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RESEARCH ARTICLE

Spectral properties of partial automorphisms of a binary rooted tree

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Abstract. We study asymptotics of the spectral measure of a randomly chosen partial automorphism of a rooted tree. To every partial automorphism x we assign its action matrix A_x. It is shown that the uniform distribution on eigenvalues of A_x converges weakly in probability to δ_0 as $n \to \infty$, where δ_0 is the delta measure concentrated at 0.

Introduction

We consider a semigroup of partial automorphisms of a binary n-level rooted tree. Throughout the paper by a partial automorphism we mean root-preserving injective tree homomorphism defined on a connected subtree. This semigroup was studied, in particular, in [4, 5].

We are interested in spectral properties of this semigroup. There is a lot of paper dealing with spectrum of action matrices for the action of finitely generated groups on a regular rooted tree. The exhaustive research on spectra of fractal groups is provided in [1]. The eigenvalues of random wreath product of symmetric group were studied by Evans in [2]; he assigned equal probabilities to the eigenvalues of a randomly chosen automorphism of a regular rooted tree, and considered the random measure Θ_n on the unit circle C. He has shown that Θ_n converges weakly

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in probability to λ as $n \to \infty$, where λ is the normalized Lebesgue measure on the unit circle.

Let $B_n = \{v_i^n \mid i = 1, \dots, 2^n\}$ be the set of vertices of the nth level of the n-level binary rooted tree. To a randomly chosen partial automorphism x, we assign the action matrix $A_x = \left(\mathbf{1}_{\{x(v_i^n)=v_j^n\}}\right)_{i,j=1}^{2^n}$. Let

$$\Xi_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \delta_{\lambda_k}$$

be the uniform distribution on eigenvalues of A_x. We show that Ξ_n converges weakly in probability to δ_0 as $n \to \infty$, where δ_0 is the delta measure concentrated at 0. This result can be generalized to a regular rooted tree, however, this generalization is not straightforward and will be studied elsewhere.

The remaining of the paper is organized as follows. Section 2 contains basic facts on a partial wreath product of semigroup and its connection with a semigroup of partial automorphisms of a regular rooted tree. The main result is stated and proved in Section 3.

1. Preliminaries

For a set $X = \{1, 2\}$ consider the set $\mathcal{I}_2$ of all partial bijections. List all of them using standard tableax representation:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & \varnothing \end{pmatrix} \right\}.$$

This set forms an inverse semigroup under natural composition law, namely, $f \circ g : \operatorname{dom}(f) \cap f^{-1} \operatorname{dom}(g) \ni x \mapsto g(f(x))$ for $f,\ g \in \mathcal{I}_2$. Obviously, $\mathcal{I}_2$ is a particular case of the well-known inverse symmetric semigroup. Detailed description of it can be found in [3, Chapter 2].

Recall the definition of a partial wreath product of semigroups. Let S be an arbitrary semigroup. For functions $f\colon\ \operatorname{dom}(f) \to S$, $g\colon\ \operatorname{dom}(g) \to S$ define the product fg as:

$$\operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g),\ \ (fg)(x) = f(x)g(x) \text{ for all } x \in \operatorname{dom}(fg).$$

For $a \in \mathcal{I}_2$, $f\colon\ \operatorname{dom}(f) \to S$, define f^a as:

$$(f^a)(x) = f(x^a),\ \operatorname{dom}(f^a) = \{x \in \operatorname{dom}(a); x^a \in \operatorname{dom}(f)\}.$$

Definition 1. The partial wreath square of the semigroup $\mathcal{I}_2$ is the set

$$\{(f, a) \mid a \in \mathcal{I}_2, f\colon \operatorname{dom}(a) \to \mathcal{I}_2\}$$

with composition defined by

$$(f, a) \cdot (g, b) = (fg^a, ab)$$

Denote it by $\mathcal{I}_2 \wr_p \mathcal{I}_2$.

The partial wreath square of $\mathcal{I}_2$ is a semigroup, moreover, it is an inverse semigroup [6, Lemmas 2.22 and 4.6]. We may recursively define any partial wreath power of the finite inverse symmetric semigroup. Denote by $\mathcal{P}_n$ the nth partial wreath power of $\mathcal{I}_2$.

