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Type conditions of stable range for identification of qualitative generalized classes of rings

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Abstract. This article deals mostly with the following question: when the classical ring of quotients of a commutative ring is a ring of stable range 1? We introduce the concepts of a ring of (von Neumann) regular range 1, a ring of semihereditary range 1, a ring of regular range 1, a semihereditary local ring, a regular local ring. We find relationships between the introduced classes of rings and known ones, in particular, it is established that a commutative indecomposable almost clean ring is a regular local ring. Any commutative ring of idempotent regular range 1 is an almost clean ring. It is shown that any commutative indecomposable almost clean Bezout ring is an Hermite ring, any commutative semihereditary ring is a ring of idempotent regular range 1. The classical ring of quotients of a commutative Bezout ring $Q_{Cl}(R)$ is a (von Neumann) regular local ring if and only if R is a commutative semihereditary local ring.

Throughout, all rings are assumed to be associative with unit and $1 \neq 0$. The set of nonzero divisors (also called regular elements) of R is denoted by $\mathfrak{R}(R)$, the set of units by $U(R)$ and the set of idempotents by $\mathfrak{B}(R)$. The Jacobson radical of a ring R is denoted by $J(R)$. The classical ring of quotients of ring R is denoted by $Q_{CL}(R)$.

A ring R is called indecomposable if $\mathfrak{B}(R) = \{0, 1\}$. A ring is called clean if every its element is the sum of a unit and idempotents, and it

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is called almost clean if each element of a ring is the sum of a regular element and an idempotent [5]. An element a of a ring R is called (von Neumann) regular element, if $axa = a$ for some element $x \in R$. An element a of a ring R is called a left (right) semihereditary element if Ra (aR) is projective. A ring R is *a ring of stable range 1*, if for any elements $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $(a + bt)R = R$. A ring R is *a ring of stable range 2*, if for any $a, b, c \in R$ such that $aR + bR + cR = R$ there exist such $x, y \in R$ that $(a + cx)R + (b + cy)R = R$ (see [6]).

Following Kaplansky [4] a commutative ring is said to be an elementary divisor ring if every matrix A over R is equivalent to a diagonal matrix, i.e. for A there exist such invertible matrices P and Q of appropriate sizes that PAQ is diagonal matrix (d_{ij}) (i.e $d_{ij} = 0$ whenever $i \neq j$) with the property that $Rd_{i+1,i+1}R \subseteq d_{ii}R \cap Rd_{ii}$. If every 1 by 2 and 2 by 1 matrix over R is equivalent to a diagonal matrix then the ring called an Hermite ring.

Obviously, an elementary divisor ring is Hermite, and it is easy to see that an Hermite ring is Bezout ring [5]. Examples, that neither implication is revertible, are provided by Gillmann and Henriksen in [2]. We have the following result.

Theorem 1 ([6]). *A commutative Bezout ring R is an Hermite ring if and only if the stable range of R is equal 2.*

Contessa in [1] called a ring (von Neumann) regular local if for each $a \in R$ either a or $(1 - a)$ is a (von Neumann) regular element.

Proposition 1. *Let R be a commutative Bezout ring. If $\varphi \in \mathfrak{B}(Q_{CL}(R))$ then $\varphi \in \mathfrak{B}(R)$.*

Proof. Let $\varphi \in \mathfrak{B}(Q_{CL}(R))$ and $\varphi = \frac{e}{s}$, where s is a regular element of R. Let $eR + sR = \delta R$, then $e = e_0\delta$, $s = s_0\delta$ and $eu + sv = \delta$ for some elements $e_0, s_0, u, v \in R$. Since s is a regular element, δ is a regular element as a divisor of s. Since $eu + sv = \delta$, then $\delta(e_0u + s_0v - 1) = 0$. Since $\delta \neq 0$ and δ is a regular element of R, we have $e_0u + s_0v - 1 = 0$. Then $\frac{e}{s} = \frac{e_0}{s_0}$, where $e_0R + s_0R = R$. Since $\frac{e_0}{s_0} \in \mathfrak{B}(Q_{CL}(R))$, then $e_0^2s_0 = e_0s_0^2$ and $s_0(e_0^2 - e_0s_0) = 0$. Since $s_0 \neq 0$ and so s_0 is a regular element of R as a divisor of s, we have $e_0^2 = e_0s_0$.

