches $T[Q', H_i']$ for $i = 1, 2$ such that $G_{T[H_1',H_2',Q]}$ is not an interval graph, and if $T[H_i, Q]$ are the branches of T for $i \in \{1, \ldots, n\}$ then $G_{T[H_i,H_j,Q']}$ are not interval graphs for $i, j \in \{1, \ldots, n\}$, $i \neq j$.*

Proof. 1), 2) Since G is a minimal non extended star graph each simplical vertex of G verifies that if we remove this vertex, the graph obtained has lower number of maximal cliques than G. Let s be a simplicial vertex of G. Clearly, there is a maximal clique $Q' \neq N[s]$ such that $N(s) \subset Q'$. Since G is a minimal non extended star graph, $G \setminus s$ is an extended star graph. By Lemma 1 $l(G) \geqslant 4$, and since s is a simplicial vertex it follows that $l(G \setminus s) \geqslant 3$. Let T' be an extended star model of $G \setminus s$, and Q be the vertex of degree at least three of T'. Clearly $T = T' + N[s]Q'$ is a model of G, and since G is not an extended star graph so $Q' \neq Q$ and Q' is not a leaf of T'. Observe that T has only two vertices of degree at least three Q and Q'. Let $H \neq N[s]$ be the leaf of T such that $Q' \in T[Q, H]$. In case that $G_{T[Q,N[s],H]}$ is an interval graph, there is an interval model T_1' of $G_{T[Q,N[s],H]}$. Let $T_1 = T - T(Q, N[s], H] + T_1'$. Clearly T_1 is an extended star model of G, a contradiction. Hence $G_{T[Q,N[s],H]}$ is not an interval graph, so there is an asteroidal triple, and clearly s must be a vertex of it.

3) Among all the trees in the condition 2), choice that has minimum leafage, and maximum degree in Q' (recall that Q' is a vertex of degree at least three such that there is at least two branches $T[H_i', Q']$ for $i = 1, 2$ such that $G_{T[H_1', H_2', Q]}$ is not an interval graph). If for some $i, j \in \{1, \dots n\}$, $i \neq j$, $G_{T[H_i, H_j, Q']}$ is an interval graph then there is an interval model T_1 of $G_{T[H_i, H_j, Q']}$. Let $T' = T - T[H_i, H_j, Q') + T_1$. Clearly T' is a model of G which has exactly two vertices of degree at least three, a leaf is incident to Q' and $G_{T[Q,N[s],H]}$ is not an interval graph. Moreover, if Q' is a leaf of T_1 then in T' the degree of Q' is the same that in T but $ln(T') < ln(T)$, a contradiction. If Q' is not a leaf of T_1 then $ln(T') = ln(T)$ but the degree of Q' in T' is greater than the degree of Q' in T, a contradiction. Hence for $i, j \in \{1, \dots n\}$, $i \neq j$, $G_{T[H_i, H_j, Q']}$ is not an interval graph. $\square$

The following algorithm is a technical tool necessary in the proof of characterization of extended star graph by forbidden asteroids.

Algorithm

Input: A model T that has minimum number of leaves, exactly two vertices Q, Q' of degree at least three at distance greater than one, $Q^* \in T(Q, Q')$ and $T[H_i, Q]$ the branches incident to Q for $i \in \{1, \dots, n\}$.

Output: A model T' that has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'; or it has at least two vertices Q, Q' of degree at least three and at most three vertices Q, Q', Q^* of degree at least three, Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \dots, n-2\}$ such that $G_{T'[\overline{H}_l, H_{l+2}, Q^*]}$ is not an interval graph.

If $G_{T[H_1, H_2, Q^*]}$ is not an interval graph **Then**

RETURN: $T' = T$

Else

Take T_1 an interval model of $G_{T[H_1, H_2, Q^*]}$ and build a model $\overline{T}_1 = T - T[H_1, H_2, Q^*) + T_1$.

If $n = 2$ **Then**

RETURN: $T' = \overline{T}_1$

Else

Let $\overline{T}_1[\overline{H}_1, Q]$ and $\overline{T}_1[H_i, Q]$ be the branches incident to Q for $i \in \{3, \ldots, n\}$.