Definition 2. The partial wreath n-th power of semigroup $\mathcal{I}_2$ is defined as a semigroup

$$\mathcal{P}_n = \big(\mathcal{P}_{n-1}\big) \wr_p \mathcal{I}_2 = \{(f, a) \mid a \in \mathcal{I}_2,\ f\colon \operatorname{dom}(a) \to \mathcal{P}_{n-1}\}$$

with composition defined by

$$(f, a) \cdot (g, b) = (fg^a, ab),$$

where $\mathcal{P}_{n-1}$ is the partial wreath $(n-1)$-th power of semigroup $\mathcal{I}_2$

Proposition 1. *Let N_n be the number of elements in the semigroup $\mathcal{P}_n$. Then $N_n = 2^{2^{n+1}-1} - 1$*

Proof. We proceed by induction.

If $n = 1$, then $2^{2^2-1} - 1 = 7$. This is exactly the number of elements in $\mathcal{I}_2$.

Assume that $N_{n-1} = 2^{2^n-1} - 1$. Then

$$\begin{aligned} N_n &= |\{(f, a) \mid a \in \mathcal{I}_2, f\colon \operatorname{dom}(a) \to N_{n-1}\}| \\ &= \sum_{a\in\mathcal{I}_2} N_{n-1}^{|\operatorname{dom}(a)|} = \sum_{a\in\mathcal{I}_2} \big(2^{2^n-1} - 1\big)^{|\operatorname{dom}(a)|} \\ &= 1 + 4 \cdot (2^{2^n-1} - 1) + 2 \cdot (2^{2^n-1} - 1)^2 \\ &= 1 + 4 \cdot 2^{2^n-1} - 4 + 2 \cdot 2^{2^{n+1}-2} - 4 \cdot 2^{2^n-1} + 2 = 2^{2^{n+1}} - 1. \end{aligned}$$ □

Remark 1. Let T be an n-level binary rooted tree. We define a partial automorphism of a tree T as an isomorphism $x : \Gamma_1 \to \Gamma_2$ of subtrees Γ_1 and Γ_2 of T containing a root. Denote $\operatorname{dom}(x) := \Gamma_1$, $\operatorname{ran}(x) := \Gamma_2$ domain and image of x respectively. Let $\operatorname{PAut} T$ be the set of all partial automorphisms of T. Obviously, $\operatorname{PAut} T$ forms a semigroup under natural composition law. It was proved in [4, Theorem 1] that the partial wreath power $\mathcal{P}_n$ is isomorphic to $\operatorname{PAut} T$.

2. Asymptotic behaviour of a spectral measure of a binary rooted tree

We identify $x \in \mathcal{P}_n$ with a partial automorphism from $\operatorname{PAut} T$. Recall, that B_n denotes the set of vertices of the nth level of T. Clearly, $|B_n| = 2^n$. Let us enumerate the vertices of B_n by positive integers from 1 to 2^n:

$$B_n = \{v_i^n \mid i = 1, \dots, 2^n\}.$$

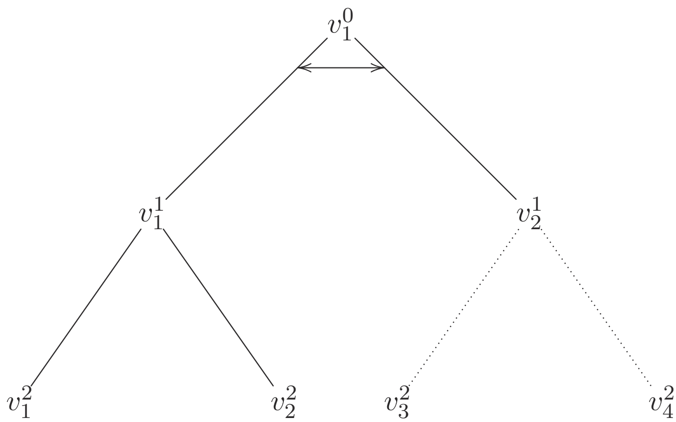
To a randomly chosen transformation $x \in \mathcal{P}_n$, we assign the matrix

$$A_x = \left(\mathbf{1}_{\{x(v_i^n)=v_j^n\}}\right)_{i,j=1}^{2^n}.$$

In other words, (i, j)th entry of A_x is equal to 1, if a transformation x maps v_i^n to v_j^n, and 0, otherwise.

