

## Exact sequences of graphs

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**ABSTRACT.** In this paper, exact sequences of graphs are defined and investigated. Considering some functors on the category of graphs, we study some conditions to determine exactness of functors.

### Introduction

In [6], we have introduced the torsion-unitary Cayley graph of modules and study the exact sequence of Cayley graphs. In this paper, we define an exact sequence of graphs in the category of graphs. Inspired by some homological algebra, we introduce some homological graph theory.

The null graph is the unique graph having no vertices and the complete graph on  $n$  vertices is denoted by  $K_n$ . In particular, let  $K_0$  and  $K_1^\circ$  be the null graph and the singleton graph with a loop, respectively. In section 1, considering the *quasi – kernel* and *unfaithful* sets of graph homomorphisms, we introduce an exact sequence of graphs and graph homomorphisms in the form  $K_0 \rightarrow \Gamma_1 \xrightarrow{\varphi} \Gamma_2 \xrightarrow{\psi} \Gamma_3 \rightarrow K_1^\circ$ . In order to extend these notions to more general cases, we introduce two other exact sequences as the first and the second kind which the first kind exact sequences could be used in the category of sets. We state a version of the Short Five Lemma in the category of graphs (see Theorem 2). Moreover, by studying special diagrams of graphs and homomorphisms regarding

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short exact sequences, we determine properties of  $\Gamma_2$  in view of  $\Gamma_1$  and  $\Gamma_3$  (see Theorem 3 and 4).

In section 2, using the notions of Cartesian product, direct product and map graph, we consider some induced functors and some conditions to determine exactness of them.

Throughout the paper, all graphs are undirected and do not have multiple edges, but may have loops, also they may be infinite. Let  $\Gamma$  be a graph with the vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . Two (induced) subgraphs  $\Gamma_1$  and  $\Gamma_2$  of  $\Gamma$  are said to be disjoint if  $\Gamma_1$  and  $\Gamma_2$  have no common vertices and no vertex of  $\Gamma_1$  is adjacent (in  $\Gamma$ ) to any vertex of  $\Gamma_2$ . Let  $[x, x'] \in E(\Gamma)$ , then  $x$  and  $x'$  are adjacent, denoted by  $x \sim x'$ , also if they are not adjacent, denoted by  $x \approx x'$ .

Let  $\Gamma$  and  $\Upsilon$  be graphs. A function  $\phi : V(\Gamma) \rightarrow V(\Upsilon)$  is a homomorphism from  $\Gamma$  to  $\Upsilon$  if it preserves edges, that is, if for any edge  $[x, x']$  of  $\Gamma$ ,  $[\phi(x), \phi(x')]$  is an edge of  $\Upsilon$ . A homomorphism  $\phi : \Gamma \rightarrow \Upsilon$  is called *faithful* if  $\phi(\Gamma)$  is an induced subgraph of  $\Upsilon$ . A homomorphism will be called *strong* whenever  $[x, x'] \in E(\Gamma)$  if and only if  $[\phi(x), \phi(x')] \in E(\Upsilon)$ . A surjective homomorphism (on vertices) is often called an epimorphism, an injective one (on vertices) is called a monomorphism and a bijective homomorphism is sometimes called a bimorphism. In other words, a homomorphism  $\phi : \Gamma \rightarrow \Upsilon$  is faithful when there is an edge between any two pre-images  $\phi^{-1}(u)$  and  $\phi^{-1}(u')$  such that  $[u, u']$  is an edge of  $\Upsilon$ . When a faithful homomorphism  $\phi$  is bijective, it is strong since each  $\phi^{-1}(u)$  is a singleton, and so  $[\phi^{-1}(u), \phi^{-1}(u')]$  is an edge in  $\Gamma$  if and only if  $[u, u']$  is an edge of  $\Upsilon$ . Thus a faithful bimorphism is an isomorphism and in this case we write  $\Gamma \cong \Upsilon$ . Note that unlike in group theory, the inverse of a bimorphism of graph need not be a homomorphism. For example, any bimorphism from the complement of  $K_n$  to  $K_n$ . For more information on graph homomorphisms, we refer the reader to [2].

Let  $\mathcal{P} = \{V_1, \dots, V_k\}$  be a partition of the vertex set of  $\Gamma$  into non-empty classes. The quotient  $\Gamma/\mathcal{P}$  of  $\Gamma$  by  $\mathcal{P}$  is the graph whose vertices are the sets  $V_1, \dots, V_k$  and whose edges are the pairs  $[V_i, V_j]$  such that there are  $u_i \in V_i, u_j \in V_j$  with  $[u_i, u_j] \in E(\Gamma)$ . The mapping  $\pi_{\mathcal{P}} : V(\Gamma) \rightarrow V(\Gamma/\mathcal{P})$  defined by  $\pi_{\mathcal{P}}(u) = V_i$  such that  $u \in V_i$ , is the natural map for  $\mathcal{P}$ . Quotients often provide a way of deriving the structure of an object from the structure of a larger one. Observe that  $\pi_{\mathcal{P}}$  is a homomorphism and it is automatically faithful. A graph  $\Gamma$  is called bipartite if its vertex set can be represented as the union of two disjoint sets  $V_1$  and  $V_2$ , such that every edge of  $\Gamma$  connects a vertex of  $V_1$  to one of  $V_2$ . In these circumstances, we call  $V_1, V_2$  a bipartition of  $V(\Gamma)$ .

The *empty graph*  $\Gamma$  is a graph with  $E(\Gamma) = \emptyset$ . The coproduct or sum of two graphs  $\Gamma_1$  and  $\Gamma_2$ , denoted  $\Gamma_1 + \Gamma_2$ , is a disjoint union with the vertex set  $V(\Gamma_1) \cup V(\Gamma_2)$  and the edge set  $E(\Gamma_1) \cup E(\Gamma_2)$ . The inclusion maps  $i_1 : \Gamma_1 \hookrightarrow \Gamma_1 + \Gamma_2$  and  $i_2 : \Gamma_2 \hookrightarrow \Gamma_1 + \Gamma_2$  are strong monomorphisms. Let  $\varphi_1$  be a homomorphism from  $\Gamma_1$  to  $\Gamma$  and let  $\varphi_2$  be a homomorphism from  $\Gamma_2$  to  $\Gamma$ , then the homomorphism  $\varphi_1 + \varphi_2 : \Gamma_1 + \Gamma_2 \rightarrow \Gamma$  maps vertices of  $\Gamma_1$  by  $\varphi_1$  and vertices of  $\Gamma_2$  by  $\varphi_2$ . On the other hand, every homomorphism  $\varphi : \Gamma_1 + \Gamma_2 \rightarrow \Gamma$  is a natural sum of homomorphisms  $\varphi_1 = \varphi|_{\Gamma_1}$  and  $\varphi_2 = \varphi|_{\Gamma_2}$ . Also,  $\sum_{i=1}^k \Gamma_i$  is denoted by  $k\Gamma$ . If  $\Gamma_1, \Gamma_2, \dots, \Gamma_k$  are graphs, then their *Cartesian product* is the graph, denoted by  $\Gamma_1 \square \Gamma_2 \square \dots \square \Gamma_k$  with vertex set  $\{(x_1, x_2, \dots, x_k) \mid x_i \in V(\Gamma_i)\}$  which two vertices  $(x_1, x_2, \dots, x_k)$  and  $(x'_1, x'_2, \dots, x'_k)$  are adjacent whenever  $[x_i, x'_i] \in E(\Gamma_i)$  for exactly one index  $1 \leq i \leq k$ , and  $x_j = x'_j$  for each index  $j \neq i$ . The *direct product* of  $\Gamma$  and  $\Upsilon$  is a graph, denoted by  $\Gamma \times \Upsilon$ , with vertex set  $V(\Gamma) \times V(\Upsilon)$ , such that vertices  $(x, y)$  and  $(x', y')$  are adjacent precisely if  $[x, x'] \in E(\Gamma)$  and  $[y, y'] \in E(\Upsilon)$ . Other names for the direct product that have appeared in the literature are *tensor product*, *Kronecker product* or *categorical product*. We know that the Cartesian product and direct product are commutative and associative up to isomorphism.

The map (exponential) graph  $\Gamma^\Upsilon$  has the set of functions from  $V(\Upsilon)$  to  $V(\Gamma)$  as its vertices; two such functions  $f$  and  $g$  are adjacent in  $\Gamma^\Upsilon$  if and only if whenever  $u$  and  $y$  are adjacent in  $\Upsilon$ , the vertices  $f(u)$  and  $g(y)$  are adjacent in  $\Gamma$ . A vertex in  $\Gamma^\Upsilon$  has a loop if and only if the corresponding function is a homomorphism. Suppose that  $\varphi$  is a homomorphism from  $\Gamma_1$  to  $\Gamma_2$ . If  $f$  is a function from  $V(\Upsilon)$  to  $V(\Gamma_1)$ , then the composition  $\varphi f$  is a function from  $V(\Upsilon)$  to  $V(\Gamma_2)$ . Hence  $\varphi$  determines a map from the vertices of  $\Gamma_1^\Upsilon$  to  $\Gamma_2^\Upsilon$  which is denoted by  $\hat{\varphi}$ . Also, if  $g$  is a function from  $V(\Gamma_2)$  to  $V(\Upsilon)$ , then the composition  $g\varphi$  is a function from  $V(\Gamma_1)$  to  $V(\Upsilon)$ . Hence  $\varphi$  determines a map from the vertices of  $\Upsilon^{\Gamma_2}$  to  $\Upsilon^{\Gamma_1}$  which is denoted by  $\check{\varphi}$ . For more information on the map graph and the other graph constructions, we refer the reader to [1] and [3–5].

