

On the spectrum of Cayley graphs

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Communicated by I. Ya. Subbotin

ABSTRACT. The set of eigenvalues of the adjacency matrix of a graph is called the spectrum of it. In the present paper, we introduce the spectrum of Cayley graphs of order pqr in terms of character table, where p, q, r are prime numbers. We also, establish some properties of Cayley graphs of non-abelian groups with a normal symmetric connected subset.

Introduction

By investigating Cayley graphs, even more detailed information about a group can be obtained. In this paper, we study the spectral properties of Cayley graphs via the character table of underlying group.

Computing the spectrum of Cayley graphs was started by a paper of Babai [2] in 1979 and recently, this exciting research topic is received increasing attention by mathematician, see for example [4, 6, 19, 22]. Ghorbani and Nowroozi in a series of articles computed the spectrum of normal and normal edge-transitive Cayley graphs of order n where $n \in \{p^3, p^2q, pqr, 2pq\}$ and p, q, r are prime numbers, see [9, 11–15]. But in general the spectrum of Cayley graphs of these groups is still an open problem. Here, we compute the spectrum of such Cayley graphs by means of character table. We also compute the spectrum of Cayley graphs of non-abelian groups by constructing the circulant matrices, see [9].

2010 MSC: 20D15.

Key words and phrases: Cayley graphs, symmetric set, semi-direct product, characteristic polynomial.

In the next section, we give the necessary definitions and some preliminary results. In section three, we establish a formula for computing the spectra of normal Cayley graphs. Finally, in section four, we compute the spectrum of Cayley graphs of order pqr , in general. Here, our notation is standard and mainly taken from the standard books such as [16].

1. Definitions and preliminaries

Here, we introduce some basic notation and terminology used throughout the paper. All graphs considered here are finite and simple. A simple graph is a graph Γ without loops and multiple edges. The vertex set and the edge set of graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. When two vertices u and v are endpoints of an edge, we say that they are adjacent and write $u \sim v$ to indicate this. The adjacency matrix A is an $n \times n$ matrix whose xy -th entry is 1 if $xy \in E$ and zero otherwise.

For given graphs Γ_1 and Γ_2 their Cartesian product $\Gamma_1 \square \Gamma_2$ is defined as the graph on the vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if and only if either ($[u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2)]$) or ($[u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1)]$). It is well-known that $A(\Gamma_1 \square \Gamma_2) = A(\Gamma_1) \otimes I + I \otimes A(\Gamma_2)$, where \otimes denotes the Tensor (Kronecker) product see [5].

For the finite group G , the generating subset S is symmetric if $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = \text{Cay}(G, S)$ on G with respect to S has the vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, sg) | g \in G, s \in S\}$. By this definition the Cayley graph $\text{Cay}(G, S)$ always is connected.

Theorem 1 ([1]). *Let $\Gamma_1 = \text{Cay}(G, S_1)$ and $\Gamma_2 = \text{Cay}(H, S_2)$ be two Cayley graphs. Then the Cartesian product $\Gamma_1 \square \Gamma_2$ is the Cayley graph of the direct product $G \times H$ with the generating subset $(S_1, 1) \square (1, S_2)$.*

The characteristic polynomial $\chi(\Gamma)$ of graph Γ with adjacency matrix A is defined as $\chi(\Gamma) = \det(xI - A)$. It is a monic polynomial of degree n . The roots of the characteristic polynomial are eigenvalues of Γ and form the spectrum of Γ . Since all considered graphs are undirected, the adjacency matrix A is symmetric. Consequently, all eigenvalues are real.

Theorem 2 ([3]). *Let A and B is square matrices of orders m and n , respectively. If $\lambda_1, \dots, \lambda_m$ are eigenvalues of A and μ_1, \dots, μ_n are the eigenvalues of B , then all eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ ($\leq i \leq m, 1 \leq j \leq n$).*

It is well-known that if A is a matrix of order m with eigenvalues λ_i and B is a matrix of order n with eigenvalues μ_j , then all eigenvalues of $aA \otimes I_n + bI_m \otimes B$ are $a\lambda_i + b\mu_j$, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

A circulant matrix is a square matrix generated from a vector such as $[a_0, \dots, a_{n-1}]$ as the first row denoted by $[[a_0, \dots, a_{n-1}]]$. Successive rows use the same elements as the first row but each such row is circularly shifted by one element. All eigenvalues of this circulant matrix are $\lambda_\omega = \sum_{i=0}^{n-1} a_i \omega^i$, where ω is an n -th root of unity.

