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RESEARCH ARTICLE

Some properties of the nilradical and non-nilradical graphs over finite commutative ring $\mathbb{Z}_n$

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Abstract. Let $\mathbb{Z}_n$ be the finite commutative ring of residue classes modulo n with identity and $\Gamma(\mathbb{Z}_n)$ be its zero-divisor graph. In this paper, we investigate some properties of nilradical graph, denoted by $N(\mathbb{Z}_n)$ and non-nilradical graph, denoted by $\Omega(\mathbb{Z}_n)$ of $\Gamma(\mathbb{Z}_n)$. In particular, we determine the Chromatic number and Energy of $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ for a positive integer n. In addition, we have found the conditions in which $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ graphs are planar. We have also given MATLAB coding of our calculations.

Introduction

The concept of zero-divisor graph was introduced by I. beck in 1988 but the most common definition of zero-divisor graph given by D. F. Anderson and P. S. Livingston in 1999 is as follows: "Let R be a commutative ring (with 1) and let $Z(R)$ be its set of zero-divisors. We associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R)$ is the empty graph if and only if R is an integral domain." We have derived some results for the ring $\mathbb{Z}_n$.

A complete graph is a graph (without loops and multiple edges) in which every vertex is adjacent to any other vertices of the graph. A graph in which all vertices have the same degree is said to be a regular graph. A complete bipartite graph is a graph whose vertices can be divided into

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two sets such that every vertex in one set is connected to every vertex in the other, and no vertex is connected to any other vertices in the same set. A star graph is a complete bipartite graph in which at least one of the two vertex sets contains only one vertex. That one vertex is called the center of the star graph. A vertex of a graph is isolated if there is no edge incident on it. A graph is almost connected if there exists a path between any two non-isolated vertices. A proper coloring of a graph $\mathbb{Z}_n$ is a function that assigns a color to each vertex such that no any two adjacent vertices have the same color. The chromatic number of $\mathbb{Z}_n$, denoted by $\chi(\mathbb{Z}_n)$, is the smallest number of colors required for proper coloring. A planar graph is a graph that can be embedded in the plane, i.e, it can be drawn on the plane in such a way that its edges intersect only at their endpoints and we will repeatedly use Kuratowski's theorem, which states that *a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$*. The energy of a graph is the sum of absolute value of all eigenvalues of the adjacency matrix. The adjacency matrix corresponding to a zero divisor graph is defined as $A = [a_{i.j}]$, where $a_{i,j} = 1$, if v_i & v_j represent zero divisor, i.e., $v_i.v_j = 0$ and $a_{i,j} = 0$ otherwise, where v_i and v_j are vertices of the graph.

Nilradical and non-nilradical graphs

Definition 1.1. The nilradical graph of $\mathbb{Z}_n$, denoted by $N(\mathbb{Z}_n)$, is the graph whose vertices are the nonzero nilpotent elements of $\mathbb{Z}_n$ and any two vertices are connected by an edge if and only if their product is 0.

Definition 1.2. The non-nilradical graph of $\mathbb{Z}_n$, denoted by $\Omega(\mathbb{Z}_n)$, is the graph whose vertices are the non-nilpotent zero-divisors of $\mathbb{Z}_n$ and any two vertices are connected by an edge if and only if their product is 0.

1. Chromatic number and planarity of nilradical and non-nilradical graphs

Theorem 1. *If p and q are distinct prime numbers and n is a positive integer, then*

(1) $\chi(N(\mathbb{Z}_n)) = 0$ *if* $n = pq$;
(2) $\chi(N(\mathbb{Z}_n)) = p - 1$ *if* $n = p^2$;
(3) $\chi(N(\mathbb{Z}_n)) = pq - 1$ *if* $n = p^2q^2$;
(4) $\chi(N(\mathbb{Z}_n)) = p$ *if* $n = p^3$;
(5) $\chi(N(\mathbb{Z}_n)) = p - 1$ *if* $n = p^2q$.