Since $e_0u + s_0v = 1$, we have $e_0^2u + e_0s_0v = e_0$ and $s_0(e_0u + s_0v) = e_0$. Hence $\frac{e_0}{s_0} \in R$. □

Proposition 2. *Let R be a commutative ring and a is a (von Neumann) regular element of R. Then $a = eu$, where $e \in \mathfrak{B}(R)$ and $u \in U(R)$.*

Proof. Let $axa = a$. This implies that $axax = ax$, i.e. $e = ax \in \mathfrak{B}(R)$ and $e \in aR$. Since $axa = a$, then $ea = a$, i.e. $a \in eR$ and we have $aR = eR$.

Consider an element $u = (1 - e) + a$. Since $u(1 - e) = 1 - e$, we have $uR + eR = R$. We proved that $eR = aR$, then $uR + aR = R$. Since $ue = ((1 - e) + a)e = ae = a$, then $aR \subset uR$. Obviously, the equality $uR + aR = R$ and inclusion $aR \subset uR$ in a commutative ring are possible if $u \in U(R)$.

Then we have $ue = a$. □

Proposition 3. *Let R be a commutative ring. Then a be a semihereditary element if and only if $a = er$, where $e \in \mathfrak{B}(R)$ and $r \in \mathfrak{R}(R)$.*

Proof. Let $\varphi R = \{\, x \mid xa = 0\}$ and $\varphi \in \mathfrak{B}(R)$. Since $\varphi a = 0$, we have $(1 - \varphi)a = a$. Let $r = a - \varphi$ and $rx = 0$.

Since $ax = \varphi x$ and $(1 - \varphi)a = a$, we have $(1 - \varphi)ax = \varphi x$ and $(1 - \varphi)\varphi x = 0$. Then $\varphi x = 0$ and $ax = 0$. Since $ax = 0$, we have $x \in \varphi R$, i.e. $x = x\varphi$. Since $x\varphi = 0$, so $x = 0$. Then we see that r is a regular element of R. Since

$$r(1 - \varphi) = a(1 - \varphi) - \varphi(1 - \varphi) = a(1 - \varphi) = a,$$

then $a = r(1 - \varphi)$. Put $1 - \varphi = e$, we have $a = re$, where $e \in \mathfrak{B}(R)$ and $r \in \mathfrak{R}(R)$. Obviously, $\{x|x(re) = 0\} = (1 - e)R$. □

Definition 1. A ring R is said to have a (von Neumann) regular range 1, if for any such $a, b \in R$ that $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a (von Neumann) regular element of R.

Obviously, an example of ring (von Neumann) regular range 1 is a ring of stable range 1. Moreover, we have the following result.

Proposition 4. *A commutative ring of (von Neumann) regular range 1 is a ring of stable range 1.*

Proof. Let R be a ring of (von Neumann) regular range 1 and $aR+bR = R$. Then there exists an element such $y \in R$ that $a+by = r$ is a (von Neumann) regular element of R. By Proposition 2, we have $a + by = r = ek$, where $e \in \mathfrak{B}(R)$ and $k \in U(R)$.

Note that, since $aR+bR = R$, we have $eR+bR = R$. Then $eu+bv = 1$ for some elements $u, v \in R$. Since $1 - e = (1 - e)eu + (1 - e)bv$, we have $1 - e = (1 - e)bv$, and $e + b(1 - e)v = 1$. Then $ek + b(1 - e)kv = k$.

Thus, we have $a+bs=k$ for some element $s\in R$, i.e. $(a+bs)R=R$. We have that R is a ring of stable range 1. □

Then we obtain the following result.

Theorem 2. *For a commutative ring the following conditions are equivalent:*
1) *R is a ring of stable range 1;*
2) *R is a ring of (von Neumann) regular range 1.*

Definition 2. A ring R is said to have a semihereditary range 1, if for any such elements $a,b\in R$ that $aR+bR=R$ there exists such $y\in R$ that $a+by$ is a semihereditary right element of R.

Obviously, an example of a ring of semihereditary range 1 is a ring of stable range 1 and a commutative semihereditary ring.