Theorem 3 ([2]). *Let $(A_{ij}, 1 \leq i, j \leq l)$ be square matrices of order n that have the complete set of eigenvectors $\{V_1, \dots, V_n\}$ with $A_{ij}V_k = \alpha_{ij}^k V_k$. Let also, $B_k = [\alpha_{ij}^k]$ be square matrices of order l , each with a complete set of eigenvectors $\{U_1^k, \dots, U_l^k\}$ satisfying $B_k U_j^k = \beta_j^k U_j^k$ for $1 \leq j \leq l$. Then a complete set of eigenvectors $\{W_1, W_2, \dots, W_{nl}\}$ for the square matrix*

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1l} \\ A_{21} & A_{22} & \dots & A_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{l1} & A_{l2} & \dots & A_{ll} \end{pmatrix}$$

is given by $W_{(k-1)l+j} = U_j^k \otimes V_k$, for $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, l$. The corresponding eigenvalues are $\lambda_{(k-1)l+j} = \beta_j^k$.

In continuing of this paper, we use above results to compute the spectrum of Cayley graphs. First, we compute the spectrum of a group of order $8n$ denoted by V_{8n} with the following presentation:

$$V_{8n} = \langle a, b, a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle,$$

where n is an odd number. By using the relations of this group, we conclude that this group has exactly $8n$ elements as follows:

$$\{b^r a^j, 1 \leq r \leq 4, 1 \leq j \leq 2n\}. \tag{1}$$

It is clear that in a Cayley graph the vertex 1 correspond to identity is adjacent with all elements of S .

Theorem 4. *The adjacency matrix of the Cayley graph $\Gamma = \text{Cay}(V_{8n}, S)$ where $S = \{ba^r, ba^s, b^3a^k\}$ ($1 \leq i \leq 2n - 1$) is as follows:*

- a^i is adjacent with

$$\begin{cases} ba^{r-i}, ba^{s-i}, b^3a^{t-i} & 2 \mid i \\ ba^{t-i}, b^3a^{s-i}, b^3a^{r-i} & 2 \nmid i \end{cases}$$

- ba^i is adjacent with

$$\begin{cases} a^{t-i}, b^2a^{s-i}, b^2a^{r-i} & 2 \mid i \\ b^2a^{t-i}, a^{s-i}, a^{r-i} & 2 \nmid i \end{cases},$$

- b^2a^i is adjacent with

$$\begin{cases} ba^{t-i}, b^3a^{s-i}, b^3a^{r-i} & 2 \mid i \\ ba^{r-i}, ba^{s-i}, b^3a^{t-i} & 2 \nmid i \end{cases},$$

- b^3a^i is adjacent with

$$\begin{cases} b^2a^{t-i}, a^{s-i}, a^{r-i} & 2 \mid i \\ a^{t-i}, b^2a^{s-i}, b^2a^{r-i} & 2 \nmid i \end{cases},$$

Proof. The proof is straightforward. □

Corollary 1. *Let G be a group V_{8n} . The adjacency matrix of the cubic Cayley graph $\Gamma = \text{Cay}(G, S)$, where $S = \{ba^r, ba^s, b^3a^k\}$ is the following circulant matrix*

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

where $A = [[0, B, 0, C]] = B \otimes C + E \otimes F$ and B, C, E, F are the following circulant matrices:

$$\begin{aligned} B &= [[0, \dots, 0, \overbrace{1}^r, 0, \dots, 0, \overbrace{1}^s, 0, \dots, 0]], \\ C &= [[0, 1, 0, 0]], \\ E &= [[0, \dots, 0, \overbrace{1}^t, 0, \dots, 0]], \\ F &= [[0, 0, 0, 1]]. \end{aligned}$$

Theorem 5. *Suppose $\omega = e^{\frac{\pi}{n}i}$, then*

- 1) *All eigenvalues of matrix B are $\lambda_j = \omega^{(r-1)j} + \omega^{(s-1)j}$ in which $j = 0, \dots, 2n - 1$,*
- 2) *The eigenvalues of C and F are $\pm i, \pm 1$,*
- 3) *The eigenvalues of E are $\lambda_j = \omega^j$ where $j = 0, \dots, 2n - 1$.*

One can apply Theorem 5 to compute all eigenvalues of Cayley graph $\Gamma = \text{Cay}(V_{8n}, S)$. For example, in the following example, we compute all eigenvalues on Cayley graph $\text{Cay}(V_{24}, S)$.