Proof. (1) Let $n = pq$, where p and q are distinct primes. Then $N(\mathbb{Z}_n)$ is an empty graph. So, there is no need of any color for coloring the graph. Hence, chromatic number is zero.

(2) Let $n = p^2$, where p is a prime number. If $p = 2$, then $N(\mathbb{Z}_n)$ has only one vertex. This implies the chromatic number is one. If $p \geqslant 3$, then the number of nilpotent elements which are divisible by p^2 are $(p-1)$. Also, these $(p-1)$ nilpotent elements form a complete graph. So, $(p-1)$ colors are required for coloring the graph and these $(p-1)$ colors are minimum in numbers. Therefore, chromatic number is $(p-1)$.

(3) Let $n = p^2q^2$, where p and q are prime numbers and $p \neq q$. Then the nilpotent elements are multiple of pq and number of nilpotent elements are $pq-1$. Also, these $pq-1$ elements are connected to each other. Thus, $pq-1$ elements form a complete graph with $pq-1$ vertices. Therefore, $(pq-1)$ colors are required for coloring the graph. Hence, chromatic number of $N(\mathbb{Z}_{p^2q^2})$ is $(pq-1)$.

(4) If $n = p^3$, where p is a prime number, then $N(\mathbb{Z}_n)$ is a complete p-partite graph with (p^2-1) vertices. Therefore, we required p colors for proper coloring. Hence, chromatic number of $N(\mathbb{Z}_n)$ is p.

(5) Let $n = p^2q$, where p and q are distinct prime numbers. Then the nilpotent elements are multiple of pq, and the number of nilpotent elements are $(p-1)$. These $(p-1)$ elements are connected to each other and form a complete graph with $(p-1)$ vertices. Therefore, $(p-1)$ colors are required for coloring the graph $N(\mathbb{Z}_{p^2q})$. Hence, chromatic number of $N(\mathbb{Z}_{p^2q})$ is $(p-1)$. □

Theorem 2. *Let p and q be two distinct prime numbers and n a positive integer. Then*

(1) $\chi(\Omega(\mathbb{Z}_n)) = m$ *if* $n = p_1p_2p_3\ldots p_m$, $m \geqslant 1$, *where* $p_1, p_2, \ldots, p_m$ *are distinct primes;*
(2) $\chi(\Omega(\mathbb{Z}_n)) = 0$ *if* $n = p^2$*;*
(3) $\chi(\Omega(\mathbb{Z}_n)) = 0$ *if* $n = p^3$*;*
(4) $\chi(\Omega(\mathbb{Z}_n)) = 2$ *if* $n = p^2q$, *for* $q = 2$ *or* 3.

Proof. (1) Let $n = p_1\, p_2\, p_3 \ldots p_m$, for some positive integer m, such that all p_i are distinct prime numbers. Then $\Omega(\mathbb{Z}_n)$ is equal to $\Gamma(\mathbb{Z}_n)$ and since $\Gamma(\mathbb{Z}_n)$ is m-partite graph, therefore $\Omega(\mathbb{Z}_n)$ is also m-partite graph. In this case, m distinct colors are needed for proper coloring of the graph $\Omega(\mathbb{Z}_n)$. Thus, Chromatic number of graph $\Omega(\mathbb{Z}_n)$ is m.

(2) Let $n = p^2$, where p is a prime number. Then clearly $\Omega(\mathbb{Z}_n)$ is an empty graph. Hence, there is no need of any color for coloring the graph $\Omega(\mathbb{Z}_n)$. Hence, chromatic number is zero.

(3) Let $n = p^3$, where p is a prime number. Then $\Omega(\mathbb{Z}_n)$ is an empty graph. Hence, there is no need of any color for coloring the graph $\Omega(\mathbb{Z}_n)$. So, chromatic number is zero.