Remark 2. In an automorphism group of a tree such a matrix describes completely the action of an automorphism. Unfortunately, for a semigroup this is not the case.

Example 1. Consider the partial automorphism $x \in \mathcal{P}_2$, which acts in the following way



(dotted lines mean that these edges are not in domain of x).

Then the corresponding matrix for x is

$$A_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that if v_2^1 were not in the $\mathrm{dom}(x)$ with action on other vertices preserved, then the corresponding matrix would be the same.

Let $\chi_x(\lambda)$ be the characteristic polynomial of A_x and $\lambda_1, \ldots, \lambda_{2^n}$ be its roots respecting multiplicity. Denote

$$\Xi_n = \frac{1}{2^n} \sum_{k=1}^{2^n} \delta_{\lambda_k}$$

the uniform distribution on eigenvalues of A_x.

Theorem 1. *For any function $f \in C(D)$, where $D = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ is a unit disc,*

$$\int_D f(x)\, \Xi_n(dx) \xrightarrow{\mathbb{P}} f(0), \quad n \to \infty. \tag{1}$$

In other words, Ξ_n converges weakly in probability to δ_0 as $n \to \infty$, where δ_0 is the delta-measure concentrated at 0.

Remark 3. Evans [2] has studied asymptotic behaviour of a spectral measure of a randomly chosen element σ of n-fold wreath product of the symmetric group $\mathcal{S}_d$.

He considered the random measure Θ_n on the unit circle C, assigning equal probabilities to the eigenvalues of σ.

Evans has shown that if f is a trigonometric polynomial, then

$$\lim_{n\to\infty} \mathbb{P}\left\{ \int_C f(x)\, \Theta_n(dx) \neq \int f(x)\, \lambda(dx) \right\} = 0,$$

where λ is the normalized Lebesgue measure on the unit circle. Consequently, Θ_n converges weakly in probability to λ as $n \to \infty$.

In fact, Theorem 1 speaks about the number of non-zero roots of characteristic polynomial $\chi_x(\lambda)$. Let us find an alternative description for them. Denote

$$S_n(x) = \bigcap_{m\geqslant 1} \mathrm{dom}(x^m) = \left\{ v_j^n \mid v_j^n \in \mathrm{dom}(x^m) \text{ for all } m \geqslant 1 \right\}$$

the vertices of the nth level, which "survive" under the action of x, and define the *ultimate rank* of x by $\mathrm{rk}_n(x) = |S_n(x)|$. Let R_n denote the total number of these vertices over all $x \in \mathcal{P}_n$, that is

$$R_n = \sum_{x\in\mathcal{P}_n} \mathrm{rk}_n(x).$$

We call the number R_n the *total ultimate rank.*

Lemma 1. *For $x \in \mathcal{P}_n$ the number of non-zero roots of χ_x with regard for multiplicity is equal to the ultimate rank* $\mathrm{rk}_n(x)$ *of x.*

Proof. Let $x \in \mathcal{P}_n$ and A_x be its action matrix. Consider A_x as a matrix in a standard basis. Let w be some basis vector. It follows from the definition of A_x that there are two possibilities: if the vertex v corresponding to w is in domain of x, then A_x sends w to another basis vector, otherwise, to the zero vector. Since x is a partial bijection, applying A_x repeatedly, we can either get the same vector or the zero vector; $A_x^n w = 0$ means that $v \notin \operatorname{dom} x^n$. In the first case, the vector w corresponds to a non-zero root of χ_x (some root of unity), and the vertex v contributes to the ultimate rank. In the second case, the vector is a root vector for the zero eigenvalue, so it corresponds to a zero root of A_x, while the corresponding vector does not contribute to the ultimate rank. □

Denote $\mathrm{rank}_n(x) = |\operatorname{dom}(x) \cap B_n|$ and define the *total rank*

$$R'_n = \sum_{x \in \mathcal{P}_n} \mathrm{rank}_n(x).$$

Remark 4. Clearly, for $x = (f, a)$, where $a \in \mathcal{I}_2$, $f\colon \operatorname{dom}(a) \to \mathcal{P}_{n-1}$,

$$\mathrm{rank}_n(x) = \sum_{y \in \operatorname{dom}(a)} \mathrm{rank}_{n-1}(f(y)) \tag{2}$$

if $\operatorname{dom}(a) \neq \varnothing$ and $\mathrm{rank}_n(x) = 0$ otherwise.