Example 1. Consider the tetravalent Cayley graph $\Gamma = \text{Cay}(V_{24}, S)$, where $S = \{ba, ba^5, b^3a^5\}$. The adjacency matrix Γ is

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

where $A = [[0, B, 0, C]]$, $B = [[1, 0, 0, 0, 1]]$ and $C = [[0, 1, 0, 0]]$. By using Theorem 5, all eigenvalues are:

$$\{-3^2, -\sqrt{3}^4, -1^6, 1^6, \sqrt{3}^4, 3^2\}.$$

2. Main results and discussions

A general linear group $GL(V)$ of vector space V is the set of all $A \in \text{End}(V)$, where A is invertible. A representation of a group G is a homomorphism $\alpha : G \rightarrow GL(V)$ and the degree of α is equal to the dimension of V . A trivial representation is a homomorphism $\alpha : G \rightarrow \mathbb{C}^*$, where $\alpha(g) = 1$, for all $g \in G$. Let $\varphi : G \rightarrow GL(V)$ be a representation with $\varphi(g) = \varphi_g$, the character $\chi_\varphi : G \rightarrow \mathbb{C}$ of φ is defined as $\chi_\varphi(g) = \text{tr}(\varphi_g)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1) = 1$. We denote the set of all irreducible characters of G by $\text{Irr}(G)$.

A character table is a matrix whose rows and columns correspond to the irreducible characters and the conjugacy classes of G , respectively.

Proposition 1 ([3]). *Let G and H be two finite groups with character tables $\mathcal{M}(G)$ and $\mathcal{M}(H)$, respectively. Then the character table of direct product group $G \times H$ is*

$$\mathcal{M}(G \times H) = \mathcal{M}(G) \otimes \mathcal{M}(H).$$

The study of spectrum of Cayley graphs is closely related to irreducible characters of G . If G is abelian, then the spectrum of $X = \text{Cay}(G, S)$ can easily be determined as follows.

Theorem 6 ([21]). *Let S be a symmetric subset of abelian group G . Then the eigenvalues of adjacency matrix of $\text{Cay}(G, S)$ are given by*

$$\lambda_\varphi = \sum_{s \in S} \varphi(s),$$

where $\varphi \in \text{Irr}(G)$.

Let G be a finite group with symmetric subset S . We recall that S is a normal subset if and only if $S^g = g^{-1}Sg = S$, for all $g \in G$. The following theorem is implicitly contained in [7, 21].

Theorem 7. *Let G be a finite group with a normal symmetric subset S . Let A be the adjacency matrix of the graph $X = \text{Cay}(G, S)$. Then the eigenvalues of A are given by $[\lambda_\varphi]^{\varphi(1)^2}$, where $\lambda_\varphi = \frac{1}{\varphi(1)} \sum_{s \in S} \varphi(s)$ and $\varphi \in \text{Irr}(G)$.*

Example 2. Consider the dihedral group D_8 with the following presentation

$$D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

The character table of dihedral group D_8 is reported in Table 1. Let $S = \{a, a^{-1}\}$, then by using Theorem 7, all eigenvalues of D_8 are:

$$\lambda_{\chi_1} = \lambda_{\chi_3} = 2, \lambda_{\chi_2} = \lambda_{\chi_4} = -2 \text{ and } \lambda_{\chi_5} = 0.$$

In other words, the spectrum of Cayley graph on group D_8 , where $S = \{a, a^{-1}\}$ is as follows:

$$\{[-2]^2, [0], [2]^2\}.$$

| $\mathcal{M}(D_8)$ | 1 | g_1 | g_2 | g_3 | g_4 |
|--------------------|---|-------|-------|-------|-------|
| χ_1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 | 1 | -1 |
| χ_3 | 1 | 1 | -1 | 1 | -1 |
| χ_4 | 1 | -1 | -1 | 1 | 1 |
| ψ_5 | 2 | 0 | 0 | -2 | 0 |

TABLE 1. The character table of D_8 .