Two distinct elements  $x, y \in V(\Gamma)$  is denoted by  $x \neq y$ . Consider a graph homomorphism  $\phi : \Gamma \rightarrow \Upsilon$ . Let  $\Gamma'$  and  $\Upsilon'$  be graphs where  $V(\Gamma) = V(\Gamma')$  and  $V(\Upsilon) = V(\Upsilon')$ . Our mean of  $\phi' : \Gamma' \rightarrow \Upsilon'$  is a graph homomorphism such that  $\phi(x) = \phi'(x)$  for all  $x \in V(\Gamma) = V(\Gamma')$ . Let  $A \subseteq V(\Gamma)$ , then  ${}_A\Gamma$  is a subgraph of  $\Gamma$  induced by  $V(\Gamma) \setminus A$  and a subgraph of  $\Upsilon$  induced by  $V(\Upsilon) \setminus \text{Im}(\phi)$  is denoted by  ${}_\phi\Upsilon$ . Let  $\Gamma_1$  be a subgraph of  $\Gamma$ , then  $\phi|_{\Gamma_1}$  will denote the restriction of  $\phi$  to  $\Gamma_1$ . The neighborhood of  $A$ , denoted by  $N(A)$ , is the set of all vertices outside of  $A$  adjacent to at

least one member of  $A$ . Moreover, let  $\Gamma_1$  be a subgraph of  $\Gamma$ , then  $N(\Gamma_1)$  denotes the subgraph induced by  $N(V(\Gamma_1))$ . The natural monomorphism from  $\Gamma_1$  to  $\Gamma$  is denoted by  $j$ . If  $\Gamma_1$  is an induced subgraph, then  $j$  is an inclusion map. In these circumstances, we denote  $j$  by  $i$ .

## 1. Exact sequences of graphs

In this section, we define an exact sequence of graphs and we obtain some of its properties. Before the original definition, we need to define some sets.

**Definition 1.** Let  $X$  and  $Y$  be a pair of sets and  $f : X \rightarrow Y$  be a function. *Quasi-kernel* of  $f$ , denoted by  $K(f)$ , is defined as  $K(f) = \bigcup \{f^{-1}(y) \mid |f^{-1}(y)| \geq 2 \text{ for } y \in Y\}$ .

**Definition 2.** Let  $\Gamma$  be a graph,  $A \subseteq V(\Gamma)$  and let  $\mathcal{A} = \{A, \{x_{\lambda_1}\}, \{x_{\lambda_2}\}, \dots\}$  as a partition of  $V(\Gamma)$  for all  $x_{\lambda_i} \in V(\Gamma) \setminus A$  with  $\lambda_i \in \Lambda$ . In the following manner the quotient  $\Gamma/\mathcal{A}$  is a graph induced by  $A$ . We contract  $A$  as a vertex of  $\Gamma/\mathcal{A}$  with a loop. In particular, if  $\phi : \Gamma \rightarrow \Upsilon$  is a graph homomorphism, then the partition  $\{\text{Im}(\phi), \{u_{\lambda_1}\}, \{u_{\lambda_2}\}, \dots\}$  of  $\Upsilon$  is denoted by  $\mathcal{I}_\phi$  for all  $u_{\lambda_i} \in V(\Upsilon) \setminus \text{Im}(\phi)$  with  $\lambda_i \in \Lambda$ . This partition induces a quotient graph  $\Upsilon/\mathcal{I}_\phi$ . The induced vertex of  $\text{Im}(\phi)$  in the quotient is denoted by  $\mathfrak{I}_\phi^\circ$ .

**Definition 3.** Suppose that  $A \subseteq V(\Gamma)$  and the vertex induced by  $A$  as a vertex of  $\Gamma/\mathcal{A}$  is denoted by  $\mathfrak{A}^\circ$ . If  $\mathfrak{A}^\circ$  is not isolated vertex of  $\Gamma/\mathcal{A}$ , then  $\Gamma|\Gamma_A$  is defined to be the graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma) \cup E(N)$  where  $N$  is a complete bipartite graph constructed with  $A$  and  $N(A)$  as a bipartition. Otherwise, graph  $\Gamma|\Gamma_A = \Gamma$ .

**Remark 1.** (a) Let  $\Gamma$  and  $\Upsilon$  be a pair of graphs and let  $\phi : \Gamma \rightarrow \Upsilon$  be a graph homomorphism.

(i) The homomorphism  $\phi$  is injective if and only if  $K(\phi) = \emptyset$ . Anyway,  $|K(\phi)| \neq 1$ .

(ii) The restriction of  $\phi$  to  $K(\phi)\Gamma$  is a monomorphism and there is an induced monomorphism  $\phi : \Gamma/\mathcal{K}_\phi \rightarrow \Upsilon/\mathcal{I}_{\phi|_K}$ .

(b) Let  $A \subseteq V(\Gamma)$ , then  $j : \Gamma \rightarrow \Gamma|\Gamma_A$  is a monomorphism where  $j$  is the inclusion map from  $V(\Gamma)$  to  $V(\Gamma|\Gamma_A)$ . If the subgraph induced by  $A$  is complete, then epimorphism  $\pi'_A : \Gamma|\Gamma_A \rightarrow \Gamma/\mathcal{A}$  is strong.

**Definition 4.** Let  $\phi : \Gamma \rightarrow \Upsilon$  be a graph homomorphism. *Unfaithful set* of  $\phi$ , denoted by  $F(\phi)$ , is defined as  $F(\phi) = \{x \in V(\Gamma) \mid x \approx x', \text{ if } \phi(x) \sim \phi(x') \text{ for some } x' \in V(\Gamma)\}$ .

**Definition 5.** Let  $\phi : \Gamma \rightarrow \Upsilon$  be a graph homomorphism. A full graph of  $\Gamma$  induced by  $\phi$ , denoted by  $\Gamma|_{F_\phi}$ , is defined to be the graph with vertex set  $V(\Gamma)$  and  $x \sim x'$  in  $\Gamma|_{F_\phi}$  if and only if  $\phi(x) \sim \phi(x')$  in  $\Upsilon$ .

**Remark 2.** Let  $\phi : \Gamma \rightarrow \Upsilon$  be a graph homomorphism.

(a) The homomorphism  $\phi$  is strong if and only if  $F(\phi) = \emptyset$ . Moreover,  $K(\phi) \cup F(\phi) = \emptyset$  if and only if  $\phi$  is a strong monomorphism of graphs. Moreover, if  $|K(\phi) \cup F(\phi)| = 1$ , then  $\phi$  is a monomorphism and there is a unique vertex without loop in  $\Gamma$  which maps to a vertex with loop. Hereinafter, the set  $K(\phi) \cup F(\phi)$  is denoted by  $Q(\phi)$ .

(b) The restriction of  $\phi$  to  $Q(\phi)\Gamma$  is a strong monomorphism. In particular, if  $\phi$  is an epimorphism, then  $\Gamma/Q_\phi \cong \Upsilon/\mathcal{I}_{\phi|_Q}$ . Moreover, if  $x \approx y$  in  $\Gamma$  where  $x \notin F(\phi)$  and  $y \in F(\phi)$ , then  $\phi(x) \approx \phi(y)$  (if  $\phi(x) \sim \phi(y)$ , then  $x \in F(\phi)$ , a contradiction).

(c) According to Definitions 4 and 5, the definition of full graph  $\Gamma|_{F_\phi}$  is equivalent to that  $x \sim x'$  in  $\Gamma|_{F_\phi}$  whenever  $\phi(x) \sim \phi(x')$  for  $x, x' \in F(\phi)$ . Therefore,  $\phi_F : \Gamma|_{F_\phi} \rightarrow \Upsilon$  is a strong homomorphism and  $Q(\phi_F) = K(\phi_F) = K(\phi)$ . Moreover, let  $j : \Gamma \rightarrow \Gamma|_{F_\phi}$ , then  $j$  is a monomorphism and  $Q(j) = F(j) = F(\phi)$ .

Now by the above preliminaries, we are ready to define the exact sequence of graph.

**Definition 6.** Suppose that  $\{\Gamma_i\}$  is a family of graphs and  $\{\varphi_i\}$  is a family of graph homomorphisms, where  $i \in \mathbb{Z}$ . A sequence of graphs and homomorphisms

$$\cdots \rightarrow \Gamma_{i-1} \xrightarrow{\varphi_{i-1}} \Gamma_i \xrightarrow{\varphi_i} \Gamma_{i+1} \rightarrow \cdots, \quad (1)$$

is called exact whenever  $|\text{Im}(\varphi_{i-1})| = 1$ , then either  $Q(\varphi_i) = \emptyset$  or  $\text{Im}(\varphi_{i-1}) = Q(\varphi_i)$ , otherwise  $\text{Im}(\varphi_{i-1}) = Q(\varphi_i)$  for all  $i \in \mathbb{Z}$ . In particular, the short exact sequence of graphs is an exact sequence in the form

$$K_0 \rightarrow \Gamma_1 \xrightarrow{\varphi} \Gamma_2 \xrightarrow{\psi} \Gamma_3 \rightarrow K_1^\circ. \quad (2)$$

where  $K_0$  and  $K_1^\circ$  denote the null graph and a graph which has one vertex with loop, respectively.

**Definition 7.** The sequence (1) is called *semi-exact* if  $Q(\varphi_i) \subseteq \text{Im}(\varphi_{i-1})$  and it is called a *complex* if  $\text{Im}(\varphi_{i-1}) \subseteq Q(\varphi_i)$  for all  $i \in \mathbb{Z}$ . Also, the sequence (1) is called *exact of the first kind* whenever  $|\text{Im}(\varphi_{i-1})| = 1$ , then  $K(\varphi_i) = \emptyset$  (i.e.,  $\varphi_i$  is injective), otherwise  $\text{Im}(\varphi_{i-1}) = K(\varphi_i)$  and

it is called *exact of the second kind* if  $\text{Im}(\varphi_{i-1}) = F(\varphi_i)$  for all  $i \in \mathbb{Z}$ . In particular, the short exact sequence of the second kind is an exact sequence of the second kind in the form  $K_0 \rightarrow \Gamma_1 \xrightarrow{\varphi} \Gamma_2 \xrightarrow{\psi} \Gamma_3$ , where  $\psi$  is an epimorphism.

It is necessary to observe that the concept of exact sequence of the first kind could be expressed in the category of sets which we overlook it. Let  $X$  and  $Y$  be a pair of sets and let  $f : X \rightarrow Y$  be a function. Suppose that  $A \subseteq X$ , then according to Definition 2, we can define  $X/\mathcal{A}$ ,  $Y/\mathcal{I}_f$  and  $\mathfrak{J}_f$ .