Lemma 2. *Let R'_n be the total rank of the semigroup $\mathcal{P}_n$. Then*

$$R'_n = 4R'_{n-1} + 4R'_{n-1}N_{n-1}.$$

Proof. Thanks to (2),

$$\begin{aligned}
R'_n &= \sum_{x=(f,a) \in \mathcal{P}_n} \mathrm{rank}_n(x) = \sum_{x=(f,a) \in \mathcal{P}_n} \sum_{y \in \operatorname{dom}(a)} \mathrm{rank}_{n-1}(f(y)) \\
&= \sum_{\substack{a \in \mathcal{I}_2 \\ |\operatorname{dom}(a)|=1}} \sum_{f_1 \in \mathcal{P}_{n-1}} \mathrm{rank}_{n-1}(f_1) \\
&\quad + \sum_{\substack{a \in \mathcal{I}_2 \\ |\operatorname{dom}(a)|=2}} \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} (\mathrm{rank}_{n-1}(f_1) + \mathrm{rank}_{n-1}(f_2)) \\
&= 4R'_{n-1} + 2 \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} (\mathrm{rank}_{n-1}(f_1) + \mathrm{rank}_{n-1}(f_2))\,.
\end{aligned}$$

We have

$$\sum_{f_1,f_2\in\mathcal{P}_{n-1}} \operatorname{rank}_{n-1}(f_1) = \sum_{f_1\in\mathcal{P}_{n-1}} \sum_{f_2\in\mathcal{P}_{n-1}} \operatorname{rank}(f_1)$$
$$= \sum_{f_1\in\mathcal{P}_{n-1}} N_{n-1} \operatorname{rank}(f_1) = N_{n-1}R'_{n-1}.$$

Hence, by symmetry, $R'_n = 4R'_{n-1} + 4R'_{n-1}N_{n-1}$. □

Lemma 3. *Let R'_n be the total rank of the semigroup $\mathcal{P}_n$. Then*

$$R'_n = 2^{n-1}(1+N_n) = 2^{2^n+n-2}.$$

Proof. We proceed by induction. A direct calculation gives

$$R'_1 = 8 = 1 + N_1.$$

Assuming that

$$R'_{n-1} = 2^{2^{n-1}+n-3},$$

we have, thanks to Lemma 2 and Proposition 1,

$$R'_n = 4R'_{n-1}(1+N_{n-1}) = 4\cdot 2^{2^{n-1}+n-3}\cdot 2^{2^{n-1}-1} = 2^{2^n+n-2},$$

as required. □

Lemma 4. *Let R_n be the total ultimate rank of the semigroup $\mathcal{P}_n$. Then*

$$R_n \leqslant 3R_{n-1} + 3R_{n-1}N_{n-1}.$$

Proof. Represent R_n as a sum

$$R_n = \sum_{x=(f,a)\in\mathcal{P}_n} \operatorname{rk}_n(x) = \sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ |\operatorname{dom}(a)|=1}} \operatorname{rk}_n(x) + \sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ |\operatorname{dom}(a)|=2}} \operatorname{rk}_n(x).$$

If $\operatorname{rank}(a)=1$, then we will be interested only in those a for which $a=(1)$ and $a=(2)$, since otherwise the ultimate rank of x is 0. Therefore,

$$\sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ |\operatorname{dom}(a)|=1}} \operatorname{rk}_n(x) = \sum_{a\in\{(1),(2)\}} \sum_{f_1\in\mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(f_1)$$
$$= 2\sum_{f_1\in\mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(f_1) = 2R_{n-1}.$$