If $G_{\overline{T}_1[\overline{H}_1, H_3, Q^*]}$ is not an interval graph **Then**

RETURN: $T' = \overline{T}_1$

Else

$i = 2$

$*$ Take T_i an interval model of $G_{\overline{T}_{i-1}[\overline{H}_{i-1}, H_{i+1}, Q^*]}$ and build a model $\overline{T}_i = \overline{T}_{i-1} - \overline{T}_{i-1}[\overline{H}_{i-1}, H_{i+1}, Q^*) + T_i$.

If $n > i + 1$ **Then**

Let $\overline{T}_i[\overline{H}_i, Q]$ and $\overline{T}_i[H_j, Q]$ be the branches incident to Q for $j \in \{i + 2, \ldots, n\}$

If $G_{\overline{T}_i[\overline{H}_i, H_{i+2}, Q^*]}$ is not an interval graph **Then**

RETURN: $T' = \overline{T}_i$

Else

$i = i + 1$ go to $*$

Else

RETURN: $T' = \overline{T}_i$

Observe that T_i is an interval model that does not have Q^* as a leaf, otherwise $ln(\overline{T}_i) < ln(T)$ a contradiction since T is a model of G that has minimum number of leaves.

Note that the way $\overline{T}_i$ was built assure that has at most three vertices of degree at least three Q, Q^*, Q' that appear in this order in $\overline{T}_i$, and $\overline{T}_i[\overline{H}_i, Q]$, $\overline{T}_i[H_j, Q]$ are the branches of $\overline{T}_i$ for $j \in \{i + 2, \ldots, n\}$. Also the degree in $\overline{T}_i$ of Q is $n + 1 - i$ and the degree of Q^* is $i + 2$.

We will see that the algorithm works.

Suppose that the algorithm stopped since $G_{T[H_1, H_2, Q^*]}$ is not an interval graph then $T' = T$ has exactly two vertices Q, Q' of degree at least three whose distance in T' is the same that its distance in T.

Suppose that the algorithm stopped when $i = 1$ and $n = 2$. Since T has minimum number of leaves then Q^* is not a leaf of T_1 then $ln(\overline{T}_1) = ln(T)$. Also $T' = \overline{T}_1$ has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'. If $n > 2$ and $G_{\overline{T}_1[\overline{H}_1, H_3, Q^*]}$ is not an interval graph then $T' = \overline{T}_1$ has three vertices Q, Q', Q^* of degree at least three, Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \ldots, n - 2\}$ such that $G_{T'[\overline{H}_l, H_{l+2}, Q^*]}$ is not an interval graph.

Suppose that the algorithm stopped when $2 \leqslant i < n-1$. Thus T' has three vertices Q, Q', Q^* of degree at least three; Q, Q^*, Q' appear in this order in T', and there are two branches $T'[\overline{H}_l, Q]$ and $T'[H_{l+2}, Q]$ for $l \in \{1, \ldots, n-2\}$ such that $G_{T'[\overline{H}_l, H_{l+2}, Q^*]}$ is not an interval graph.

Suppose that the algorithm stopped when $i = n-1$. Thus T' has exactly two vertices Q^*, Q' of degree at least three whose distance in T' is the same that its distance in T, and Q, Q^*, Q' appear in this order in T'

2. Forbidden asteroids characterization for extended star graphs

A pair of asteroidal triples in a graph G is *strongly linked* if it contains from two asteroidal triples a_1, a_2, a_3; b_1, b_2, b_3 satisfying the following conditions:

1) $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$.
2) Every path between a_i and b_j has vertices in $N[a_3]$ and in $N[b_3]$ for $i, j \in \{1, 2\}$.
3) Let S, M be minimal separators of G with $S \subset N[b_3]$ and $M \subset N[a_3]$. If a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$ then there is no $Q \in C(G)$ such that $M \cup S \subset Q$.

Observe that if T is a model of a graph G that has a pair of strongly linked asteroidal triples a_1, a_2, a_3; b_1, b_2, b_3 and $Q_i, Q'_i \in C(G)$ such that $a_i \in Q_i$ and $b_i \in Q'_i$ for $i = 1, 2$ then by 2, there are at least two edges $e, e' \in T[Q_i, Q'_i]$ such that $lab(e) \subset N[a_3]$ and $lab(e') \subset N[b_3]$. Also $T_{a_i} \cap T_{b_j} = \varnothing$ for $i, j \in \{1, 2\}$.

Notice that if G has a pair of strongly linked asteroidal triples by item 2 of the definition: a_i, b_j are in different connected component of $G \setminus N[a_3]$ and $G \setminus N[b_3]$ or $a_i \in N[b_3]$ or $b_j \in N[a_3]$ for $i, j \in \{1, 2\}$.