(4) Let $n = p^2q$, where p and q are distinct prime numbers. Then multiple of p, p^2 and q^2 are not adjacent to themselves. But the vertices which are multiple of p^2 are adjacent to those vertices which are multiple of q and not adjacent with multiple of p. Similarly, elements which are multiple of q are not adjacent with multiple of p. Thus, there are two disjoint sets of vertices which are adjacent from one set to other but not adjacent to each other in a set. Therefore, two colors are required for coloring the $\Omega(\mathbb{Z}_n)$ graph and also we can use one color from them for isolated vertices. Hence, chromatic number is two for $\Omega(\mathbb{Z}_n)$, when $n = p^2q$, where p, q are distinct prime numbers. □

Theorem 3. *If p and q are distinct prime numbers and n is a positive integer, then*

(1) *$N(\mathbb{Z}_n)$ is planar, where $n = pq$;*
(2) *$N(\mathbb{Z}_n)$ is planar for $p \leqslant 5$ and non-planar for $p > 5$, where $n = p^2$;*
(3) *$N(\mathbb{Z}_n)$ is planar for $p \leqslant 5$ and q is any prime number, where $n = p^2q$;*
(4) *$N(\mathbb{Z}_n)$ is planar, if $p < 5$ and non-planar for $p \geqslant 5$, where $n = p^3$;*
(5) *$N(\mathbb{Z}_n)$ is planar, where $n = 4k$, $gcd(2,k) = 1$, $p^2 \nmid k$ for any prime p and k is any positive integer;*
(6) *$N(\mathbb{Z}_n)$ is planar, where $n = 9k$, $gcd(3,k) = 1$, $p^2 \nmid k$ for any prime p and k is any positive integer.*

Proof. (1) If $n = pq$, where p and q are distinct prime numbers, then $N(\mathbb{Z}_n)$ is an empty graph. Therefore, $N(\mathbb{Z}_n)$ graph is a planar graph.

(2) If $n = p^2$, where p is a prime number, then the nilpotent elements of $(\mathbb{Z}_n)$ are multiple of p. So, there are $(p-1)$ nilpotent elements which form a complete graph with $(p-1)$ vertices and all vertices are adjacent to each other. If $p = 2$, then $N(\mathbb{Z}_n)$ has only one vertex and when $p = 3$, then $N(\mathbb{Z}_n)$ has two vertices. In this case, $N(\mathbb{Z}_n)$ is a planar graph. If $p = 5$, then $N(\mathbb{Z}_n)$ is a complete graph with 4 vertices and all vertices are adjacent to each other. Therefore, $N(\mathbb{Z}_n)$ is a planar graph.

For $p > 5$, $N(\mathbb{Z}_n)$ graph contains $K_{3,3}$ or K_5 as a proper subgraph. Hence, $N(\mathbb{Z}_n)$ is not a planar graph for $p > 5$.

(3) If $n = p^2q$, where p and q are distinct prime numbers, then $N(\mathbb{Z}_n)$ is a complete graph with $(p-1)$ vertices. Thus, $N(\mathbb{Z}_n)$ is a planar graph only when $p \leqslant 5$ and q is any prime, $p \neq q$, otherwise $N(\mathbb{Z}_n)$ contains

K_5 as a subgraph which is not planar and therefore $N(\mathbb{Z}_n)$ is a planar if $p \leqslant 5$.

(4) If $n = p^3$, where p is any prime, then $N(\mathbb{Z}_n)$ is a complete p-partite graph with $(p^2 - 1)$ vertices. Therefore, $N(\mathbb{Z}_n)$ is planar for $p < 5$ and non-planar for $p \geqslant 5$.

(5) If $n = 4k$, and $p^2 \not| \, k$, for a prime p and k is any positive integer, then $N(\mathbb{Z}_n)$ has only one vertex, hence $N(\mathbb{Z}_n)$ graph is a planar graph.