If $\operatorname{rank}(a) = 2$, then

$$\sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ |\operatorname{dom}(a)|=2}} \operatorname{rk}_n(x) = \sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ a=(1)(2)}} \operatorname{rk}_n(x) + \sum_{\substack{x=(f,a)\in\mathcal{P}_n\\ a=(12)}} \operatorname{rk}_n(x) =: S_1 + S_2.$$

Clearly, if $x = (f, a)$ with $a = (1)(2)$, then $\operatorname{rk}_n(x) = \operatorname{rk}_{n-1}(f(1)) + \operatorname{rk}_{n-1}(f(2))$, whence

$$\begin{aligned} S_1 &= \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} (\operatorname{rk}_{n-1}(f_1) + \operatorname{rk}_{n-1}(f_2)) \\ &= 2 \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(f_1) = 2R_{n-1}N_{n-1}. \end{aligned}$$

Further, if $x = (f, a)$ with $a = (12)$, then $\operatorname{rk}_n(x) = 2\operatorname{rk}_{n-1}(f(1)(f(2))$. So,

$$S_2 = 2 \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(f_1 f_2).$$

Note that every element $x \in \mathcal{P}_n$ can be represented as a product $x = e\sigma$, where e is an idempotent on $\operatorname{dom}(x)$ and σ is a permutation on the set of the tree vertices. Then

$$\sum_{x_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(x_1 x_2) = \sum_{x_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(e\sigma x_2) = \sum_{x_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(e x_2).$$

The last equality is true since transformation $x \mapsto \sigma x$ is bijective on $\mathcal{P}_{n-1}$.

It follows from above that

$$\begin{aligned} S_2 &= 2 \sum_{f_1, f_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(f_1 f_2) = 2 \sum_{f_1 \in \mathcal{P}_{n-1}} \sum_{f_2 \in \mathcal{P}_{n-1}} \operatorname{rk}_{n-1}(\operatorname{id}_{\operatorname{dom}(f_1)} f_2) \\ &\leqslant 2 \sum_{f_1 \in \mathcal{P}_{n-1}} \sum_{f_2 \in \mathcal{P}_{n-1}} |\operatorname{dom}(f_1) \cap S_{n-1}(f_2)| \\ &= 2 \sum_{f_1 \in \mathcal{P}_{n-1}} \sum_{f_2 \in \mathcal{P}_{n-1}} \sum_{j=1}^{2^{n-1}} \mathbf{1}_{\{v_j^{n-1} \in \operatorname{dom}(f_1)\}} \cdot \mathbf{1}_{\{v_j^{n-1} \in S_{n-1}(f_2)\}} \\ &= 2 \sum_{j=1}^{2^{n-1}} \sum_{f_1 \in \mathcal{P}_{n-1}} \mathbf{1}_{\{v_j^{n-1} \in \operatorname{dom}(f_1)\}} \cdot \sum_{f_2 \in \mathcal{P}_{n-1}} \mathbf{1}_{\{v_j^{n-1} \in S_{n-1}(f_2)\}}. \end{aligned}$$

Thanks to symmetry, for each j, we have

$$\sum_{x\in\mathcal{P}_n} \mathbf{1}_{\{v_j^{n-1}\in\mathrm{dom}(x)\}} = \sum_{x\in\mathcal{P}_n} \mathbf{1}_{\{v_1^{n-1}\in\mathrm{dom}(x)\}}.$$

Therefore,

$$\frac{1}{2^n}\sum_{x\in\mathcal{P}_n}\sum_{k=1}^{2^{n-1}} \mathbf{1}_{\{v_k^{n-1}\in\mathrm{dom}(x)\}} = \frac{1}{2^{n-1}}\left|\mathrm{dom}(x)\right|.$$

Thus, we can write

$$\begin{aligned} S_2 &= 2\sum_{j=1}^{2^{n-1}}\sum_{f_1\in\mathcal{P}_{n-1}} \frac{1}{2^{n-1}}\left|\mathrm{dom}(f_1)\right| \mathbf{1}_{\{v_j^{n-1}\in S_{n-1}(f_2)\}} \\ &= \frac{1}{2^{n-1}}\sum_{f_1\in\mathcal{P}_n}\left|\mathrm{dom}(f_1)\right|\sum_{j=1}^{2^n}\mathbf{1}_{\{v_j^{n-1}\in S_{n-1}(f_2)\}} \\ &= 2\cdot\frac{1}{2^{n-1}}\sum_{f_1\in\mathcal{P}_{n-1}}\left|\mathrm{dom}(f_1)\right|\cdot\sum_{f_2\in\mathcal{P}_{n-1}}\left|S_{n-1}(f_2)\right|. \end{aligned}$$