**Remark 3.** (a) Let sequence (1) be exact. Suppose that  $\Gamma_i = K_1^\circ$ ; then  $\varphi_{i-1}$  is an epimorphism, but the converse is not true in general. Also, consider the following exact sequence:

$$K_0 \rightarrow \Gamma_1 \xrightarrow{\varphi} \Gamma_2 \xrightarrow{\psi} \Gamma_3 \xrightarrow{\tau} \Gamma_4$$

where  $\psi$  is an epimorphism. Then it can be converted to a short exact sequence by replacing  $\{\pi_{\mathcal{P}_3}, K_1^\circ\}$  instead of  $\{\tau, \Gamma_4\}$  with the partition  $\mathcal{P}_3 = \{V(\Gamma_3)\}$ .

(b) In the short exact sequence (2),  $\varphi$  is a strong monomorphism by Remark 2(a). Moreover,  $\Gamma_2/\mathcal{I}_\varphi \cong \Gamma_3/\mathcal{I}_{\psi\varphi}$  by Remark 2(b). Also, if  $K(\psi) = \emptyset$ , then  $\Gamma_2|_{F_\psi} \cong \Gamma_3$ .

(c) Suppose that the sequence  $K_0 \rightarrow \Gamma \xrightarrow{\phi} \Upsilon \rightarrow K_1^\circ$  is exact. It is easy to see that  $\phi$  is an isomorphism by parts (a) and (b).

(d) Let  $\Gamma_1 \xrightarrow{\varphi_1} \Gamma_2 \xrightarrow{\varphi_2} \Gamma_3$  and  $\Gamma_3 \xrightarrow{\varphi_3} \Gamma_4 \xrightarrow{\varphi_4} \Gamma_5$  be exact sequences of graphs and homomorphisms. Obviously, the following statements hold:

- (i) The sequence  $\Gamma_1 \xrightarrow{\varphi_1} \Gamma_2 \xrightarrow{\varphi_3\varphi_2} \Gamma_4$  is exact, if  $\varphi_3$  is a strong monomorphism.
- (ii) The sequence  $\Gamma_2 \xrightarrow{\varphi_3\varphi_2} \Gamma_4 \xrightarrow{\varphi_4} \Gamma_5$  is exact, if  $\varphi_2$  is an epimorphism.

**Example 1.** (a) Consider the sequence  $K_0 \rightarrow 2K_1 \xrightarrow{\text{id}} 2K_1 \xrightarrow{j} K_2 \rightarrow K_1^\circ$ , then this sequence is exact of the second kind where  $\text{id}$  is an isomorphism and  $j$  is a bimorphism. Moreover, it is exact too.

(b) Let  $G$  be a subgraph of  $\Gamma$ , then  $K_0 \rightarrow G \xrightarrow{i} \Gamma|G \xrightarrow{\pi'_G} \Gamma/\mathcal{G} \rightarrow K_1^\circ$  is a short exact sequence. Also, the short sequence  $K_0 \rightarrow G \xrightarrow{i} \Gamma \xrightarrow{\pi_G} \Gamma/\mathcal{G} \rightarrow K_1^\circ$  is exact of the first kind.

(c) Let  $\phi : \Gamma \rightarrow \Upsilon$  be a graph homomorphism.

(i) The sequences

$$\begin{aligned} K_0 &\rightarrow Q_\phi \xrightarrow{i} \Gamma|Q_\phi \xrightarrow{\pi'_Q} \Gamma/Q_\phi \rightarrow K_1^\circ, \\ K_0 &\rightarrow \text{Im}_\phi \xrightarrow{i} \Upsilon|\text{Im}_\phi \xrightarrow{\pi'_\mathcal{I}} \Upsilon/\mathcal{I}_\phi \rightarrow K_1^\circ \end{aligned}$$

are short exact and the sequences

$$\begin{aligned} K_0 &\rightarrow Q_\phi \xrightarrow{i} \Gamma \xrightarrow{\phi} \Upsilon|\text{Im}_\phi \xrightarrow{\pi'_\mathcal{I}} \Upsilon/\mathcal{I}_\phi \rightarrow K_1^\circ, \\ K_0 &\rightarrow K_\phi \xrightarrow{i} \Gamma|_{F_\phi} \xrightarrow{\phi'_F} \Upsilon|\text{Im}_\phi \xrightarrow{\pi'_\mathcal{I}} \Upsilon/\mathcal{I}_\phi \rightarrow K_1^\circ \end{aligned}$$

are exact.

(ii) The sequences

$$\begin{aligned} K_0 &\rightarrow K_\phi \xrightarrow{i} \Gamma \xrightarrow{\pi_\mathcal{K}} \Gamma/\mathcal{K}_\phi \rightarrow K_1^\circ, \\ K_0 &\rightarrow \text{Im}_\phi \xrightarrow{i} \Upsilon \xrightarrow{\pi_\mathcal{I}} \Upsilon/\mathcal{I}_\phi \rightarrow K_1^\circ \end{aligned}$$

are short exact of the first kind and the sentence  $K_0 \rightarrow K_\phi \xrightarrow{i} \Gamma \xrightarrow{\phi} \Upsilon \xrightarrow{\pi_\mathcal{I}} \Upsilon/\mathcal{I}_\phi \rightarrow K_1^\circ$  is exact of the first kind.

(iii) The sequence  $K_0 \rightarrow F_\phi \xrightarrow{i} \Gamma \xrightarrow{j} \Gamma|_{F_\phi}$  is short exact of the second kind. Also,  $F(\phi) = Q(j)$  by Remark 2(c). Therefore,  $K_0 \rightarrow F_\phi \xrightarrow{i} \Gamma \xrightarrow{j} \Gamma|_{F_\phi} \rightarrow K_1^\circ$  is a short exact sequence.

(d) By considering the short exact sequence (2), the sequence

$$K_0 \rightarrow \Gamma_1 \xrightarrow{\varphi} \Gamma_2|\text{Im}_\varphi \xrightarrow{\psi'} \Gamma_3|\text{Im}_{\psi\varphi} \rightarrow K_1^\circ$$

is short exact.

**Theorem 1.** *Let  $\Gamma_1 \xrightarrow{\varphi_1} \Gamma_2 \xrightarrow{\varphi_2} \Gamma_3 \xrightarrow{\varphi_3} \Gamma_4$  be an exact sequence of graphs and let  $\Gamma_1 \neq \emptyset$ . If  $|Q(\varphi_3)| = 0$  or 1, then  $\varphi_1$  is an epimorphism.*

*Proof.* Suppose that  $|Q(\varphi_3)| = 1$ , then  $|\text{Im}(\varphi_2)| = 1$ . If  $|V(\Gamma_2)| = 1$ , it is clear that  $\varphi_1$  is an epimorphism since  $\Gamma_1 \neq \emptyset$ . Suppose not;  $V(\Gamma_2) = K(\varphi_2) = \text{Im}(\varphi_1)$  and so  $\varphi_1$  is an epimorphism. Let  $Q(\varphi_3) = \emptyset$ ; then  $|\text{Im}(\varphi_2)| = 0$  or 1 by the Definition 6. Let  $|\text{Im}(\varphi_2)| = 1$ ; then  $\varphi_1$  is an epimorphism, by the previous proof. If  $\text{Im}(\varphi_2) = \emptyset$ , then  $\Gamma_1 = \Gamma_2 = K_0$  and so  $\varphi_1$  is an epimorphism.  $\square$

**Definition 8.** A homomorphism of short exact sequences (of the first kind) is a triple  $\eta_1, \eta_2, \eta_3$  of graph homomorphisms such that the following diagram with exact (of the first kind) rows commutes:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ \\
 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 & & \\
 K_0 & \longrightarrow & \Gamma'_1 & \xrightarrow{\varphi'} & \Gamma'_2 & \xrightarrow{\psi'} & \Gamma'_3 & \longrightarrow & K_1^\circ
 \end{array}$$

The homomorphism is an isomorphism of short exact sequences (of the first kind) if  $\eta_1, \eta_2, \eta_3$  are all isomorphisms.

**Example 2.** Consider the short exact sequence (2). Then the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \text{Im}_\varphi & \xrightarrow{i} & \Gamma_2 | \text{Im}_\varphi & \xrightarrow{\pi_{\mathcal{I}_\varphi}} & \Gamma_2 / \mathcal{I}_\varphi & \longrightarrow & K_1^\circ \\
 & & \downarrow \varphi^{-1} & & \downarrow \text{id} & & \downarrow \bar{\psi} & & \\
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 | \text{Im}_\varphi & \xrightarrow{\bar{\psi}\pi_{\mathcal{I}_\varphi}} & \Gamma_3 / \mathcal{I}_{\psi\varphi} & \longrightarrow & K_1^\circ
 \end{array}$$

By Remark (3)(b), the induced homomorphism  $\bar{\psi}$  is an isomorphism. Therefore, this is an isomorphism of short exact sequences.

**Theorem 2.** *Let*

$$\begin{array}{ccccccccc}
 & & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ \\
 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 & & \\
 K_0 & \longrightarrow & \Gamma'_1 & \xrightarrow{\varphi'} & \Gamma'_2 & \xrightarrow{\psi'} & \Gamma'_3 & \longrightarrow & K_1^\circ
 \end{array}$$

*be a commutative diagram of graphs with semi-exact rows.*

- (1) *If  $\eta_1$  and  $\eta_3$  are monomorphism, then so is  $\eta_2$ .*
- (2) *If  $\eta_1$  and  $\eta_3$  are strong homomorphism, then so is  $\eta_2$ .*
- (3) *If  $\eta_1$  and  $\eta_3$  are epimorphism, then so is  $\eta_2$ .*
- (4) *If  $\eta_1$  and  $\eta_3$  are isomorphism, then so is  $\eta_2$ .*

*Proof.* (1) Let  $\eta_2(x_2) = \eta_2(y_2)$  where  $x_2 \neq y_2$  in  $\Gamma_2$ . Then  $\psi'\eta_2(x_2) = \psi'\eta_2(y_2)$  and so  $\eta_3\psi(x_2) = \eta_3\psi(y_2)$ . Since  $\eta_3$  is monomorphism,  $\psi(x_2) =$



$\psi(y_2)$ . Hence,  $x_2, y_2 \in K(\psi) \subseteq \text{Im}(\varphi)$  since  $x_2 \neq y_2$  and the rows are semi-exact. Then there are  $x_1 \neq y_1$  in  $\Gamma_1$  where  $\varphi(x_1) = x_2$  and  $\varphi(y_1) = y_2$ . Hence,  $\varphi'\eta_1(x_1) \neq \varphi'\eta_1(y_1)$  since  $\eta_1$  and  $\varphi'$  are monomorphisms. It is contradiction to  $\eta_2\varphi(x_1) = \eta_2(x_2) = \eta_2(y_2) = \eta_2\varphi(y_1)$ . Therefore,  $\eta_2$  is a monomorphism.

