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Parafunctions of triangular matrices and m-ary partitions of numbers

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Abstract. Using the machinavy of paradeterminants and parapermanents developed in [2] we get new relations for some number-theoretical functions natural argument that were studied in [3].

Introduction

Partition polynomials together with corresponding linear recurrent equations appear in different areas of mathematics. Therefore, it is important to develop the general theory of partition polynomials, which would unify the results obtained in this area of mathematics. One of these general approaches to studying partition polynomials and its corresponding linear recurrent equations is their study with the help of triangular matrices (tables) machinery [1, 2].

The present paper continues the study of properties and interrelations of three number-theoretical functions of a natural argument, which was started in [3]. These functions are the functions $b_m(n), \xi_m(n)$, ([3], p.68-69.) respectively generalizing the number $p(n)$ of unordered partitions of a positive integer n into positive integer summands and the sum of divisors of a positive integer $\sigma(n)$, as well as the function $d_m(n)$, which for $m = 2$ equals $(-1)^{t(n)}$, where $t(n)$ is the n-th term of the Prouhet-Thue-Morse

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sequence [4]. In [3] the authors apply the methods of combinatorial analysis (generatrix method) and linear algebra. As for us, in order to study these functions, we use the general theory of partition polynomials developed with the help of triangular matrix calculus machinery developed by the first autor. At that, we received new relations between these functions and all the proofs are considerably simplified. As the result we get a seweral new relations between functions $b_m(n), \xi_m(n)$ and $d_m(n)$ and express them via paradeterminants and parapermanents.

1. Preliminaries

This section includes some necessary notions and their properties, which will be needed in the next section.

1.1. Some notions and results concerning triangular matrices

In this section we provide basic notions and results about paradeterminants and parapermanents that will be used for the proving of the main results of the paper.

Let K be some number field.

Definition 1 ([1, 2])**.** A triangular table

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

of numbers from a number field K is called *a triangular matrix*, an element a_{11} is an upper element of this triangular matrix, and a number n is its order.

Definition 2 ([1, 2])**.** If A is the triangular matrix (1), then its paradeterminant and parapermanent are the following numbers, respectively:

$$\operatorname{ddet}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\}, \tag{2}$$

$$\operatorname{pper}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

where the summation is over the set of natural solutions of the equality $p_1 + \ldots + p_r = n$ and

$$b_{ij} = \{a_{ij}\} = \prod_{k=j}^{i} a_{ik}, \quad 1 \leqslant j \leqslant i \leqslant n.$$

For a parapermanent and paradeterminant of a matrix we will use notations shown in (15) and (16) respectively.

Definition 3 ([1,2])**.** To each element a_{ij} of the given triangular matrix (1) we correspond a triangular matrix with this element in the bottom left corner, which we call *a corner* of the given triangular matrix and denote by $R_{ij}(A)$.

It is obvious that the corner $R_{ij}(A)$ is a triangular matrix of order $(i-j+1)$. The corner $R_{ij}(A)$ includes only those elements a_{rs} of the triangular matrix (1), the indices of which satisfy the relations $j \leqslant s \leqslant r \leqslant i$.

Sometimes we extend the range of indeces in (1) from 1,...,n to 0,1,...,n+1 and agree that

$$\begin{aligned} \mathrm{ddet}(R_{01}(A)) = \mathrm{ddet}(R_{n,n+1}(A)) = \mathrm{pper}(R_{01}(A)) \\ = \mathrm{pper}(R_{n,n+1}(A)) = 1. \end{aligned} \tag{3}$$

When finding values of the paradeterminant and the parapermanent of triangular matrices, it is convenient to use *algebraic complements.*

Definition 4 ([1,2])**.** *Algebraic complements D_{ij}, P_{ij} to a factorial product* $\{a_{ij}\}$ of a key element a_{ij} of the matrix (1) are, respectively, numbers

$$D_{ij} = (-1)^{i+j} \cdot \mathrm{ddet}(R_{j-1,1}) \cdot \mathrm{ddet}(R_{n,i+1}), \tag{4}$$

$$P_{ij} = \mathrm{pper}(R_{j-1,1}) \cdot \mathrm{pper}(R_{n,i+1}), \tag{5}$$

where $R_{j-1,1}$ and $R_{n,i+1}$ are corners of the triangular matrix (1).