Theorem 1. *Let G be a chordal graph. G is an extended star graph if and only if G does not have a pair of strongly linked asteroidal triples.*

Proof. $\Rightarrow$ Suppose that G has a pair of strongly linked asteroidal triples a_1, a_2, a_3 ; b_1, b_2, b_3, and it is an extended star graph. Then there is an extended star model T of G. Since G has an asteroidal triple then $l(G) \geqslant 3$. Let Q be the vertex of degree at least three in T. Since T is an extended star model, T_{a_i} and T_{b_i} induce paths in T for $i \in \{1, 2, 3\}$. Let H_1, H_2, H_3 be leaves of T such that T_{a_i} induces a path in $T(Q, H_i]$ for $i \in \{1, 2, 3\}$.

In the follows, we prove that T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$.

Suppose that T_{b_1} induces a path in $T(Q, H_1]$.

Let $T_{a_1} = T[Q_1, Q_2]$ and $T_{b_1} = T[Q_3, Q_4]$ be such that $Q_1 \in T[Q, Q_2]$ and $Q_3 \in T[Q, Q_4]$. Since $a_1, a_2, a_3; b_1, b_2, b_3$ is a pair of strongly linked asteroidal triples it follows that $T_{a_1} \cap T_{b_1} = \varnothing$. Thus $Q, Q_3, Q_4, Q_1, Q_2, H_1$ or $Q, Q_1, Q_2, Q_3, Q_4, H_1$ appear in this order in $T[Q, H_1]$.

In case that $Q, Q_3, Q_4, Q_1, Q_2, H_1$ appear in this order in $T[Q, H_1]$, by the item 2) of the definition of a pair of strongly linked asteroidal triples, there is an edge $e \in T[Q_4, Q_1]$ such that $lab(e) \subset N[a_3]$ so each path between a_1 and a_2 in G has neighbors of a_3 contradicting that a_1, a_2, a_3 is an asteroidal triple.

In case that $Q, Q_1, Q_2, Q_3, Q_4, H_1$ appear in this order in $T[Q, H_1]$, by the item 2) of the definition of a pair of strongly linked asteroidal triples, there is an edge $e' \in T[Q_3, Q_2]$ such that $lab(e') \subset N[b_3]$. Then each path between b_1 and b_2 in G has neighbors of b_3 contradicting that b_1, b_2, b_3 is an asteroidal triple.

Following the earlier argument, we can conclude that T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$.

Finally, we prove that T_{b_3} does not induce a path in $T(Q, H_3]$.

Suppose that T_{b_3} induces a path in $T(Q, H_3]$. Let $T_{a_3} = T[Q_5, Q_6]$ and $T_{b_3} = T[Q_7, Q_8]$ be such that $Q_5 \in T[Q, Q_6]$ and $Q_7 \in T[Q, Q_8]$. Observe that $T_{a_3} \cap T_{b_3}$ may be different from $\varnothing$. Clearly Q, Q_5, Q_7, H_3 or Q, Q_7, Q_5, H_3 appear in this order in $T[Q, H_3]$. As T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$, and T_{b_3} induces a path in $T(Q, H_3]$ then there exist H_4, H_5 leaves of T such that T_{b_1} and T_{b_2} induce paths in $T(Q, H_4]$ and $T(Q, H_5]$ respectively.

In case that Q, Q_5, Q_7, H_3 appear in this order in $T[Q, H_3]$, there is an edge $e' \in T[Q_1, Q]$ such that $lab(e') \subset N[b_3]$. By the position in T of Q_5, $lab(e') \subset N[a_3]$ so each path between a_1 and a_2 in G has neighbors of a_3 contradicting that a_1, a_2, a_3 is an asteroidal triple.

In case that Q, Q_7, Q_5, H_3 appear in this order in $T[Q, H_3]$, there is an edge $e \in T[Q_3, Q]$ such that $lab(e) \subset N[a_3]$, following the earlier argument each path between b_1 and b_2 in G has neighbors of b_3 contradicting that b_1, b_2, b_3 is an asteroidal triple.

Hence T_{b_3} does not induce a path in $T(Q, H_3]$.

By before exposed, T_{b_i} does not induce a path in $T(Q, H_j]$ for $i, j \in \{1, 2\}$ and T_{b_3} does not induce a path in $T(Q, H_3]$.