Using that $|S_{n-1}(f_2)| = \mathrm{rk}_{n-1}(f_2)$, $|\mathrm{dom}(f_1)| = \mathrm{rank}(f_1)$, and applying Lemma 3, we get

$$\begin{aligned} \frac{2}{2^{n-1}} R_{n-1}R'_{n-1} &= \frac{2R_{n-1}\cdot(1+N_{n-1})2^{n-2}}{2^{n-1}} \\ &= \frac{2R_{n-1}(1+N_{n-1})}{2} = (1+N_{n-1})R_{n-1}. \end{aligned}$$

Therefore,

$$R_n \leqslant 2R_{n-1}+2R_{n-1}N_{n-1}+(1+N_{n-1})R_{n-1} = 3R_{n-1}+3R_{n-1}N_{n-1}. \quad \square$$

Lemma 5. *Let* $p_n = \dfrac{R_n}{2^n N_n}$ *for* $n\in\mathbb{N}$. *Then*

$$p_n \leqslant \frac{3}{4}p_{n-1}, \quad n\geqslant 2.$$

Proof. Using Lemma 4, we get

$$\begin{aligned} p_n &= \frac{R_n}{2^nN_n} \leqslant \frac{3R_{n-1}+3R_{n-1}N_{n-1}}{2^nN_n} = \frac{3R_{n-1}(1+N_{n-1})}{2^nN_n} \\ &= \frac{3R_{n-1}\cdot 2^{2^n-1}}{2^n(2^{2^{n+1}-1}-1)} = \frac{3}{2}\cdot\frac{R_{n-1}}{2^{n-1}(2^{2^n}-2^{1-2^n})} \leqslant \frac{3}{2}\cdot\frac{R_{n-1}}{2^{n-1}(2^{2^n}-2)} \\ &= \frac{3}{2}\cdot\frac{R_{n-1}}{2^n(2^{2^n-1}-1)} = \frac{3}{4}\cdot\frac{R_{n-1}}{2^{n-1}N_{n-1}} = \frac{3}{4}\cdot p_{n-1}. \end{aligned} \quad \square$$

Proof of Theorem 1. Note that $\int_D f(z)\Xi_n(dz) = \frac{1}{2^n}\sum_{k=1}^{2^n} f(\lambda_k)$, where $\lambda_1, \ldots, \lambda_{2^n}$ are the roots of the characteristic polynomial $\chi_x(\lambda)$.

Then, thanks to Lemma 1,

$$\left|\int_D f(z)\Xi_n(dz) - f(0)\right| = \left|\int_D (f(z) - f(0))\,\Xi(dz)\right|$$
$$\leqslant \frac{1}{2^n}\sum_{k:\lambda_k\neq 0} |(f(k) - f(0))| \leqslant 2\max_D |f| \cdot \frac{|k : \lambda_k \neq 0|}{2^n} = 2\max_D |f|\,\frac{\mathrm{rk}_n(x)}{2^n}.$$

Therefore,

$$\mathbb{E}\left|\int_D f(z)\Xi(dz) - f(0)\right| \leqslant 2\max|f| \cdot \frac{\sum_{z\in\mathcal{P}_n}\mathrm{rk}_n(x)}{2^n N_n} = 2\max|f|\cdot p_n,$$

where p_n is defined in Lemma 5. By Lemma 5, $p_n \leqslant \frac{3}{4}\cdot p_{n-1} \leqslant \ldots \leqslant (\frac{3}{4})^{n-1}p_0$, whence $p_n \to 0$, $n \to \infty$.

Consequently, $\mathbb{E}\left|\int_D f(z)\Xi(dz) - f(0)\right| \to 0$, $n \to \infty$, whence the statement follows. □

Remark 5. We can see from the proof that the rate of convergence in (1) is in some sense exponential.

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