(2) Let  $\eta_2(x_2) \sim \eta_2(y_2)$  where  $x_2 \approx y_2$  in  $\Gamma_2$ . Then  $\psi'\eta_2(x_2) \sim \psi'\eta_2(y_2)$  and so  $\eta_3\psi(x_2) \sim \eta_3\psi(y_2)$ . Since  $\eta_3$  is a strong homomorphism,  $\psi(x_2) \sim \psi(y_2)$ . Hence,  $x_2, y_2 \in F(\psi) \subseteq \text{Im}(\varphi)$  since  $x_2 \approx y_2$  and the rows are semi-exact. Then there are  $x_1 \approx y_1$  in  $\Gamma_1$ , where  $\varphi(x_1) = x_2$  and  $\varphi(y_1) = y_2$ . Hence,  $\varphi'\eta_1(x_1) \approx \varphi'\eta_1(y_1)$  since  $\eta_1$  and  $\varphi'$  are strong homomorphisms. It is contradiction to  $\eta_2\varphi(x_1) = \eta_2(x_2) \sim \eta_2(y_2) = \eta_2\varphi(y_1)$ . Therefore,  $\eta_2$  is a strong homomorphism.

(3) Let  $x'_2 \in V(\Gamma'_2)$ . Then there is  $x_2 \in V(\Gamma_2)$ , where  $\psi'(x'_2) = \eta_3\psi(x_2)$  since  $\eta_3$  and  $\psi$  are epimorphisms. If  $x'_2 = \eta_2(x_2)$ , then  $x'_2 \in \text{Im}(\eta_2)$ . Suppose not; then  $x'_2 \in K(\psi') \subseteq \text{Im}(\varphi')$  since  $\psi'(x'_2) = \psi'\eta_2(x_2)$  and the rows are semi-exact. Hence, there is  $x_1 \in V(\Gamma_1)$ , where  $\varphi'\eta_1(x_1) = x'_2$  since  $\eta_1$  is an epimorphism. So  $\eta_2\varphi(x_1) = \varphi'\eta_1(x_1) = x'_2$  and  $x'_2 \in \text{Im}(\eta_2)$ . Therefore,  $\eta_2$  is an epimorphism.

(4) By parts (1), (2) and (3), If  $\eta_1$  and  $\eta_3$  are isomorphisms, then  $\eta_2$  is an isomorphism.  $\square$

In the following theorem, part of the impact of side graphs on  $\Gamma_2$  is expressed in the short exact sequence (2).

**Theorem 3.** *Consider the short exact sequence (2). Suppose that  $\Gamma_1$  and  $\psi_\varphi\Gamma_3$  are empty graphs. Then  $\Gamma_2$  is bipartite, where  $V(\Gamma_1)$  and  $V(\psi_\varphi\Gamma_3)$  is as a bipartition of  $V(\Gamma_2)$ . If  $\Gamma_1$  and  $\Gamma_3$  are finite graphs, then so is  $\Gamma_2$ , where  $|V(\Gamma_2)| = |V(\Gamma_1)| + |V(\psi_\varphi\Gamma_3)|$ .*

*Proof.* Suppose that  $\Gamma_1$  is an empty graph. Then  $\Gamma_1 \cong \text{Im}_\varphi = Q_\psi$  is empty graph since  $\varphi$  is a strong monomorphism. Also, the restriction  $\psi|_{\varphi\Gamma_2}$  is a strong monomorphism. Hence,  $\varphi\Gamma_2 \cong \psi^{-1}(\psi_\varphi\Gamma_3)$ , where  $\psi^{-1}(\psi_\varphi\Gamma_3)$  is an empty graph. Therefore,  $\Gamma_2$  is bipartite where  $\Gamma_1$  and  $\psi_\varphi\Gamma_3$  is a bipartition of  $V(\Gamma)$ . Since the restriction  $\psi|_{\varphi\Gamma_2}$  is a monomorphism, one has  $V(\Gamma_2) = \text{Im}(\varphi) \cup \psi^{-1}(V(\Gamma_3) \setminus \text{Im}(\psi_\varphi))$ . Therefore,  $|V(\Gamma_2)| = |V(\Gamma_1)| + |V(\psi_\varphi\Gamma_3)|$  since  $\psi$  is an epimorphism and  $\text{Im}(\varphi) \cap \psi^{-1}(V(\Gamma_3) \setminus \text{Im}(\psi_\varphi)) = \emptyset$ .  $\square$

**Lemma 1.** *Consider the short exact sequence (2). Then the following sequence is short exact:*

$$K_0 \rightarrow \Gamma_1 + N(\text{Im}_{\psi_\varphi}) \xrightarrow{i} \Gamma_1 +_{\psi_\varphi} \Gamma_3 \xrightarrow{\psi_\varphi+i} \Gamma_3 \rightarrow K_1^\circ.$$

*Proof.* It is clear that the inclusion map  $i$  is a strong monomorphism. Since  $V(\Gamma_3) = \text{Im}(\psi\varphi) \cup V({}_{\psi\varphi}\Gamma_3)$ , the homomorphism  $\psi\varphi + i$  is surjective. Since  $\text{Im}(\psi\varphi) \cap V({}_{\psi\varphi}\Gamma_3) = \emptyset$ , one has  $K(\psi\varphi + i) = K(\psi\varphi)$ . Also, since  $\Gamma_1$  and  ${}_{\psi\varphi}\Gamma_3$  are disjoint and every vertex of  $\text{Im}_{\psi\varphi}$  is adjacent to some vertex of  $N(\text{Im}_{\psi\varphi})$ ,  $F(\psi\varphi + i) = F(\psi\varphi) \cup N(\text{Im}(\psi\varphi))$ . Hence,  $Q(\psi\varphi + i) = Q(\psi\varphi) \cup N(\text{Im}(\psi\varphi))$ . Let  $x_1 \in \Gamma_1$ ; then  $\varphi(x_1) \in K(\psi)$  (or  $F(\psi)$ ). Hence, there is  $\varphi(y_1) \in Q(\psi)$ , where  $\varphi(x_1) \neq \varphi(y_1)$  ( $\varphi(x_1) \approx \varphi(y_1)$ ) and  $\psi\varphi(x_1) = \psi\varphi(y_1)$  ( $\psi\varphi(x_1) \sim \psi\varphi(y_1)$ ). Since  $\varphi$  is a homomorphism,  $x_1 \neq y_1$  ( $x_1 \approx y_1$ ). Thus,  $x_1 \in Q(\psi\varphi)$ . Therefore,  $\Gamma_1 = Q_{\psi\varphi}$  and  $Q(\psi\varphi + i) = V(\Gamma_1) \cup N(\text{Im}(\psi\varphi))$ .  $\square$

**Theorem 4.** Consider the short exact sequence (2). The following conditions are equivalent:

- (1)  $\mathfrak{J}_\varphi^\circ$  is an isolated vertex of  $\Gamma_2/\mathcal{I}_\varphi$ ,
- (2)  $\mathfrak{J}_{\psi\varphi}^\circ$  is an isolated vertex of  $\Gamma_3/\mathcal{I}_{\psi\varphi}$  and  $\Gamma_3 = \text{Im}_{\psi\varphi} + {}_{\psi\varphi}\Gamma_3$ ,
- (3) The short exact sequences  $K_0 \rightarrow \Gamma_1 \xrightarrow{i} \Gamma_1 + {}_{\psi\varphi}\Gamma_3 \xrightarrow{\psi'} \text{Im}_{\psi\varphi} + {}_{\psi\varphi}\Gamma_3 \rightarrow K_1^\circ$  and (2) are isomorphic.

*Proof.* (1)  $\Rightarrow$  (2) By Remark (3)(b),  $\mathfrak{J}_{\psi\varphi}^\circ$  is an isolated vertex of  $\Gamma_3/\mathcal{I}_{\psi\varphi}$ . Hence graphs  ${}_{\psi\varphi}\Gamma_3$  and  $\text{Im}_{\psi\varphi}$  are disjoint and  $\Gamma_3 = \text{Im}_{\psi\varphi} + {}_{\psi\varphi}\Gamma_3$ .

(2)  $\Rightarrow$  (3) We define the function  $\eta_2 : \Gamma_2 \rightarrow \Gamma_1 + {}_{\psi\varphi}\Gamma_3$ , which  $\eta_2(x_2) = \varphi^{-1}(x_2)$  if  $x_2 \in \text{Im}(\varphi)$ ; otherwise,  $\eta_2(x_2) = \psi(x_2)$ . By the fact that the graphs  ${}_{\psi\varphi}\Gamma_3$  and  $\text{Im}_{\psi\varphi}$  are disjoint, one has  $x_2 \approx y_2$ , where  $x_2 \in \text{Im}(\varphi)$  and  $y_2 \in V({}_{\psi\varphi}\Gamma_3)$ . So, the map  $\eta_2$  is a homomorphism. By Lemma 1, the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ \\
 & & \downarrow \text{id}_{\Gamma_1} & & \downarrow \eta_2 & & \downarrow \text{id}_{\Gamma_3} & & \\
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{i} & \Gamma_1 + {}_{\psi\varphi}\Gamma_3 & \xrightarrow{\psi\varphi + \text{id}} & \text{Im}_{\psi\varphi} + {}_{\psi\varphi}\Gamma_3 & \longrightarrow & K_1^\circ
 \end{array}$$

Thus,  $\eta_2$  is an isomorphism by Theorem 2. Therefore, this diagram represents an isomorphism of short exact sequences.