3. Spectra of Cayley graphs of order pqr

Given two groups G, H and a group homomorphism $\varphi : H \rightarrow \text{Aut}(G)$, the semi-direct product of G and H with respect to φ denoted $G \rtimes_\varphi H$ (or, simply, $G \rtimes H$) is a new group with set $G \times H$ and multiplication operation $(g_1; h_1)(g_2; h_2) = (g'_1\varphi(h_1)g_2, h_1h_2)$. The aim of this section is to compute the spectrum of Cayley graphs of order pqr by means of semi-direct product. To do this, we first introduce the presentation of these group.

A Frobenius group of order pq where p is prime and $q|p-1$ is a group of order pq by the following presentation:

$$F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u \rangle,$$

where u is an element of order q in multiplicative group \mathbb{Z}_p^* . Let G be a group of order pqr , where $p \geq q > r$ are prime numbers. It is easy to see that the Sylow p -subgroup P of G is normal. This means that G has the following structure:

$$G \cong \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{qr} \text{ or } F_{q,r}.$$

By using the concept of semi-direct product, Hölder in [17] classified all groups of order pqr . Ghorbani and Nowroozi in [10] proved that a group of order pqr is isomorphic with one of the following groups:

- i) $p = q = r$, in this case there are five groups of order p^3 as follows:
 - $P_1 = \mathbb{Z}_{p^3}$,
 - $P_2 = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$,
 - $P_3 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$,
 - $P_4 = \mathbb{Z}_p \rtimes \mathbb{Z}_{p^2}$,
 - $P_5 = \mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.
- ii) $p > q > r$, then all groups of order pqr are
 - $G_1 = \mathbb{Z}_{pqr}$,
 - $G_2 = \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
 - $G_3 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
 - $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
 - $G_5 = F_{p,qr}(qr|p-1)$,
 - $G_{i+5} = \langle a, b, c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i} \rangle$, where $r|p-1, q-1, o(u) = r$ in \mathbb{Z}_q^* and $o(v) = r$ in \mathbb{Z}_p^* ($1 \leq i \leq r-1$).
- iii) $p < q$ and $r = p$, then all groups of order p^2q are
 - $L_1 = \mathbb{Z}_{p^2q}$,
 - $L_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
 - $L_3 = \mathbb{Z}_p \times F_{q,p}(p|q-1)$,
 - $L_4 = F_{q,p^2}(p^2|q-1)$,
 - $L_5 = \langle a, b : a^{p^2} = b^q = 1, a^{-1}ba = b^\alpha, \alpha^p \equiv 1 \pmod{q} \rangle$.
- iv) $q < p$ and $r = p$, then all groups of order p^2q , where $q|p-1$ are
 - $Q_1 = \mathbb{Z}_{p^2q}$,
 - $Q_2 = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_p$,
 - $Q_3 = \mathbb{Z}_p \times F_{p,q}(q|p-1)$,
 - $Q_4 = F_{p^2,q}(q|p^2-1)$,

- $Q_5 = \langle a, b, c : a^p = b^q = c^p = 1, ac = ca, b^{-1}ab = a^\alpha, b^{-1}cb = c^{\alpha^x}, \alpha^q \equiv 1 \pmod{p}, x = 1, \dots, q - 1 \rangle$.

The general structures of groups G_{i+5} , Q_5 and Q_6 are in terms of semi-direct product, namely $\mathbb{Z}_r \rtimes (\mathbb{Z}_p \times \mathbb{Z}_q)$. In [8] the spectra of Cayley graphs $\text{Cay}(\mathbb{Z}_m \rtimes \mathbb{Z}_n, S)$ are studied. In continuing, by using some results of representation theory, we propose the spectra of Cayley graphs of order pqr , where p, q and r are prime numbers. A class of $\mathbb{Z}_c \rtimes (\mathbb{Z}_m \times \mathbb{Z}_n)$ has a representation as follows:

$$\begin{aligned} &\mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n) \\ &= \langle x, y, z; x^m = y^n = z^c = 1, ab = ba, zxz^{-1} = x^k, zyz^{-1} = y^{k'} \rangle, \quad (2) \end{aligned}$$

where $o(k) = c$ in \mathbb{Z}_m^* , $o(k') = c$ in \mathbb{Z}_n^* and $K = (k, k')$.