Theorem 1 ([1,2])**.** *If A is the triangular matrix (1), then the parafunctions of this matrix can be decomposed by the elements of the last row. With that, the following equalities hold:*

$$\operatorname{ddet}(A)=\sum_{s=1}^{n}\{a_{ns}\}\,D_{ns}=\sum_{s=1}^{n}(-1)^{n+s}\{a_{ns}\}\cdot\operatorname{ddet}(R_{s-1,1}), \tag{6}$$

$$\operatorname{pper}(A)=\sum_{s=1}^{n}\{a_{ns}\}\,P_{ns}=\sum_{s=1}^{n}\{a_{ns}\}\cdot\operatorname{pper}(R_{s-1,1}), \tag{7}$$

where

$$b_{ij}=\{a_{ij}\}=\prod_{k=j}^{i}a_{ik},\quad 1\leqslant j\leqslant i\leqslant n.$$

Theorem 2 (Relation between parapermanent and paradeterminant [1, 2])**.** *If A is the triangular matrix* (1)*, then the following relation holds*

$$\operatorname{pper}\left(a_{ij}\right)_{1\leqslant j\leqslant i\leqslant n}=\operatorname{ddet}\left((-1)^{\delta_{ij}+1}a_{ij}\right)_{1\leqslant j\leqslant i\leqslant n}. \tag{8}$$

Corollary 1. *For any triangular matrix $(b_{ij})_{1\leqslant j\leqslant i\leqslant n}$, the following equality holds*

$$\operatorname{ddet}(b_{ij})_{1\leqslant j\leqslant i\leqslant n}=\operatorname{pper}((-1)^{\delta_{ij}+1}b_{ij})_{1\leqslant j\leqslant i\leqslant n}.$$

Theorem 3 ([5])**.** *The following is true*

$$\begin{vmatrix} a_{11} & a_1 & 0 & \dots & 0 & 0 & 0\\ a_{21} & a_{22} & a_2 & \dots & 0 & 0 & 0\\ a_{31} & a_{32} & a_{33} & \dots & 0 & 0 & 0\\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots\\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \dots & a_{n-2,n-2} & a_{n-2} & 0\\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1}\\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix}$$

$$=\left\langle\begin{matrix} a_{11} & & & & & & \\ a_1\frac{a_{21}}{a_{22}} & a_{22} & & & & & \\ a_1\frac{a_{31}}{a_{32}} & a_2\frac{a_{32}}{a_{33}} & a_{33} & & & & \\ \vdots & \dots & \dots & \ddots & & & \\ a_1\frac{a_{n-2,1}}{a_{n-2,2}} & a_2\frac{a_{n-2,2}}{a_{n-2,3}} & a_3\frac{a_{n-2,3}}{a_{n-2,4}} & \dots & a_{n-2,n-2} & & \\ a_1\frac{a_{n-1,1}}{a_{n-1,2}} & a_2\frac{a_{n-1,2}}{a_{n-1,3}} & a_3\frac{a_{n-1,3}}{a_{n-1,4}} & \dots & a_{n-2}\frac{a_{n-1,n-2}}{a_{n-1,n-1}} & a_{n-1,n-1} & \\ a_1\frac{a_{n1}}{a_{n2}} & a_2\frac{a_{n2}}{a_{n3}} & a_3\frac{a_{n3}}{a_{n4}} & \dots & a_{n-2}\frac{a_{n,n-2}}{a_{n,n-1}} & a_{n-1}\frac{a_{n,n-1}}{a_{nn}} & a_{nn} \end{matrix}\right\rangle \tag{9}$$

1.2. Some data from the general theory of partition polynomials

We will need also the following three results.