(3)  $\Rightarrow$  (1) Let  $\eta_1, \eta_2$  and  $\eta_3$  be isomorphisms and the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ \\
 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 & & \\
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{i} & \Gamma_1 + {}_{\psi\varphi}\Gamma_3 & \xrightarrow{\psi\varphi + \text{id}} & \text{Im}_{\psi\varphi} + {}_{\psi\varphi}\Gamma_3 & \longrightarrow & K_1^\circ
 \end{array}$$

Suppose that there are  $x_2 \in \text{Im}(\varphi)$  and  $y_2 \in V(\varphi\Gamma_2)$ , where  $x_2 \sim y_2$  and  $\varphi(x_1) = x_2$  (i.e.,  $\mathfrak{I}_\varphi^\circ$  is not an isolated vertex of  $\Gamma_2/\mathcal{I}_\varphi$ ). Hence,  $\eta_2\varphi(x_1) \sim \eta_2(y_2)$ , where  $\eta_2(y_2) \notin V(\Gamma_1)$ . Therefore,  $\eta_1(x_1) \sim \eta_2(y_2)$ , where  $\eta_1(x_1) \in V(\Gamma_1)$ . It is a contradiction since the subgraphs  $\Gamma_1$  and  $\psi\varphi\Gamma_3$  of  $\Gamma_1 + \psi\varphi\Gamma_3$  are disjoint.  $\square$

A homomorphism  $f$  from  $G$  to  $f(G) \subseteq H$  is called a retraction if there exists a monomorphism  $g$  from  $f(G)$  to  $G$  such that  $fg = \text{id}_{f(G)}$ . In this circumstance,  $f(G)$  is called a retract of  $G$ , and  $G$  is called a coretract of  $f(G)$  while  $g$  is called a coretraction.

**Theorem 5.** *Let  $\phi : \Gamma \rightarrow \Upsilon$  be an epimorphism of graph. The restriction  $\phi|_{F_\phi}$  is retraction if and only if  $\phi$  is a retraction. In particular, let the sequence (2) be short exact. The epimorphism  $\psi$  is a retraction if and only if the subgraph  $\text{Im}_{\psi\varphi}$  is a retract of  $\text{Im}_\varphi$ .*

*Proof.* Let  $\phi|_{F_\phi}$  is retraction with  $\phi'_1$  it's coretraction. Since the restriction of  $\phi$  to  $_{F(\phi)}\Gamma$  is a strong homomorphism and  $\phi$  is an epimorphism, there is a coretraction  $\phi'_2 : U\Upsilon \rightarrow _{F(\phi)}\Gamma$ , where  $U = \text{Im}(\phi|_{F_\phi})$ . Define the homomorphism  $\phi' : \Upsilon \rightarrow \Gamma$  which  $\phi'(u) = \phi'_1(u)$  if  $u \in \text{Im}(\phi|_{F_\phi})$ ; otherwise,  $\phi'(u) = \phi'_2(u)$ . Suppose that  $x \approx y$ , where  $x \in V(\Gamma) \setminus F(\phi)$  and  $y \in F(\phi)$ . Then  $\phi(x) \approx \phi(y)$ . Therefore,  $\phi\phi' = \text{id}_\Upsilon$  and  $\phi$  is a retraction. Conversely, it is clear. The "in particular" statement is clear since  $F(\psi) \subseteq \text{Im}(\varphi)$ .  $\square$

**Lemma 2.** *Let  $\phi : \Gamma \rightarrow \Upsilon$  be a homomorphism of graph and let  $\Theta$  be a subgraph of  $\Upsilon$  induced by  $\text{Im}(\phi) \cup N(\text{Im}(\phi))$ . Then  $\Upsilon$  is a retract of  $\Gamma + \Upsilon$  and the following sequence is short exact:*

$$\mathbb{K}_0 \rightarrow \Gamma + \Theta \xrightarrow{i} \Gamma + \Upsilon \xrightarrow{\phi + \text{id}_\Upsilon} \Upsilon \rightarrow \mathbb{K}_1^\circ.$$

*Proof.* It is clear that  $i$  and  $\phi + \text{id}_\Upsilon$  are strong monomorphism and epimorphism, respectively. Obviously,  $K_{\phi + \text{id}_\Upsilon} = (\phi + \text{id}_\Upsilon)^{-1}(\text{Im}_\phi) = \Gamma + \text{Im}_\phi$ . Let  $y \in N(\text{Im}(\phi))$ , where  $N(\text{Im}_\phi)$  is a subgraph of  $\Upsilon$ . Then there is  $\phi(x_1) \in \text{Im}(\phi)$  such that  $\phi(x_1) \sim y$  in  $\Upsilon$  and  $x_1 \approx y$  in  $\Gamma + \Upsilon$ . Therefore,  $Q_{\phi + \text{id}_\Upsilon} = \Gamma + \Theta$  since  $\text{Im}_\phi$  and  $_{V(\Theta)}\Upsilon$  are disjoint.  $\square$

## 2. Graph functors

In this section, we introduce some functors induced by operations on graphs with an observation to exactness of them.

**Definition 9.** Let  $\Gamma$  be a graph. For every graph  $\Gamma_1$  define the Cartesian product functor  $\vee_{\Gamma}(\Gamma_1) = \Gamma_1 \square \Gamma$  which is a covariant endofunctor. It is easily verified that if  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  is a homomorphism, then  $\vee_{\Gamma}(\varphi) = \varphi \times \text{id}_{\Gamma}$  given by  $\vee_{\Gamma}(\varphi)(x_1, x) = (\varphi(x_1), x)$  is a homomorphism.

Let  $C$  and  $D$  be categories. A functor  $F : C \rightarrow D$  preserves a property  $\mathfrak{P}$  of a morphism  $f$  in  $C$  if  $F(f)$  in  $D$  also has the property  $\mathfrak{P}$ . We say that  $F$  reflects a property  $\mathfrak{P}$  if  $f$  has  $\mathfrak{P}$  in  $C$  whenever  $F(f)$  has  $\mathfrak{P}$  in  $D$ . Analogous definitions can be made with respect to properties of objects. It is clear that every functor preserves commutative diagrams. According to the definition of the functor  $\vee$ , we have the following corollary.

**Corollary 1.** *The functor  $\vee$  preserves and reflects injective mappings, surjective mappings, retractions and coretractions.*

Note that in general, the functor  $\vee$  does not preserve strong homomorphisms. In the next theorem, it will be determined that the functor  $\vee$  is exact.

**Theorem 6.** *Let  $\Gamma$  be a graph. The sequence (2) is exact if and only if the sequence*

$$K_0 \rightarrow \Gamma_1 \square \Gamma \xrightarrow{(\varphi, \text{id}_{\Gamma})} \Gamma_2 \square \Gamma \xrightarrow{(\psi, \text{id}_{\Gamma})} \Gamma_3 \square \Gamma \rightarrow K_1^{\circ} \quad (3)$$

*is exact.*

*Proof.* By Corollary 1,  $(\varphi, \text{id}_{\Gamma})$  is a monomorphism and  $(\psi, \text{id}_{\Gamma})$  is an epimorphism. Suppose that  $(\varphi(x_1), x) \sim (\varphi(y_1), y)$  in  $\Gamma_2 \square \Gamma$ . If  $\varphi(x_1) = \varphi(y_1)$  and  $x \sim y$ , then  $(x_1, x) \sim (y_1, y)$  since  $\varphi$  is a monomorphism. If  $\varphi(x_1) \sim \varphi(y_1)$  and  $x = y$ , then  $(x_1, x) \sim (y_1, y)$  since the homomorphism  $\varphi$  is strong. Therefore,  $(\varphi, \text{id}_{\Gamma})$  is a strong homomorphism. Now, let  $(\varphi(x_1), x) \in \text{Im}(\varphi \times \text{id}_{\Gamma})$ , then  $\varphi(x_1) \in Q(\psi)$  since  $\text{Im}(\varphi) = Q(\psi)$ . Suppose that  $\varphi(x_1) \in K(\psi)$ ; there is  $y_2 \in K(\psi)$  where  $y_2 \neq \varphi(x_1)$  and  $\psi(y_2) = \psi\varphi(x_1)$ . Hence  $(\varphi(x_1), x), (y_2, x) \in K(\psi \times \text{id}_{\Gamma})$ . Suppose that  $\varphi(x_1) \in F(\psi)$ . By the same way with replacing “=” by “ $\sim$ ”, one has  $(\psi\varphi(x_1), x) \sim (\psi(y_2), x)$  in  $\Gamma_3 \square \Gamma$  and so,  $(\varphi(x_1), x) \in F(\psi \times \text{id}_{\Gamma})$ . Therefore,  $\text{Im}(\varphi \times \text{id}_{\Gamma}) \subseteq Q(\psi \times \text{id}_{\Gamma})$ . Let  $(x_2, x) \in K(\psi \times \text{id}_{\Gamma})$ , then there is  $(y_2, y) \in K(\psi \times \text{id}_{\Gamma})$ , where  $(y_2, y) \neq (x_2, x)$  and  $\psi \times \text{id}_{\Gamma}(y_2, y) = \psi \times \text{id}_{\Gamma}(x_2, x)$ . Hence,  $x = y$  and  $\psi(x_2) = \psi(y_2)$ , where  $x_2 \neq y_2$ . So,  $x_2 \in K(\psi)$  and there is  $x_1 \in \Gamma_1$ , where  $\varphi(x_1) = x_2$  since  $\text{Im}(\varphi) = Q(\psi)$ . Therefore,  $(x_2, x) \in \text{Im}(\varphi \times \text{id}_{\Gamma})$ . If  $(x_2, x) \in F(\psi \times \text{id}_{\Gamma})$ , then there is  $(y_2, y) \in F(\psi \times \text{id}_{\Gamma})$ , where  $(y_2, y) \approx (x_2, x)$  in  $\Gamma_2 \square \Gamma$  and  $\psi \times \text{id}_{\Gamma}(y_2, y) \sim \psi \times \text{id}_{\Gamma}(x_2, x)$  in

$\Gamma_3 \square \Gamma$ . Thus either  $y = x$  and  $\psi(x_2) \sim \psi(y_2)$  or  $y \sim x$  and  $\psi(x_2) = \psi(y_2)$ . Suppose that  $y = x$ ; then  $x_2 \approx y_2$  since  $(y_2, y) \approx (x_2, x)$ . Since  $\psi(x_2) \sim \psi(y_2)$ ,  $x_2 \in F(\psi)$  and there is  $x_1 \in \Gamma_1$  where  $\varphi(x_1) = x_2$ . Therefore,  $(\varphi(x_1), x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$ . Now, suppose that  $\psi(x_2) = \psi(y_2)$ . Then  $x_2 \neq y_2$  since  $y \sim x$  and  $(y_2, y) \approx (x_2, x)$  in  $\Gamma_2 \square \Gamma$ . Hence,  $x_2 \in K(\psi)$  and so there is  $x_1 \in \Gamma_1$  where  $\varphi(x_1) = x_2$ . Thus  $(\varphi(x_1), x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$  and  $Q(\psi \times \text{id}_\Gamma) \subseteq \text{Im}(\varphi \times \text{id}_\Gamma)$ . Therefore,  $Q(\psi \times \text{id}_\Gamma) = \text{Im}(\varphi \times \text{id}_\Gamma)$  and the sequence (3) is short exact.