(6) If $n = 9k$, $p^2 \not| \, k$, for all prime p and k is any positive integer, then $N(\mathbb{Z}_n)$ has two vertices which are adjacent to each other. Thus, $N(\mathbb{Z}_n)$ is a planar graph. □

Theorem 4. *If p and q are distinct prime numbers and n is a positive integer, then*

(1) $\Omega(\mathbb{Z}_n)$ *is not planar, for $n = pq$, (specially $p \geqslant 5$ and $q \geqslant 3$);*
(2) $\Omega(\mathbb{Z}_n)$ *is planar, for $n = p^2$;*
(3) $\Omega(\mathbb{Z}_n)$ *is planar, for $n = p^3$;*
(4) $\Omega(\mathbb{Z}_n)$ *is planar for $k \leqslant 6$ and non-planar for all $k > 6$, where $n = 4k$, $gcd(2, k) = 1$ and $p^2 \not| \, k$, for a prime p and k is any positive integer;*
(5) $\Omega(\mathbb{Z}_n)$ *is a planar for $k \leqslant 4$ and non-planar for $k \geqslant 5$, where $n = 9k$, $gcd(3, k) = 1$ and $p^2 \not| \, k$, for a prime p and k is any positive integer;*
(6) $\Omega(\mathbb{Z}_n)$ *is planar for $q = 2$ and 3, and p is any prime number, where $n = p^2 q$.*

Proof. (1) Let $n = pq$, such that p and q are distinct primes. Then clearly $\Omega(\mathbb{Z}_n)$ is a bi-partite graph. If, we take $n = pq$ where $p = 2$ and q is any prime number, then $\Omega(\mathbb{Z}_n)$ is a star graph. We know that star graph is a planar graph. Hence, $\Omega(\mathbb{Z}_n)$ is a planar graph in this case. If $p = 3$ and q is any prime number, then $\Omega(\mathbb{Z}_n)$ is a complete bi-partite graph, which is a planar graph. If $p \geqslant 5$ and q is any prime number which is greater than 3, then $\Omega(\mathbb{Z}_n)$ is not a planar graph. Because, in this case, $\Omega(\mathbb{Z}_n)$ graph contain $K_{3,3}$ as a subgraph. Therefore, $\Omega(\mathbb{Z}_n)$ is not a planar graph for $n = pq$.

(2) Let $n = p^2$, where p is any prime number. Then, there are no non-nilpotent elements of $\mathbb{Z}_n$ in $\Omega(\mathbb{Z}_n)$. Therefore, $\Omega(\mathbb{Z}_n)$ is an empty graph. Hence, $\Omega(\mathbb{Z}_n)$ is a planar graph.

(3) Let $n = p^3$, where p is any prime number. Then, there is no non-nilpotent element of $\mathbb{Z}_n$ in $\Omega(\mathbb{Z}_n)$. Therefore, $\Omega(\mathbb{Z}_n)$ is an empty graph. Hence, $\Omega(\mathbb{Z}_n)$ is a planar graph.

(4) Let $n = 4k$, where $p^2 \not| \, k$, for a prime p and k is any positive integer. Then, $\Omega(\mathbb{Z}_n)$ is planar for $k \leqslant 6$. If we take k is any prime number,

then $\Omega(\mathbb{Z}_n)$ is always complete bi-partite graph. We know that complete bi-partite graph is planar graph. Therefore, $\Omega(\mathbb{Z}_n)$ is the planar graph for the prime k. On the other hand, if $k > 6$, then $\Omega(\mathbb{Z}_n)$ graph contains $K_{3,3}$ or K_5 as a subgraph. Thus, for $k > 6$, $\Omega(\mathbb{Z}_n)$ graph is not a planar.

(5) Let $n = 9k$, where $p^2 \nmid k$, for a prime p and k is any positive integer. Then $\Omega(\mathbb{Z}_n)$ is a planar graph for $k \leqslant 4$. For $k \geqslant 5$, $\Omega(\mathbb{Z}_n)$ graph contains $K_{3,3}$ as a subgraph. Therefore, graph is not a planar for $k \geqslant 5$.