Theorem 4 ([2], Theorem 2.5.3)**.** *The following three equalities are equipotent:*

$$A_n = \left\langle \begin{matrix} x(1) & & & & \\ \frac{x(2)}{x(1)} & x(1) & & & \\ \vdots & \dots & \ddots & & \\ \cdot\frac{x(n-1)}{x(n-2)} & \frac{x(n-2)}{x(n-3)} & \dots & x(1) & \\ \frac{x(n)}{x(n-1)} & \frac{x(n-1)}{x(n-2)} & \dots & \frac{x(2)}{x(1)} & x(1) \end{matrix} \right\rangle,$$

$$A_n = x_1 A_{n-1} - x_2 A_{n-2} + \ldots + (-1)^{n-1} x_n A_0, \quad A_0 = 1,$$

$$A_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k} \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} \cdot \ldots x_n^{\lambda_n},\ k = \lambda_1 + \ldots + \lambda_n.$$

Theorem 5 ([7])**.** *Let polynomials* $y_n(x_1, x_2, \ldots, x_n), n = 0, 1, \ldots,$ *satisfy the recurrence relation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} a_n x_n y_0, \quad (10)$$

where $y_0 = 1$. *Then the relations*

$$y_n = \operatorname{ddet} \begin{pmatrix} a_1 x_1 & & & \\ a_2 \frac{x_2}{x_1} & x_1 & & \\ \vdots & \dots & \ddots & \\ a_n \frac{x_n}{x_{n-1}} & \dots & \frac{x_2}{x_1} & x_1 \end{pmatrix}, \quad (11)$$

$$y_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k} \Big(\sum_{i=1}^{n} \lambda_i a_i \Big) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n}, \quad (12)$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, *hold.*

Theorem 6 ([2], Theorem 3.6.1)**.** *The following formulae of inversion of partition polynomials written as parafunctions of triangular matrices are valid:*

$$1) \qquad b_i = \left\langle \tau_{sr} \frac{a_{s-r+1}}{a_{s-r}} \right\rangle_{1\leqslant r\leqslant s\leqslant i}, \tag{13}$$

$$a_i = \left\langle \tau^{-1}_{s,s-r+1} \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1\leqslant r\leqslant s\leqslant i}, \quad i = 1,2,\ldots; \tag{14}$$

$$2) \qquad b_i = \left[\tau_{sr} \frac{a_{s-r+1}}{a_{s-r}} \right]_{1\leqslant r\leqslant s\leqslant i},$$

$$a_i = (-1)^{i-1} \left\langle \tau^{-1}_{s,s-r+1} \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1\leqslant r\leqslant s\leqslant i}, \quad i = 1,2,\ldots,$$

where a_i, b_i are arbitrary real variables, τ_{rs} are rational numbers.

2. Parafunctions of triangular matrices and *m*-ary partitions of numbers

Our first theorem show how functions $b_m(n)$, $\xi_m(n)$, $d_m(n)$, studied in [3] can be expressed via paradeterminant and parapermanent.

Theorem 7. *The following equalities hold:*

$$b_m(n) = \left[\begin{array}{ccccc} \xi_m(1) & & & & \\ \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{2}\xi_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ \frac{\xi_m(n-1)}{\xi_m(n-2)} & \frac{\xi_m(n-2)}{\xi_m(n-3)} & \ldots & \frac{1}{n-1}\xi_m(1) & \\ \frac{\xi_m(n)}{\xi_m(n-1)} & \frac{\xi_m(n-1)}{\xi_m(n-2)} & \ldots & \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{n}\xi_m(1) \end{array}\right], \tag{15}$$