An embedding map is a homomorphism $\psi : G \rightarrow GL_n(\mathbb{F})$, and we say that the representation is faithful if ψ is injective. Here, we suppose that the field \mathbb{F} is complex number \mathbb{C} . Let G is a finite group and for $g \in G$, $A_g = A(\text{Cay}(G, \{g\}))$ is the adjacency matrix of related a Cayley graph. If $S \subseteq G$, then $A_S = A(\text{Cay}(G, S))$ and according to [8, Theorem 3.2], we have $A_S = \sum_{s \in S} A_s$.

Theorem 8 ([8]). *Given a group G and an element $g \in G$, consider the set $\Gamma = \{A_g | g \in G\}$ and the map $\psi : G \rightarrow \Gamma$ given by $\psi(g) = A_g$. Then, ψ gives a faithful representation for G in $GL_{|G|}(\mathbb{Q})$.*

The adjacency representation of group G is a representation given by ψ . This is called the regular representation in the literature [24].

Definition 1. Let C_h be the $h \times h$ matrix with entries

$$c_{ij} = \begin{cases} 1, & j - i \equiv 1 \pmod{h} \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is a circulant matrix or cyclic, since it is the adjacency matrix of Cayley graph \mathbb{Z}_h with $S = \{x\}$ where $x^h = id$.

Definition 2. Suppose m, n and k satisfies in relation $m^k \equiv 1 \pmod{n}$. Let Ω_h be the $m \times m$ matrix with entries

$$\Omega_{ij} = \begin{cases} \omega^{hk^{i-1}} & \text{when } (i = j), \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega = e^{\frac{2\pi i}{n}}$ is a primitive n -th root of unity. Clearly, for $a \in \mathbb{Z}$, $(\Omega_h)^a = \Omega_{ha}$. On the other hand, the matrix $(C_h)^l$ is given by

$$c_{ij} = \begin{cases} 1, & j - i \equiv l \pmod{h} \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $m^k \equiv 1 \pmod{c}$ and $n^{k'} \equiv 1 \pmod{c}$. Suppose X, Y and Z are three $mn \times mn$ block matrices with the following blocks with $c \times c$ order:

$$X_{c \times c} = \begin{pmatrix} \Omega_1 & & 0 \\ & \ddots & \\ 0 & & \Omega_{mn} \end{pmatrix}, \quad Y_{c \times c} = \begin{pmatrix} \Omega'_1 & & 0 \\ & \ddots & \\ 0 & & \Omega'_{mn} \end{pmatrix},$$

$$\text{and } Z_{c \times c} = \begin{pmatrix} C_c & & 0 \\ & \ddots & \\ 0 & & C_c \end{pmatrix}.$$

It is simple to show $X^n = I, Y^m = I$ and $Z^c = I$. If x, y and z are generators of $\mathbb{Z}_m, \mathbb{Z}_n$ and \mathbb{Z}_c , respectively then we have:

Theorem 9. *Let G be a finite group and X, Y, Z be block matrices of order $mn \times mn$ with blocks of order $c \times c$. Three matrices X, Y and Z are generators of a faithful representation of $G = \mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n)$, where $K = (k, k')$. Then G has a representation in (2), φ is injection and $\varphi(x^a y^b z^d) = X^a Y^b Z^d$.*

Proof. See [8, Theorem 3.4]. □

The representation φ with conditions of Theorem 8 or 9 is called a natural representation.