Conversely, let the sequence (3) be exact for some  $\Gamma$ . Since  $\varphi \times \text{id}_\Gamma$  is a strong monomorphism and  $\psi \times \text{id}_\Gamma$  is an epimorphism,  $\varphi$  is a strong monomorphism and  $\psi$  is an epimorphism in the sequence (2). Let  $x_2 \in \text{Im}(\varphi)$ , where  $\varphi(x_1) = x_2$ . Then  $(\varphi(x_1), x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$ . Since  $\text{Im}(\varphi \times \text{id}_\Gamma) = Q(\psi \times \text{id}_\Gamma)$ ,  $(\varphi(x_1), x) \in Q(\psi \times \text{id}_\Gamma)$ . By the same way in the previous part (Concerning  $Q(\psi \times \text{id}_\Gamma) \subseteq \text{Im}(\varphi \times \text{id}_\Gamma)$ ), It is easy to show that  $\varphi(x_1) = x_2 \in Q(\psi)$  and so  $\text{Im}(\varphi) \subseteq Q(\psi)$ . Let  $x_2 \in Q(\psi)$ . By the same way in the previous part (Concerning  $\text{Im}(\varphi \times \text{id}_\Gamma) \subseteq Q(\psi \times \text{id}_\Gamma)$ ), It is easy to show that  $x_2 \in \text{Im}(\varphi)$ . Therefore, the sequence (2) is short exact.  $\square$

**Definition 10.** Let  $\Gamma$  be a graph. For every graph  $\Gamma_1$  define the direct product functor  $\wedge_\Gamma(\Gamma_1) = \Gamma_1 \times \Gamma$  which is a covariant endofunctor. It is easily verified that if  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  is a homomorphism, then  $\wedge_\Gamma(\varphi) = \varphi \times \text{id}_\Gamma$  given by  $\wedge_\Gamma(\varphi)(x_1, x) = (\varphi(x_1), x)$  is a homomorphism.

Let  $(\varphi(x_1), x) \sim (\varphi(y_1), y)$ ; then  $\varphi(x_1) \sim \varphi(y_1)$  and  $x \sim y$ . If the homomorphism  $\varphi$  is strong, then  $\wedge(\varphi)$  is a strong homomorphism. Moreover, according to the definition of the functor  $\wedge$  the following corollary holds.

**Corollary 2.** *The direct product functor  $\wedge$  preserves and reflects injective mappings, surjective mappings, strong homomorphisms, retractions and coretractions.*

In the next theorem, it will be determined that the functor  $\wedge$  is exact if  $\Gamma$  is a graph of minimum degree  $k \geq 1$ .

**Theorem 7.** *Consider the short exact sequence (2). Let  $\Gamma$  be a graph of minimum degree  $k \geq 1$ , then*

$$K_0 \rightarrow \Gamma_1 \times \Gamma \xrightarrow{(\varphi, \text{id}_\Gamma)} \Gamma_2 \times \Gamma \xrightarrow{(\psi, \text{id}_\Gamma)} \Gamma_3 \times \Gamma \rightarrow K_1^0 \quad (4)$$

*is a short exact sequence of graphs. In particular, if  $\Gamma$  is a connected graph, then the functor  $\wedge$  is exact.*

*Proof.* By Corollary 2,  $(\varphi, \text{id}_\Gamma)$  is a strong monomorphism and  $(\psi, \text{id}_\Gamma)$  is an epimorphism. Let  $(\varphi(x_1), x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$ ; then  $\varphi(x_1) \in Q(\psi)$  since  $\text{Im}(\varphi) = Q(\psi)$ . Suppose that  $\varphi(x_1) \in K(\psi)$ . By the similar way as the proof of Theorem 6,  $(\varphi(x_1), x) \in K(\psi \times \text{id}_\Gamma)$ . Suppose that  $\varphi(x_1) \in F(\psi)$ , then there is  $y_2 \in F(\psi)$  where  $y_2 \approx \varphi(x_1)$  and  $\psi(y_2) \sim \psi\varphi(x_1)$ . Since  $\text{deg}(x) \geq 1$ , it has an adjacent  $y \in \Gamma$  where  $(\psi(y_2), y) \sim (\psi\varphi(x_1), x)$  in  $\Gamma_3 \times \Gamma$ . Therefore,  $(\varphi(x_1), x) \in F(\psi \times \text{id}_\Gamma)$  since  $(\varphi(x_1), x) \approx (y_2, y)$  in  $\Gamma_2 \times \Gamma$ . Let  $(x_2, x) \in K(\psi \times \text{id}_\Gamma)$ . By the same reason in the proof of Theorem 6,  $(x_2, x) \in \text{Im}(\psi \times \text{id}_\Gamma)$ . Now, if  $(x_2, x) \in F(\psi \times \text{id}_\Gamma)$ , then there is  $(y_2, y) \in F(\psi \times \text{id}_\Gamma)$ , where  $(y_2, y) \approx (x_2, x)$  in  $\Gamma_2 \times \Gamma$  and  $\psi \times \text{id}_\Gamma(y_2, y) \sim \psi \times \text{id}_\Gamma(x_2, x)$  in  $\Gamma_3 \times \Gamma$ . Hence,  $x \sim y$  and  $\psi(x_2) \sim \psi(y_2)$ . It is clear from  $(y_2, y) \approx (x_2, x)$ , that  $x_2 \approx y_2$  and so  $x_2 \in F(\psi)$ . Hence, there is  $x_1 \in \Gamma_1$  where  $\varphi(x_1) = x_2$ . Thus  $(x_2, x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$ . Therefore,  $Q(\psi \times \text{id}_\Gamma) = \text{Im}(\varphi \times \text{id}_\Gamma)$  and the sequence (4) is short exact. The ‘‘in particular’’ statement is clear.  $\square$

**Theorem 8.** *Let  $\Gamma$  be a non-empty graph. If the sequence (4) is exact, then the sequence (2) is exact.*

*Proof.* By Corollary 2,  $\varphi$  and  $\psi$  are strong monomorphism and epimorphism, respectively. By the same way as in the proof of Theorem 7 (Concerning  $Q(\psi \times \text{id}_\Gamma) \subseteq \text{Im}(\varphi \times \text{id}_\Gamma)$ ), It is easy to see that if  $x_2 \in \text{Im}(\varphi)$ , then  $x_2 \in Q(\psi)$ . By the similar way as in the proof of Theorem 7 (Concerning  $\text{Im}(\varphi \times \text{id}_\Gamma) \subseteq Q(\psi \times \text{id}_\Gamma)$ ), It is easy to see that if  $x_2 \in K(\psi)$ , then  $x_2 \in \text{Im}(\varphi)$ . Let  $x_2 \in F(\psi)$ , then there is  $y_2 \in F(\psi)$  where  $y_2 \approx x_2$  and  $\psi(y_2) \sim \psi(x_2)$ . Since  $E(\Gamma) \neq \emptyset$ , there are  $x, y \in \Gamma$  where  $x \sim y$ . Hence  $(x_2, x) \approx (y_2, y)$  in  $\Gamma_2 \times \Gamma$  where  $\psi \times \text{id}_\Gamma(y_2, y) \sim \psi \times \text{id}_\Gamma(x_2, x)$  in  $\Gamma_3 \times \Gamma$ . Thus  $(x_2, x) \in F(\psi \times \text{id}_\Gamma)$  and so  $(x_2, x) \in \text{Im}(\varphi \times \text{id}_\Gamma)$  since  $F(\psi \times \text{id}_\Gamma) = \text{Im}(\varphi \times \text{id}_\Gamma)$ . Hence  $x_2 \in \text{Im}(\varphi)$  and  $F(\psi) \subseteq \text{Im}(\varphi)$ . Therefore, the sequence (2) is short exact.  $\square$

**Remark 4.** (a) Let  $\Gamma = \Gamma_1 \square \Gamma_2 \square \cdots \square \Gamma_k$ , then the inclusion map  $\iota_{Y_i} : \Gamma_i \rightarrow \Gamma$  with respect to  $Y_i$  given by

$$x_i \mapsto (y_1, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_k),$$

is a monomorphism where  $Y_i = (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_{k-1}) \in V(\Gamma_1 \square \cdots \square \Gamma_{i-1} \square \Gamma_{i+1} \square \cdots \square \Gamma_{k-1})$ . Further, let  $\Upsilon = \Upsilon_1 \times \Upsilon_2 \times \cdots \times \Upsilon_k$ . By simple rewording of the definitions, each projection  $p_i : \Upsilon \rightarrow \Upsilon_i$  is a homomorphism. Furthermore, given a graph  $T$  and a collection of homomorphisms  $\tau_i : T \rightarrow \Upsilon_i$ , for  $1 \leq i \leq k$ , observe that the map

$\tau : x \mapsto (\tau_1(x), \tau_2(x), \dots, \tau_k(x))$  is a homomorphism  $T \rightarrow \Upsilon$ . From the two facts just mentioned, we see that every homomorphism  $\tau : T \rightarrow \Upsilon$  has the form  $\tau : x \mapsto (\tau_1(x), \tau_2(x), \dots, \tau_k(x))$ , for homomorphisms  $\tau_i : T \rightarrow \Upsilon_i$ , where  $\tau_i = p_i\tau$ . Clearly  $\tau$  is uniquely determined by  $p_i$  and  $\tau_i$ .