Suppose that T_{b_1} does not induce a path in $T(Q, H_j]$ for $j \in \{1, 2, 3\}$.

Let H_4 be a leaf different from H_1, H_2, H_3 such that T_{b_1} induces a path in $T(Q, H_4]$. We can assume that T_{b_3} does not induce a path in $T[H_1, Q]$. By the item 2) of the definition of a pair of strongly linked asteroidal triples, there are edges e, e', $e \in T[H_1, Q]$ and $e' \in T[Q, H_4]$ such that $lab(e) \subset N[b_3]$ and $lab(e') \subset N[a_3]$. Let $S = lab(e)$ and $M = lab(e')$. Clearly S and M are minimal separators of G such that a_1, a_2 are in different connected components of $G \setminus S$, and b_1, b_2 are in different connected components of $G \setminus M$. By the position in T of the maximal cliques $N[b_3]$ and $N[a_3]$ it follows that $S \cup M \subset Q$, contradicting the item 3) of the definition of a pair of strongly linked asteroidal triples.

Thus the pair of strongly linked asteroidal triples do not have way of being located on an extended star model. Therefore, G is not an extended star graph.

$\Leftarrow$ Suppose that G is a minimally non extended star graph. By Lemma 1, $l(G) \geqslant 4$ and by Lemma 4. 3), there is a model T of G that has exactly two vertices Q, Q' of degree at least three. Let $H_1, \ldots, H_n$ be the leaves of T such that $T[H_i, Q]$ are branches of T for $i = 1, \ldots, n$, and let $H'_1, \ldots, H'_m$ be the leaves of T such that $T[H'_j, Q']$ are branches of T for $j = 1, \ldots, m$. Moreover, by Lemma 4. 3), Q' has maximum degree and there are at least two leaves H'_k, H'_l for $k \neq l$, $k, l \in \{1, \ldots, m\}$ such that $G_{T[H'_k, H'_l, Q]}$ is not an interval graph. Also for all $i \neq j$, $i, j \in \{1, \ldots, n\}$ $G_{T[H_i, H_j, Q']}$ are not interval graphs. Recall that T has minimum leafage. Among all the trees in these conditions choice one that minimizing the distance in T between Q and Q'.

- In case that the distance in T between Q and Q' is greater than one we analyze two situations:

Case 1. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T; or Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T.

Case 2. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, and applying the Algorithm to Tconsidering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$, in both cases it does not output T.

Observe that applying the Algorithm to T considering $Q^* \in T(Q,\, Q')$, the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, and by our election of T, which minimizing the distance in T between Q and Q', if the Algorithm outputs a tree with exactly two vertices of degree at least three then it must be T. More clearly, if it outputs a tree T' with exactly two vertices of degree at least three, which are not Q and Q', then they must be Q^* and Q'. Also by the way T' was built $l(T') = ln(T)$, and the distance between Q^* and Q' in T' is the same that its distance in T, and it is lower that the distance in T between Q and Q', contradicting this way the election of T that has exactly two vertices of degree at least three to minimum distance.

Case 1. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T; or applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T.

Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$, it outputs T. In this case we can assume that $G_{T[H_1, H_2, Q^*]}$ is not an interval graph. We will analyze two situations: applying the Algorithm considering $Q^* \in T(Q, Q')$, and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T or not.

Case 1.1. Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T. Also suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$ and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it outputs T. In this case we can assume that $G_{T[H'_1, H'_2, Q^*]}$ is not an interval graph.

Since $G_{T[H_1, H_2, Q^*]}$ is not an interval graph then there is an asteroidal triple a_1, a_2, a_3. Analogously, there is an asteroidal triple b_1, b_2, b_3 in $G_{T[H'_1, H'_2, Q^*]}$.

Suppose that $a_3 \in Q_3$ with $Q_3 \in T(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

Case 1.2. Suppose that applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ it outputs T. Let T_0 be the connected component of $T - T(Q^*, Q')$ that contains Q and Q^*.

Also, assume that applying the Algorithm to T considering $Q^* \in T(Q, Q')$ and the branches $T[H'_j, Q']$ for $j = 1, \ldots, m$ it does not output T. Let $\overline{T'}$ be the tree outputs by the Algorithm, and $\overline{T_0}$ be the connected component of $\overline{T'} - \overline{T'}(Q, Q^*)$ that contains Q' and Q^*.

Let $T'' = T_0 + \overline{T_0}$. Clearly T'' is a model of G.