Theorem 10. *The natural representation and adjacency representation of $\mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n)$, where $K = (k, k')$ are isomorphic.*

Proof. We show $\text{tr}(A_{x^a y^b z^d}) = \text{tr}(X^a Y^b Z^d)$. $x^a y^b z^d = id$ iff $X^a Y^b Z^d = I$ and so $\text{tr}(A_{id}) = \text{tr}(I) = mnc$. Let $x^a y^b z^d \neq id$. The adjacency representation of this element is $A_{x^a y^b z^d}$. The Cayley graph $\text{Cay}(\mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n), \{x^a y^b z^d\})$ is simple and so there is no a non-zero element in the main diagonal. This means that $\text{tr}(A_{x^a y^b z^d}) = 0$. The natural representation of $x^a y^b z^d$ is $X^a Y^b Z^d$, where

$$(X^a Y^b Z^d)_{ij} = \begin{cases} (\Omega_i)^a (\Omega'_j)^b (C_c)^d, & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If $d \nmid c$, all diagonal elements in the block matrix $X^a Y^b Z^d$ are zero and then $\text{tr}(A_{x^a y^b z^d}) = 0$. Let $d \mid c$, if $a \nmid n$ and $b \nmid m$ then the block $\Omega_{ca} \Omega'_{cb}$ is

$$\text{tr}(\Omega_{ca} \Omega'_{cb}) = \omega^a \cdot \omega'^b (1 + \omega^k \cdot \omega'^{k'} + \dots + \omega^{k^{c-1}} \cdot \omega'^{k'^{c-1}}).$$

Then, for matrix $X^a Y^b$ we yield

$$\begin{aligned} \text{tr}(X^a Y^b) &= \omega^a \cdot \omega'^b \alpha + \omega^{2a} \cdot \omega'^{2b} \alpha + \dots + \omega^{mna} \cdot \omega'^{mnb} \alpha \\ &= (\omega^a \cdot \omega'^b + \omega^{2a} \cdot \omega'^{2b} + \dots + \omega^{mna} \cdot \omega'^{mnb}) \alpha \\ &= \omega^a \cdot \omega'^b \frac{(\omega^{mna} \cdot \omega'^{mnb} - 1)}{\omega^a \cdot \omega'^b - 1} \alpha, \end{aligned}$$

where $\alpha = (1 + \omega^k \cdot \omega'^{k'} + \dots + \omega^{k^{c-1}} \cdot \omega'^{k'^{c-1}})$. So $\text{tr}(X^a Y^b) = 0$ and these two group representations are isomorphic. \square

Theorem 11. *The characteristic polynomial of Cayley graph $\text{Cay}(G, S)$, where G is as represented in Eq. (2), is as follows:*

$$\chi(A(\mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n), S)) = \prod_{i=0}^{mn-1} \chi\left(\sum_{x^a y^b z^d} \Omega_{ia} \Omega'_{ib} (C_c)^d\right).$$

Proof. Let $G = \mathbb{Z}_c \rtimes_K (\mathbb{Z}_m \times \mathbb{Z}_n)$, where $K = (k, k')$ and

$$\begin{aligned} &\mathbb{Z}_c \rtimes (\mathbb{Z}_m \times \mathbb{Z}_n) \\ &\langle x, y, z; x^m = y^n = z^c = 1, ab = ba, zxz^{-1} = x^k, zyz^{-1} = y^{k'} \rangle, \end{aligned}$$

where $o(k) = c$ in \mathbb{Z}_m^* and $o(k') = c$ in \mathbb{Z}_n^* . It yields that

$$\chi(A(\text{Cay}(G, S))) = \chi\left(\sum_{s \in S} A(\text{Cay}(G, \{s\}))\right) = \chi\left(\sum_{s \in S} A_s\right).$$

The element $s \in G$ can be written uniquely as $x^a y^b z^d$, where $0 \leq a < n$, $0 \leq b < m$ and $0 \leq d < c$. Then

$$\chi\left(\sum_{x^a y^b z^d} A_{x^a y^b z^d}\right) = \chi\left(\sum_{x^a y^b z^d} X^a Y^b Z^d\right).$$

Therefore

$$\prod_{i=0}^{mn-1} \chi\left(\sum_{x^a y^b z^d \in S} (\Omega_i)^a (\Omega'_i)^b (C_c)^d\right) = \prod_{i=0}^{mn-1} \chi\left(\sum_{x^a y^b z^d \in S} (\Omega_{ia})(\Omega'_{ib})(C_{cd})\right). \square$$

By this formula, we can determine the characteristic polynomials of Cayley graphs of groups G_{i+5} , L_5 , P_5 and Q_5 .