$$d_m(n) = (-1)^n \left\langle\begin{array}{ccccc} \xi_m(1) & & & & \\ \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{2}\xi_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ \frac{\xi_m(n-1)}{\xi_m(n-2)} & \frac{\xi_m(n-2)}{\xi_m(n-3)} & \ldots & \frac{1}{n-1}\xi_m(1) & \\ \frac{\xi_m(n)}{\xi_m(n-1)} & \frac{\xi_m(n-1)}{\xi_m(n-2)} & \ldots & \frac{\xi_m(2)}{\xi_m(1)} & \frac{1}{n}\xi_m(1) \end{array}\right\rangle, \tag{16}$$

$$\xi_m(n) = (-1)^{n-1} \left\langle\begin{array}{ccccc} b_m(1) & & & & \\ 2\cdot\frac{b_m(2)}{b_m(1)} & b_m(1) & & & \\ \vdots & \ldots & \ddots & & \\ (n-1)\cdot\frac{b_m(n-1)}{b_m(n-2)} & \frac{b_m(n-2)}{b_m(n-3)} & \ldots & b_m(1) & \\ n\cdot\frac{b_m(n)}{b_m(n-1)} & \frac{b_m(n-1)}{b_m(n-2)} & \ldots & \frac{b_m(2)}{b_m(1)} & b_m(1) \end{array}\right\rangle, \tag{17}$$

$$\xi_m(n) = (-1)^n \left\langle \begin{array}{ccccc} d_m(1) & & & & \\ 2\cdot\frac{d_m(2)}{d_m(1)} & d_m(1) & & & \\ \vdots & \dots & \ddots & & \\ (n-1)\cdot\frac{d_m(n-1)}{d_m(n-2)} & \frac{d_m(n-2)}{d_m(n-3)} & \dots & d_m(1) & \\ n\cdot\frac{d_m(n)}{d_m(n-1)} & \frac{d_m(n-1)}{d_m(n-2)} & \dots & \frac{d_m(2)}{d_m(1)} & d_m(1) \end{array} \right\rangle, \quad (18)$$

$$b_m(n) = (-1)^n \left\langle \begin{array}{ccccc} d_m(1) & & & & \\ \frac{d_m(2)}{d_m(1)} & d_m(1) & & & \\ \vdots & \dots & \ddots & & \\ \frac{d_m(n-1)}{d_m(n-2)} & \frac{d_m(n-2)}{d_m(n-3)} & \dots & d_m(1) & \\ \frac{d_m(n)}{d_m(n-1)} & \frac{d_m(n-1)}{d_m(n-2)} & \dots & \frac{d_m(2)}{d_m(1)} & d_m(1) \end{array} \right\rangle, \quad (19)$$

$$d_m(n) = (-1)^n \left\langle \begin{array}{ccccc} b_m(1) & & & & \\ \frac{b_m(2)}{b_m(1)} & b_m(1) & & & \\ \vdots & \dots & \ddots & & \\ \cdot\frac{b_m(n-1)}{b_m(n-2)} & \frac{b_m(n-2)}{b_m(n-3)} & \dots & b_m(1) & \\ \frac{b_m(n)}{b_m(n-1)} & \frac{b_m(n-1)}{b_m(n-2)} & \dots & \frac{b_m(2)}{b_m(1)} & b_m(1) \end{array} \right\rangle. \quad (20)$$

Proof. Relations (15), (16) follows from recurrent relations of Theorem 3 (from [3], p. 70). Indeed, each of these equalities is a result of expansion of the paradeterminants on the right side of (15) or (16) by elements of the last raw. Relations (17), (18) can be obtained by inversion of (15), (16) using Theorem 6; (19), (20) follows directly from Theorem 2 in [3], p. 69, and the above Theorem 3 on the relation between paradeterminants and determinants. □

The following theorem gives recurrent relations between functions $b_m(n), \xi_m(n), d_m(n)$.