(b) Consider the short exact sequence (2). By Theorem 6 and part (a), the following diagram with exact rows determines a homomorphism of short exact sequences:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ \\
 & & \downarrow \iota_y & & \downarrow \iota_y & & \downarrow \iota_y & & \\
 K_0 & \longrightarrow & \Gamma_1 \square \Gamma & \xrightarrow{\varphi \times \text{id}_\Gamma} & \Gamma_2 \square \Gamma & \xrightarrow{\psi \times \text{id}_\Gamma} & \Gamma_3 \square \Gamma & \longrightarrow & K_1^\circ
 \end{array}$$

where  $\iota_y : \Gamma_i \rightarrow \Gamma_i \square \Gamma$  with  $1 \leq i \leq 3$  for some  $y \in V(\Gamma)$ .

(c) Let  $\Gamma$  be a graph with  $E(\Gamma) \neq \emptyset$  and let the sequence (4) be exact. By Theorem 8 and part (a), the following diagram with exact rows determines a homomorphism of short exact sequences:

$$\begin{array}{ccccccccc}
 K_0 & \longrightarrow & \Gamma_1 \times \Gamma & \xrightarrow{\varphi \times \text{id}_\Gamma} & \Gamma_2 \times \Gamma & \xrightarrow{\psi \times \text{id}_\Gamma} & \Gamma_3 \times \Gamma & \longrightarrow & K_1^\circ \\
 & & \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\
 K_0 & \longrightarrow & \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\psi} & \Gamma_3 & \longrightarrow & K_1^\circ
 \end{array}$$

where  $p_i : \Gamma_i \times \Gamma \rightarrow \Gamma_i$  for  $1 \leq i \leq 3$ .

**Lemma 3.** *Let  $\Upsilon$  be a graph and  $\varphi$  is a homomorphism from  $\Gamma_1$  to  $\Gamma_2$ .*

- (1)  $\hat{\varphi}$  is a homomorphism from  $\Gamma_1^\Upsilon$  to  $\Gamma_2^\Upsilon$ .
- (2)  $\check{\varphi}$  is a homomorphism from  $\Upsilon^{\Gamma_2}$  to  $\Upsilon^{\Gamma_1}$ .

*Proof.* (1) Suppose that  $f_1$  and  $g_1$  are adjacent vertices of  $\Gamma_1^\Upsilon$ . Then  $f_1(u) \sim g_1(y)$  where  $u, y \in V(\Upsilon)$  and  $u \sim y$ . Hence  $\varphi f_1(u) \sim \varphi g_1(y)$  for every  $u \sim y$ . Therefore,  $\hat{\varphi}(f_1) \sim \hat{\varphi}(g_1)$ .

- (2) It follows from [1, Theorem 6.4.1].

□

Based on the above lemma, we define a map functor as follows.

**Definition 11.** Let  $\Gamma$  be a graph. For every graph  $\Gamma_1$  define the map functor  $\text{Map}(\Gamma, \Gamma_1) = \Gamma_1^\Gamma$  which is a covariant endofunctor. Let  $\varphi : \Gamma_1 \rightarrow$

$\Gamma_2$  be a homomorphism, then

$$\text{Map}(\Gamma, \varphi) = \hat{\varphi} : \text{Map}(\Gamma, \Gamma_1) \rightarrow \text{Map}(\Gamma, \Gamma_2)$$

given by  $f_1 \mapsto \varphi f_1$  is a homomorphism. Also, the map functor  $\text{Map}(-, \Gamma) = \Gamma^-$  is defined as a contravariant endofunctor and

$$\text{Map}(\varphi, \Gamma) = \hat{\varphi} : \text{Map}(\Gamma_2, \Gamma) \rightarrow \text{Map}(\Gamma_1, \Gamma)$$

given by  $f_2 \mapsto f_2 \varphi$  is a homomorphism.

By the above definition, it is easy to check that the following corollary holds.

**Corollary 3.** *The map functor  $\text{Map}(\Gamma, -)$  preserves and reflects retractions and coretractions. Moreover, the map functor  $\text{Map}(-, \Gamma)$  maps retraction and coretraction to coretraction and retraction, respectively.*

In the next theorem, it will be determined the map functor  $\text{Map}(\Gamma, -)$  preserves and reflects injective mappings, surjective mappings and strong homomorphisms. Further, it preserves the complexes if  $\psi(F_\psi)$  is a clique of  $\Gamma_3$  and it reflects the semi-exact sequences.

**Theorem 9.** *Let  $\Gamma, \Gamma_1, \Gamma_2$  and  $\Gamma_3$  be graphs where  $E(\Gamma) \neq \emptyset$ .*

(1) *Let the sequence  $\Gamma_1 \xrightarrow{\varphi} \Gamma_2 \xrightarrow{\psi} \Gamma_3$  be a complex. Suppose that  $\psi(F_\psi)$  is a clique (with loop) of  $\Gamma_3$ , then the following sequence is a complex:*

$$\Gamma_1^\Gamma \xrightarrow{\hat{\varphi}} \Gamma_2^\Gamma \xrightarrow{\hat{\psi}} \Gamma_3^\Gamma. \quad (5)$$

*Moreover, if  $\varphi$  is a strong monomorphism, then  $\hat{\varphi}$  is a strong monomorphism. If  $\psi$  is an epimorphism, then so is  $\hat{\psi}$ .*

(2) *Let the following sequence is semi-exact*

$$K_0 \rightarrow \Gamma_1^\Gamma \xrightarrow{\hat{\varphi}} \Gamma_2^\Gamma \xrightarrow{\hat{\psi}} \Gamma_3^\Gamma \rightarrow K_1^\circ, \quad (6)$$

*then the sequence (2) is semi-exact. In particular, if the sequence (6) is exact for every  $\Gamma$ , then the sequence (2) is exact.*

*Proof.* (1) Let  $f_2 \in \text{Im}(\hat{\varphi})$ , then there is  $f_1 \in V(\Gamma_1^\Gamma)$  where  $f_2 = \varphi f_1$ . Since  $\text{Im}(\varphi) \subseteq Q(\psi)$ ,  $\text{Im}(f_2) \subseteq Q(\psi)$ . Suppose that  $f_2(y) \in K(\psi)$  for some  $y \in V(\Gamma)$ , then there is  $x_2 \in \Gamma_2$  where  $x_2 \neq f_2(y)$  and  $\psi(x_2) = \psi f_2(y)$ . Define  $g_2 : \Gamma \rightarrow \Gamma_2$  with  $g_2(y) = x_2$  and  $g_2(x) = f_2(x)$  for all elements  $x$  other than  $y$ . Hence  $g_2 \neq f_2$  and  $\psi f_2 = \psi g_2$ . Therefore,  $\hat{\psi}(f_2) = \hat{\psi}(g_2)$



and  $f_2 \in K(\hat{\psi})$ . Let  $f_2(x) \in F(\psi)$  for all  $x \in V(\Gamma)$ . Since  $E(\Gamma) \neq \emptyset$ , there are  $y, z \in V(\Gamma)$  where  $y \sim z$ . On the other hand, there is  $x_2 \in \Gamma_2$ , where  $x_2 \approx f_2(y)$  and  $\psi(x_2) \sim \psi f_2(y)$ . Define  $g_2(x) = x_2$  for all  $x \in V(\Gamma)$ . Since  $y \sim z$  and  $f_2(y) \approx g_2(z)$ , we have  $f_2 \approx g_2$ . By the fact that  $\psi|_{F_\psi}$  is a complete graph, one has  $\psi f_2(x) \sim \psi g_2(x)$ . Hence,  $\hat{\psi}(f_2) \sim \hat{\psi}(g_2)$  where  $f_2 \approx g_2$  and so  $f_2 \in F(\hat{\psi})$ . Therefore,  $\text{Im}(\hat{\varphi}) \subseteq Q(\hat{\psi})$  and the sequence (5) is a complex. Moreover, suppose that  $\hat{\varphi}(f_1) = \hat{\varphi}(g_1)$ , then  $\varphi f_1 = \varphi g_1$ . Hence,  $f_1 = g_1$  since  $\varphi$  is a monomorphism. If  $\hat{\varphi}(f_1) \sim \hat{\varphi}(g_1)$ , then  $\varphi f_1(y) \sim \varphi g_1(z)$  for every  $y \sim z$  in  $\Gamma$ . Hence,  $f_1(y) \sim g_1(z)$  for every  $y \sim z$  in  $\Gamma$  since the homomorphism  $\varphi$  is strong. Therefore,  $f_1 \sim g_1$  and  $\hat{\varphi}$  is a strong homomorphism. Now let  $f_3 \in V(\Gamma_3^\Gamma)$ . Since  $\psi$  is an epimorphism, function  $f_2$  can be defined  $f_2(x) = x_2$  for some  $x_2 \in \psi^{-1}(f_3(x))$ . Hence,  $\psi f_2(x) = f_3(x)$ . Therefore,  $\hat{\psi}(f_2) = f_3$  and so  $\hat{\psi}$  is an epimorphism.