Corollary 2. *For the Frobenius group $F_{p,q}$, we have*

$$\chi(A(\mathbb{Z}_q \rtimes_k \mathbb{Z}_p, S)) = \prod_{i=0}^{q-1} \chi\left(\sum_{x^a y^b} \Omega_{ia}(C_q)^b\right).$$

This result gives characteristic polynomials of $\text{Cay}(G, S)$, where $G \in \{G_5, L_4, P_4, Q_4\}$. The spectrum of Cayley graphs of cyclic groups was computed in [20] and this subject is a corollary of Theorem 11.

Proposition 2 ([20]). *All eigenvalues of $\text{Cay}(\mathbb{Z}_n; S)$ are given by*

$$\left\{ \lambda \mid \lambda = \sum_{s \in S} \omega^{xs}, x \in \mathbb{Z}_n, 1 \leq x \leq n \right\}.$$

Hence, the characteristic polynomials of Cayley graphs of groups G_1 , L_1 , P_1 and Q_1 are as follows:

Corollary 3. *Let x be a generator of cyclic group \mathbb{Z}_n and y be a generator of \mathbb{Z}_m . The eigenvalues of $\text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_m, S)$, where $S = \{x, y\}$ are*

$$\{\lambda \mid \lambda = \omega_n^i + \omega_m^j, 0 \leq i < n, 0 \leq j < m\}.$$

By using Corollary 3, we can determine the spectra of Cayley graphs of groups L_2 , P_2 , P_3 and Q_2 .

Let $\Gamma_1 = \text{Cay}(\mathbb{Z}_c, S_1)$ and $\Gamma_2 = \text{Cay}(\mathbb{Z}_n \rtimes \mathbb{Z}_m, S_2)$ be two Cayley graphs, then

$$\Gamma_1 \square \Gamma_2 \cong \text{Cay}(\mathbb{Z}_c \times (\mathbb{Z}_n \rtimes \mathbb{Z}_m), T),$$

where $T = \{(s, 1)(1, t) \mid s \in S_1, t \in S_2\}$.

Corollary 4.

$$\chi(\text{Cay}(\Gamma_1 \times \Gamma_2, T)) = \chi(\text{Cay}(\Gamma_1, S_1)) \cdot \chi(\text{Cay}(\Gamma_2, S_2)).$$

By using Corollary 4 one can find the characteristic polynomial of Cayley graph $\text{Cay}(G, S)$, where $G \in \{G_2, G_3, G_4, L_3, Q_3\}$.

Example 3. Let

$$P_4 = \langle a, b : a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle \quad \text{and} \quad S = \{a, a^1, b, b^{-1}\},$$

then the adjacency matrix of $\text{Cay}(P_4, S)$ is a $p \times p$ block matrix as following form:

$$A = [L_{p^2}, I_{p^2}, \overbrace{0, \dots, 0}^{p-3}, I_{p^2}] = L_{p^2} \otimes I_p + I_{p^2} \otimes [0, 1, 0, \dots, 0, 1],$$

where $L_{p^2} = [0, 1, 0, \dots, 0, 1]$. Then all eigenvalues of this matrix are $2 \cos(\frac{2k\pi}{p^2}) + 2 \cos(\frac{2k'\pi}{p})$, where $1 \leq k \leq p^2$ and $1 \leq k' \leq p$.

Example 4. Let

$$P_5 = \langle a, b, c : a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$$

and

$$S = \{a, a^1, b, b^{-1}\},$$

then the adjacency matrix of $\text{Cay}(P_5, S)$ is a $p^2 \times p^2$ block matrix, where

$$A = [L_p, I, \overbrace{0, \dots, 0}^{p-3}, I, \overbrace{0, \dots, 0}^{p^2-p}] = L_p \otimes I_{p^2} + I_p \otimes [0, 1, 0, \dots, 0, 1]_{p^2},$$

and $L = [0, 1, 0, \dots, 0, 1]$. So all eigenvalues of this matrix are $\lambda_k + \mu_{k'}$, where $1 \leq k \leq p$, $1 \leq k' \leq p^2$ and

$$\lambda_k = e^{\frac{2\pi ik}{p}} + e^{-\frac{2\pi ik}{p}}, \quad 1 \leq k \leq p, \quad \mu_{k'} = e^{\frac{2\pi ik'}{p}} + e^{-\frac{2\pi ik'}{p}}, \quad 1 \leq k' \leq p^2.$$

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Received by the editors: 03.10.2017.