Theorem 8. *The following equalities hold:*

$$\xi_m(n) = -\Big(b_m(1)\xi_m(n-1) + b_m(2)\xi_m(n-2) + \ldots + b_m(n-1)\xi_m(1) - nb_m(n)\xi_m(0)\Big), \quad (21)$$

$$\xi_m(n) = -\Big(d_m(1)\xi_m(n-1) + d_m(2)\xi_m(n-2) + \ldots + d_m(n-1)\xi_m(1) + nd_m(n)\xi_m(0)\Big), \quad (22)$$

$$b_m(n) = -\Big(d_m(1)b_m(n-1) + d_m(2)b_m(n-2) + \ldots + d_m(n-1)b_m(1) + d_m(n)b_m(0)\Big), \tag{23}$$

$$d_m(n) = -\Big(b_m(1)d_m(n-1) + b_m(2)d_m(n-2) + \ldots + b_m(n-1)d_m(1) + b_m(n)d_m(0)\Big), \tag{24}$$

where $b_m(0) = d_m(0) = \xi_m(0) = 1$.

Proof. To prove (21) multiply both sides of (17) by $(-1)^{n-1}$ and expand paradeterminant on the right side of the obtained equality by elements of the last row. As the result, we get

$$\begin{aligned}(-1)^{n-1}\xi_m(n) = b_m(1)(-1)^{n-2}\xi_m(n-1) - b_m(2)(-1)^{n-3}\xi_m(n-2) \\ + \ldots + (-1)^{n-2}b_m(n-1)(-1)^0\xi_m(n-(n-1)) \\ + (-1)^{n-1}b_m(n)(-1)^{-1}\xi_m(n-n)\end{aligned}$$

and hence (21). Similarly, one can prove the relation (22) using (18). Relations (23), (24) can be obtained from (19) and (20) respectively. Let us prove, for example, (23). Multiply both sides of (19) by $(-1)^n$ and expand paradeterminant on the right side of obtained equality by elements of the last row. Then

$$\begin{aligned}(-1)^n b_m(n) = d_m(1)(-1)^{n-1}b_m(n-1) - d_m(2)(-1)^{n-2}b_m(n-2) \\ + \ldots + (-1)^{n-2}d_m(n-1)(-1)^{1)}b_m(n-(n-1)) \\ + (-1)^{n-1}d_m(n)(-1)^0 b_m(n-n),\end{aligned}$$

and the required relation follow immediately. □

In the next theorem, we describe partition polynomials as defined in [6] presenting m-ary numbers $b_m(n), \xi_m(n), d_m(n)$.

Theorem 9. *The following equalities hold:*

$$d_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{\xi_m^{\lambda_1}(1)\cdot\ldots\cdot\xi_m^{\lambda_n}}{\lambda_1!\cdot\ldots\cdot\lambda_n! 1^{\lambda_1}\cdot\ldots\cdot n^{\lambda_n}}, \tag{25}$$

$$\xi_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{k-1} \frac{n(k-1)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\cdot b_m^{\lambda_1}(1)\cdot\ldots\cdot b_m^{\lambda_n}(n), \tag{26}$$

$$\xi_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{n(k-1)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\cdot d_m^{\lambda_1}(1)\cdot\ldots\cdot d_m^{\lambda_n}(n), \tag{27}$$

$$b_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} \cdot d_m^{\lambda_1}(1) \cdot \ldots \cdot d_m^{\lambda_n}(n), \tag{28}$$

$$d_m(n) = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^k \frac{k!}{\lambda_1! \cdot \ldots \cdot \lambda_n!} \cdot b_m^{\lambda_1}(1) \cdot \ldots \cdot b_m^{\lambda_n}(n), \tag{29}$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof. Partition polynomial corresponding to parapermanent (15), were described by Kachi and Tzermias [3, Theorem 1, p. 68]. Paradeterminant of the same matrix corresponds to the partition polynomial that differs from the previous one only by sign $(-1)^{n-k}$. Thus (25) holds. The relations for partition polynomials (26), (27) and (28), (29) follow directly from theorems 5 and 4 respectively. □

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