By the way T'' was built Q, Q^*, Q' appear in this order in T'', T'' has three vertices Q, Q^*, Q' of degree at least three. Also there are four branches in T'', $T''[H_1, Q] = T_0[H_1, Q] = T[H_1, Q]$, $T''[H_2, Q] = T_0[H_2, Q] = T[H_2, Q]$, $T''[H'_j, Q'] = \overline{T_0}[H'_j, Q'] = \overline{T'}[H'_j, Q']$, $T''[\overline{H'_l}, Q'] = \overline{T_0}[\overline{H'_l}, Q'] = \overline{T'}[\overline{H'_l}, Q']$ for $j \neq l$, $j, l \in \{1, \ldots, m\}$ such that $G_{T''[H_1, H_2, Q^*]}$ and $G_{T''[H'_j, \overline{H'_l}, Q^*]}$ are not interval graphs. Suppose that $j = 1$ and $l = 2$.

In each situations describing before, we can assume that there is an asteroidal triple a_1, a_2, a_3 in $G_{T''[H_1, H_2, Q^*]}$ and there is an asteroidal triple b_1, b_2, b_3 in $G_{T''[H'_1, \overline{H'_2}, Q^*]}$. Suppose that $a_3 \in Q_3$ with $Q_3 \in T''(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T''[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T''(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

Case 2. Applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_i, Q]$ for $i = 1, \ldots, n$ and applying the Algorithm to T considering $Q^* \in T(Q, Q')$, and the branches $T[H_j, Q']$ for $i = j, \ldots, m$, in both cases it does not output T. Let T' and $\overline{T'}$ be the subtrees obtained respectively. By our assumption $T' \neq T$ and $\overline{T'} \neq T$.

Let T_0 be the connected component of $T' - T'(Q^*, Q')$ that contains Q and Q^*, and $\overline{T_0}$ be the connected component of $\overline{T'} - \overline{T'}(Q, Q^*)$ that contains Q' and Q^*. Let $T'' = T_0 + \overline{T_0}$.Clearly T'' is a model of G.

By the way T'' was built Q, Q^*, Q' appear in this order in T'', T'' has at least two vertices Q, Q' of degree at least three and at most three vertices Q, Q^*, Q' of degree at least three. Also there are four branches in T'', $T''[H_i, Q] = T_0[H_i, Q] = T'[H_i, Q]$, $T''[\overline{H_k}, Q] = T_0[\overline{H_k}, Q] = T'[\overline{H_k}, Q]$, $T''[H'_j, Q'] = \overline{T_0}[H'_j, Q'] = \overline{T'}[H'_j, Q']$, $T''[\overline{H'_l}, Q'] = \overline{T_0}[\overline{H'_l}, Q'] = \overline{T'}[\overline{H'_l}, Q']$ for $i \neq k$, $j \neq l$, $i, k \in \{1, \ldots, n\}$ and $j, l \in \{1, \ldots, m\}$ such that $G_{T''[H_i, \overline{H_k}, Q^*]}$ and $G_{T''[H'_j, \overline{H'_l}, Q^*]}$ are not interval graphs. Suppose that $i = 1$, $k = 2$, $j = 1$ and $l = 2$.

We can assume that there is an asteroidal triple a_1, a_2, a_3 of $G_{T''[H_1,\overline{H_2},Q^*]}$ and there is an asteroidal triple b_1, b_2, b_3 of $G_{T''[H'_1,\overline{H'_2},Q^*]}$. Suppose that $a_3 \in Q_3$ with $Q_3 \in T''(Q, Q^*]$, and $b_3 \in Q'_3$ with $Q'_3 \in T''[Q^*, Q')$. Thus $|\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\}| \leqslant 1$. Then the item 1) of the definition of a pair of strongly linked asteroidal triples was checked.

Given that $Q_3, Q'_3 \in T''(Q, Q')$ each path between a_i and b_j must have vertices in Q_3 and Q'_3 for $i, j \in \{1, 2\}$. So each path between a_i and b_j has neighbors of a_3 and b_3 for $i, j \in \{1, 2\}$. Then the item 2) of the definition of a pair of strongly linked asteroidal triples was checked.

Finally, by our choice of a_1, a_2, a_3; b_1, b_2, b_3, there are not minimal separators $S \subset N[b_3]$, $M \subset N[a_3]$ satisfying a_1, a_2 are in different connected components of $G \setminus S$ and b_1, b_2 are in different connected components of $G \setminus M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 are a pair of strongly linked asteroidal triples.