(6) Let $n = p^2q$, where p and q are distinct primes. If $q = 2$ and p is any prime number, then $\Omega(\mathbb{Z}_n)$ graph is a star graph. Therefore, $\Omega(\mathbb{Z}_n)$ graph is planar. If $q = 3$ and p is any prime number, then $\Omega(\mathbb{Z}_n)$ graph is a complete bi-partite graph. Therefore, $\Omega(\mathbb{Z}_n)$ is planar graph. For $q \geqslant 5$ and p is any prime greater than 2 (and 3), $\Omega(\mathbb{Z}_n)$ graph contains $K_{3,3}$ or K_5 as a subgraph. Thus, $\Omega(\mathbb{Z}_n)$ is non-planar. □

Lemma 1. *If $n = pq$, where p and q are primes, then there is no isolated vertex in $\Omega(\mathbb{Z}_n)$ graph.*

Proof. If $n = pq$, where p and q are distinct primes, then $\Omega(\mathbb{Z}_n)$ is a complete bi-partite graph. Hence, there is no isolated vertex. When $n = p^2$, for any prime p, then there is no vertex in $\Omega(\mathbb{Z}_n)$. Hence, graph is empty. Thus, in this case again we have no isolated vertex. □

Lemma 2. *If $n = p^3$, for any prime p, then $\Omega(\mathbb{Z}_n)$ graph has no isolated vertex.*

Proof. If $n = p^3$, then zero divisor graph has $p^2 - 1$ elements in which all elements are nilpotent and no element is non-nilpotent. Also all nilpotent elements are adjacent with nilpotent elements, but in $\Omega(\mathbb{Z}_n)$, there are no non-nilpotent elements. Thus, $\Omega(\mathbb{Z}_n)$ is an empty graph. Therefore, $\Omega(\mathbb{Z}_n)$ graph has no isolated vertex. □

Observation 1. *If $n = p^2q$, for p and q are distinct prime numbers, then $\Omega(\mathbb{Z}_n)$ graph has $(p-1)(q-1)$ isolated vertices.*

2. Energy of nilradical and non-nilradical graphs

Theorem 5. *If $n = p^2$, for prime p, then $E(N(\mathbb{Z}_n))$ is $(2p-4)$ and $E(\Omega(\mathbb{Z}_n))$ is zero ($E(\Omega(\mathbb{Z}_n))$ is zero also for p^3).*

Proof. When $n = p^2$, $N(\mathbb{Z}_n)$ is a complete graph with $p-1$ vertices. Then $f(\lambda) = \mid \lambda I_{p-1} - M(N(\mathbb{Z}_n)) \mid = (\lambda - 1)^{p-2}(\lambda + p - 2)$ by [2], where M

is a matrix of order $(p-1)$. If $f(\lambda)=0$, then $\lambda=1,2-p$. Therefore, $\Sigma_{i=1}^{p-1}\mid\lambda_i\mid=2p-4$.

When $n=p^2$, then $\Omega(\mathbb{Z}_n)$ graph is an empty graph. Hence, it has zero energy.

When $n=p^3$, then $\Omega(\mathbb{Z}_n)$ is an empty graph and hence, it has zero energy. □

Theorem 6. *If $n=pq$, where p and q are distinct primes, then energy of $\Omega(\mathbb{Z}_n)$ is $2\sqrt{(p-1)(q-1)}$ and energy of $N(\mathbb{Z}_n)$ is zero.*

Proof. Let $n=pq$, where p and q are two distinct prime. Then $\Omega(\mathbb{Z}_n)$ is a bi-partite graph. Also, its eigen polynomial $f(\lambda)=\mid\lambda I_{p+q-2}$ - $M(\Omega(\mathbb{Z}_n))\mid=(\lambda)^{p+q-4}(\lambda^2-(p-1)(q-1))$, where M is a matrix of order $(p+q-2)$. Thus, nonzero eigenvalues are $\pm\sqrt{(p-1)(q-1)}$ and so $E(\Omega(\mathbb{Z}_n))=2\sqrt{(p-1)(q-1)}$. Also, $N(\mathbb{Z}_n))$ graph has no vertices for distinct primes p and q. Thus, $E(N(\mathbb{Z}_n))$ has no energy. □