(2) First it is shown that  $\varphi$  and  $\psi$  are monomorphism and epimorphism, respectively. Let  $\varphi(x_1) = \varphi(y_1)$ , then there are  $f_1, g_1 \in V(\Gamma_1^\Gamma)$  where  $f_1(x) = x_1$  and  $g_1(x) = y_1$  for all  $x \in V(\Gamma)$  since  $\Gamma \neq K_0$ . Hence  $\hat{\varphi}(f_1) = \hat{\varphi}(g_1)$ . Since  $\hat{\varphi}$  is a monomorphism,  $f_1(x) = x_1 = y_1 = g_1(x)$  and so  $\varphi$  is a monomorphism. Let  $x_3 \in V(\Gamma_3)$ , then there is  $f_3 \in V(\Gamma_3^\Gamma)$  where  $f_3(x_0) = x_3$  for some  $x_0 \in V(\Gamma)$ . Since  $\hat{\psi}$  is an epimorphism, there is  $f_2 \in V(\Gamma_2^\Gamma)$  where  $\psi f_2(x) = f_3(x)$  for every  $x \in V(\Gamma)$ . Hence  $\psi(x_2) = f_3(x_0) = x_3$  where,  $f_2(x_0) = x_2$  and so  $\psi$  is an epimorphism. Now let  $x_2 \in Q(\psi)$ , then there is  $y_2 \in Q(\psi)$  where either  $y_2 \neq x_2$  and  $\psi(x_2) = \psi(y_2)$  or  $y_2 \approx x_2$  and  $\psi(x_2) \sim \psi(y_2)$ . Define  $f_2(x) = x_2$  and  $g_2(x) = y_2$  where  $f_2, g_2 \in V(\Gamma_2^\Gamma)$ . Hence, either  $f_2 \neq g_2$  and  $\hat{\psi}(f_2) = \hat{\psi}(g_2)$  or  $f_2 \approx g_2$  and  $\hat{\psi}(f_2) \sim \hat{\psi}(g_2)$  since  $E(\Gamma) \neq \emptyset$  and  $\psi(x_2) \sim \psi(y_2)$ . So  $f_2 \in Q(\hat{\psi}) \subseteq \text{Im}(\hat{\varphi})$  since the sequence (6) is semi-exact. Thus there is  $f_1 \in V(\Gamma_1^\Gamma)$  where  $\hat{\varphi}(f_1) = f_2$ . Hence,  $\varphi f_1(x) = f_2(x) = x_2$  and so  $x_2 \in \text{Im}(\varphi)$ . Therefore,  $Q(\psi) \subseteq \text{Im}(\varphi)$  and the sequence (2) is semi-exact. For the ‘‘in particular’’ statement, it is sufficient to set  $\Gamma = K_1^0$ .  $\square$

**Lemma 4.** *Let  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  be a homomorphism of graphs. Consider the homomorphism  $\tilde{\varphi} : \Gamma^{\Gamma_2} \rightarrow \Gamma^{\Gamma_1}$  where  $|V(\Gamma)| \geq 2$ .*

- (1)  *$\varphi$  is not surjective if and only if  $K_{\tilde{\varphi}} = Q_{\tilde{\varphi}} = \Gamma^{\Gamma_2}$ .*
- (2) *If  $\varphi$  is an epimorphism, then  $\tilde{\varphi}$  is a strong monomorphism. If  $\tilde{\varphi}$  is a monomorphism, then  $\varphi$  is an epimorphism.*
- (3)  *$\varphi$  is injective if and only if  $\tilde{\varphi}$  is an epimorphism.*

*Proof.* (1) Let  $f_2 \in V(\Gamma^{\Gamma_2})$ . Obviously, there is  $g_2 \in V(\Gamma^{\Gamma_2})$  where  $g_2 \neq f_2$  and  $g_2\varphi = f_2\varphi$  since  $\text{Im}(\varphi) \neq V(\Gamma_2)$  and  $|V(\Gamma)| \geq 2$ . Therefore,  $f_2 \in K(\tilde{\varphi})$ . Conversely, Let  $f_2 \in V(\Gamma^{\Gamma_2}) = K(\tilde{\varphi})$ . Hence there is  $g_2 \in V(\Gamma^{\Gamma_2})$  where

$g_2 \neq f_2$  and  $g_2\varphi = f_2\varphi$ . Now if  $\varphi$  is an epimorphism, then  $f_2 = g_2$ , a contradiction. Therefore, the homomorphism  $\varphi$  is not surjective.

(2) Let  $\check{\varphi}(f_2) = \check{\varphi}(g_2)$ , then  $f_2\varphi(x_1) = g_2\varphi(y_1)$  for all  $x_1 = y_1$  in  $\Gamma_1$ . By the fact that  $\varphi$  is an epimorphism,  $f_2 = g_2$  for all  $x_1 = y_1$  in  $\Gamma_1$  and so  $\check{\varphi}$  is a monomorphism. By the same way with replacing “=” by “ $\sim$ ”, the homomorphism  $\check{\varphi}$  is strong. By part (1), if the homomorphism  $\varphi$  is not surjective, then  $\Gamma^{\Gamma_2} = K_{\check{\varphi}} \neq \emptyset$ . Therefore, the homomorphism  $\check{\varphi}$  is not injective.

(3) Let  $f_1 \in V(\Gamma^{\Gamma_1})$ . Obviously, there is  $f_2 \in V(\Gamma^{\Gamma_2})$  where  $f_1(x_1) = f_2\varphi(x_1)$  for every  $x_1 \in V(\Gamma_1)$ . So,  $\check{\varphi}(f_2) = f_1$  and  $\check{\varphi}$  is an epimorphism. Since  $|V(\Gamma)| \geq 2$ , there is  $f_1 \in V(\Gamma^{\Gamma_1})$  such that  $f_1(x_1) \neq f_1(y_1)$ . Suppose that the homomorphism  $\varphi$  is not injective, then there are  $x_1 \neq y_1$  in  $\Gamma_1$  where  $\varphi(x_1) = \varphi(y_1)$ . Since  $\check{\varphi}$  is an epimorphism, there is  $f_2 \in V(\Gamma^{\Gamma_2})$  where  $\check{\varphi}(f_2) = f_1$ . It is a contradiction to  $f_2\varphi(x_1) = f_1(x_1) \neq f_1(y_1) = f_2\varphi(y_1)$  since  $\varphi(x_1) = \varphi(y_1)$ . Therefore,  $\varphi$  must be a monomorphism.  $\square$

**Theorem 10.** *Consider the sequence (2) where,  $\Gamma_3 \neq K_0$  and let  $|V(\Gamma)| \geq 2$ . The following conditions are equivalent:*

- (1) *The map functor  $\text{Map}(-, \Gamma)$  is exact.*
- (2) *The homomorphism  $\varphi$  is not surjective and  $\psi$  is a monomorphism.*
- (3)  *$\text{Im}_{\check{\psi}} = \Gamma^{\Gamma_2} = K_{\check{\varphi}}$ .*

*Proof.* (1)  $\Rightarrow$  (2) Suppose that  $\varphi$  is an epimorphism, then  $\check{\varphi}$  is a strong monomorphism by Lemma 4(2). Hence  $Q(\check{\varphi}) = \emptyset$  and so  $|\text{Im}(\check{\psi})| = 0$  or 1 since the map functor  $\text{Map}(-, \Gamma)$  is exact. Hence  $|\Gamma^{\Gamma_3}| = 0$  or 1 since  $\check{\psi}$  is a strong monomorphism. It is contradiction to  $|V(\Gamma)| \geq 2$  and  $\Gamma_3 \neq K_0$ . Hence the homomorphism  $\varphi$  is not surjective and so  $K_{\check{\varphi}} = Q_{\check{\varphi}} = \Gamma^{\Gamma_2}$  by Lemma 4(1). Since the map functor  $\text{Map}(-, \Gamma)$  is exact,  $\text{Im}_{\check{\psi}} = Q_{\check{\varphi}} = \Gamma^{\Gamma_2}$  and so  $\check{\psi}$  is an epimorphism. Therefore,  $\psi$  is a monomorphism by Lemma 4(3).

(2)  $\Rightarrow$  (3) By Lemma 4(1)(2),  $K_{\check{\varphi}} = Q_{\check{\varphi}} = \Gamma^{\Gamma_2}$  and  $\check{\psi}$  is an epimorphism. Hence  $\text{Im}_{\check{\psi}} = Q_{\check{\varphi}} = \Gamma^{\Gamma_2}$ .

(3)  $\Rightarrow$  (1) Since  $\psi$  is an epimorphism and  $\varphi$  is a strong monomorphism,  $\check{\psi}$  and  $\check{\varphi}$  are strong monomorphism and epimorphism by Lemma 4(2)(3), respectively. Therefore, the map functor  $\text{Map}(-, \Gamma)$  is exact.  $\square$

Let  $\Gamma_i$ 's be graphs with  $1 \leq i \leq k$ . Then  $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_k$  and  $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$  are denoted by  $\sum_{i=1}^k \Gamma_i$  and  $\times_{i=1}^k \Gamma_i$ , respectively.

**Theorem 11.** *Let  $\Gamma_i$ 's be graphs with  $1 \leq i \leq k$  and let  $\Gamma$  be a finite graph. The following statements hold:*

$$\begin{aligned}
(1) \quad (a) \quad \vee_{\Gamma}(\sum_{i=1}^k \Gamma_i) &= \sum_{i=1}^k \vee_{\Gamma}(\Gamma_i). \\
(b) \quad \wedge_{\Gamma}(\sum_{i=1}^k \Gamma_i) &= \sum_{i=1}^k \wedge_{\Gamma}(\Gamma_i). \\
(2) \quad (a) \quad \text{Map}(\sum_{i=1}^k \Gamma_i, \Gamma) &= \prod_{i=1}^k \text{Map}(\Gamma_i, \Gamma). \\
(b) \quad \text{Map}(\Gamma, \prod_{i=1}^k \Gamma_i) &= \prod_{i=1}^k \text{Map}(\Gamma, \Gamma_i). \\
(3) \quad \text{Map}(\prod_{i=1}^k \Gamma_i, \Gamma) &= \text{Map}(\Gamma_k, \text{Map}(\prod_{i=1}^{k-1} \Gamma_i, \Gamma)).
\end{aligned}$$

*Proof.* Part (1) follows from [5, Theorem 4.5.3], and part (2),(3) follow from [3, Chapter 9.4].  $\square$

So for the direct product functor and the map functor, here is a behavior similar to the tensor product functor and Hom functor in algebra.

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