- In case that the distance in T between Q and Q' is one.

By our election of T, we can assume that there is an asteroidal triple a_1, a_2, a_3 of $G_{T[H_1,H_2,Q']}$ and there is an asteroidal triple b_1, b_2, b_3 of $G_{T[H'_1,H'_2,Q]}$. Clearly $a_3 \in Q'$ and $b_3 \in Q$. It is easy to verify that a_1, a_2, a_3; b_1, b_2, b_3 satisfy the items 1), 2) of the definition of a pair of strongly linked asteroidal triples.

Finally, we check the item 3) of the definition of a pair of strongly asteroidal triples. Let $Q_1, Q_2 \in T[H_1, H_2]$ be such that minimizing the distance to Q and $a_i \in Q_i$ for $i = 1, 2$. Observe that each minimal separator $S \subset N[b_3]$, which satisfies a_1, a_2 are in different connected components of $G \setminus S$, is the label of an edge in $T[H_1, H_2]$. Moreover it is in $T[Q_1, Q_2]$. Analogously, each minimal separator $M \subset N[a_3]$, which satisfies b_1, b_2 are in different connected components of $G \setminus M$, is the label of an edge in $T[H_3, H_4]$, and it is in $T[Q_3, Q_4]$ with $Q_3, Q_4 \in T[H'_1, H'_2]$ minimizing the distance to Q and $b_i \in Q_{i+2}$ for $i \in \{1, 2\}$. Suppose that there is Q^* such that $S \cup M \subset Q^*$. Let T_1, T_2 be subtrees of T such that $T_1 + T_2 + T[Q, Q'] = T$, $T_1 \cap T_2 = \varnothing$, $T_1 \cap T[Q, Q'] = \{Q\}$, $T_2 \cap T[Q, Q'] = \{Q'\}$. Suppose that $Q^* \in T_1$.It is clear that Q^*, Q, Q' appear in this order in T. Since $M \subset N[a_3]$, there is an edge $e' \in T_2$ such that $lab(e') = M \subset Q^*$. Given that $e' \in T[Q_3, Q_4]$ and by the order in that appear Q^*, Q in T it follows that $lab(e') \subset Q$. As $b_3 \in Q$, $Q \subset N[b_3]$. It follows that $lab(e') \subset N[b_3]$. Thus each path between b_1 and b_2 in G has vertices in $N[b_3]$ contradicting that b_1, b_2, b_3 is an asteroidal triple of G. Hence $Q^* \notin T_1$. Suppose that $Q^* \in T_2$. Following an argument similar to the previous one, we arrive to a contradiction since a_1, a_2, a_3 is an asteroidal triple of G.

Hence there is no $Q^* \supset S \cup M$. Therefore a_1, a_2, a_3; b_1, b_2, b_3 is a pair of strongly linked asteroidal triples. $\square$

Corollary 1. *Let G be a minimal non extended star graph. Then $l(G) = 4$*

Proof. Suppose that $l(G) > 4$. Thus each model of G has at least five leaves. As a consequence of the proof of Theorem 1, there are a model T of G and H_1, H_2, H_3, H_4 four leaves of T such that $G_{T[H_1,H_2,H_3,H_4]} \neq G$ has a pair of strongly linked asteroidal triples contradicting that G is a minimal non extended star graph. $\square$

Conclusions

The characterization of interval graphs given by Lekkerkerker-Boland, related chordal non interval graphs with asteroidal triples. This kind of characterization is given by Cameron, Hoáng and Lévêque for chordal non directed path graphs. In this paper we have defined a subclass of chordal graphs, extended star graphs, and we related chordal non extended star graphs with asteroids. For this purpose we defined a particular type of asteroidal triple to obtain a characterization of this class by forbidden asteroids. On the other hand, this class is hereditary so it admits a characterization by forbidden induced subgraphs. Our result is useful to build forbidden induced subgraphs, it may be choice two forbidden induced subgraphs for interval graphs whose asteroidal triples are a_1, a_2, a_3 and b_1, b_2, b_3 and add a path between a_3 and b_3 or identify a_3 and b_3.

On the other hand, it is known that for path graphs and directed path graphs there is a model that reaches the leafage. But it is not true for rooted directed path graphs. An interesting questions is if for extended star graphs there is a model that reaches the leafage or if it is possible to build a model with minimum number of leaves.

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