Theorem 7. *For $n=p^2q$, energy of $N(\mathbb{Z}_n)$ is $2p-4$, for all distinct primes p and q.*

Proof. Same as above Theorem (5). □

Observation 2. *If $n=p^2q$, then energy of $\Omega(\mathbb{Z}_n)$ is:*
(1) $2\sqrt{pq-2}$, *for $p=2$ and q is any prime number;*
(2) $2\sqrt{pq+p(q-2)}$, *for $p=3$ and q is any prime number;*
(3) $2\sqrt{2pq+2p(q-2)}$, *for $p=5$ and q is any prime number.*

3. Computer program

Now, we offer three algorithms for calculating energy with MATLAB software. These algorithms include several sub-algorithms. It is enough to input n. In the first algorithm at the first stage, we obtain $M(N(\mathbb{Z}_n))$ and plot $N(\mathbb{Z}_n)$ by function `nil_radical_zn2(p)`. At the second stage, we calculate Energy index by using `energy`.

In the second algorithm at the first stage, we obtain $\Omega(N(\mathbb{Z}_n))$ and plot $\Omega(\mathbb{Z}_n)$ by function `non_nil_radical_zn2(p)`. At the second stage, we calculate Energy index by using `energy`.

In third algorithm, we put the value of n and call above two functions together.

First algorithm

```
function Nz=nil_radical_zn2(p)
n=p;
M=[];
for i=1:n-1
for j=1:n-1
if mod(i*i,n)==0
 M=[M, i];
 break;
end
end
end
M
n=length(M);
for i=0:n-1
axes(i+1,:)=[cos(2*pi*i/n),sin(2*pi*i/n)];
end
Nz=zeros(n);
hold on
for i=1:n
plot(axes(i,1),axes(i,2),'*')
if mod(M(i)^2,p)==0
Nz(i,i)=1;
plot(axes(i,1),axes(i,2),'ro')
end
end
for i=1:n-1
for j=i+1:n
if mod(M(i)*M(j),p)==0
Nz(i,j)=1; Nz(j,i)=1;
plot(axes([i,j],1),axes([i,j],2));
end
end
end
eg=eig(Nz)
E=sum(abs(eg))
```

Second algorithm

```
function NNz=non_nil_radical_zn2(p)
n=p;
M=[];
for i=1:n-1
for j=1:n-1
if mod(i*j,n)==0
```

```
 if mod(i*i,n)~=0
 M=[M, i];
 break;
end
end
end
end
M
n=length(M);
for i=0:n-1
axes(i+1,:)=[cos(2*pi*i/n),sin(2*pi*i/n)];
end
NNz=zeros(n);
hold on
for i=1:n
plot(axes(i,1),axes(i,2),'*')
if mod(M(i)^2,p)==0
NNz(i,i)=1;
plot(axes(i,1),axes(i,2),'ro')
end
end
for i=1:n-1
for j=i+1:n
if mod(M(i)*M(j),p)==0
NNz(i,j)=1; NNz(j,i)=1;
plot(axes([i,j],1),axes([i,j],2));
end
end
end
eg=eig(NNz)
E=sum(abs(eg))
```

Third algorithm

```
p=n;
Nz=nil_radical_zn2(p)
figure;
NNz=non_nil_radical_zn2(p)
figure;
```

All above algorithms are also useful for p^3. If we use the formula "if mod(i*j,n)==0" at the place of sixth line in the first algorithm, then it will give fruitful result for p^3.

n	$E(N(\mathbb{Z}_n))$	$E(\Omega(\mathbb{Z}_n))$
27	7.2111	0
45	2	9.7980
77	0	15.4919
121	18	0
225	26.00	21.9089
343	32.3110	0

TABLE 1. The values of $E(N(\mathbb{Z}_n))$ and $E(\Omega(\mathbb{Z}_n))$ for $n = 27, 45, 77, 121, 225$ and 343.

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