A special place in the class of rings of semihereditary range 1 is taken by semihereditary local rings.

Definition 3. A commutative ring R is a semihereditary local ring if for any such $a,b\in R$ that $aR+bR=R$, either a or b is a semihereditary element of R.

Obviously, an example of a semihereditary local ring is a (von Neumann) regular local ring and a semihereditary ring. A commutative domain (which is not a local ring) is a semihereditary local ring which is not a (von Neumann) regular local ring.

Proposition 5. *A commutative semihereditary local ring is a ring of semihereditary range 1.*

Proof. Let R be a commutative semihereditary local ring and $aR+bR=R$. If a is a semihereditary element, then representation $a+b0$ is as tequired. If a is not semihereditary, by condition $aR+(a+b)R=R$, the element $a+b1$ is semihereditary. □

The ring $\mathbb{Z}_{36}$ is not a semihereditary local ring, but $\mathbb{Z}_{36}$ is a ring of semihereditary range 1 (see [1]).

Definition 4. A ring R is said to have regular range 1 if for any such $a,b\in R$ that $aR+bR=R$ there exists such $y\in R$ that $a+by$ is a regular element of R.

Theorem 3. *For a commutative ring R the following conditions are equivalent:*

1) *R is a ring of regular range 1;*
2) *R is a ring of semihereditary range 1.*

Proof. A regular element is a semihereditary element and then if R is a ring of regular range 1 then R is a ring of semihereditary range 1.

Let R be a ring of semihereditary range 1 and $aR + bR = R$. Then there exists such $y \in R$ that $a+by = er$, where $e \in \mathfrak{B}(R)$, $r \in \mathfrak{R}(R)$. Since $aR+bR = R$, we have $eR+bR = R$. Then $eu+bv = 1$ for some elements $u, v \in R$. Since $1-e = (1-e)eu + (1-e)bv$ we have $e + b(1-e)v = 1$ and $er + br(1-e)v = r$. Since $a + by = er$ and $er + br(1-e)v = r$, we have $a + bs = r$ for some element $s \in R$. Then R is a ring of regular range 1. □

Proposition 6. *A classical ring of quotients $Q_{CL}(R)$ of a commutative Bezout ring R of regular range 1 is a ring of stable range 1.*

Proof. Let

$$\frac{a}{s}Q_{CL}(R) + \frac{b}{s}Q_{CL}(R) = Q_{CL}(R).$$

Then $au+bv = t$. where $u, v \in R$ and $t \in \mathfrak{R}(R)$. Since R is a commutative Bezout ring, we have $aR + bR = dR$ for some element $d \in R$. Then $a = a_0d$, $b = b_0d$ and $ax + by = d$ for some elements $a_0, b_0, x, y \in R$. Since $au + bv = t$, we have $d(a_0u + b_0v) = t$. Then d is a regular element as the divisor of a regular element t.

Since $d(a_0x + b_0y - 1) = 0$ and $d \neq 0$, we have $a_0x + b_0y - 1 = 0$ i.e. $a_0R + b_0R = R$. Since R is a ring of regular range 1, we have $a_0 + b_0k = r$ regular element of R for some element $k \in R$. Then $a + bk = rd \in \mathfrak{R}(R)$. So we have $\frac{a}{s} + \frac{b}{s}k = \frac{rd}{s}$.

Since $\frac{rd}{s} \in U(Q_{CL}(R))$ we have $(\frac{a}{s} + \frac{b}{s}k)Q_{CL}(R) = Q_{CL}(R)$ i.e. $Q_{CL}(R)$ is a ring of stable range 1. □

Here are some examples of rings of regular range 1.

Definition 5. A commutative ring R is a regular local ring if for any $a \in R$ either a or $(1-a)$ is a regular element.

Proposition 7. *A commutative regular local Bezout ring is a ring of stable range 2.*

Proof. Let R be a regular local Bezout ring. Let a, b be nonzero elements of R. Since R is a commutative Bezout ring, we have $aR+bR = dR$. Then we have $au + bv = d$, $a = a_0d$, $b = b_0d$ for some elements $a_0, b_0, u, v \in R$.

Since $d(a_0u+b_0v-1)=0$, by the definition of a ring R we see that either a_0u+b_0v or a_0u+b_0v-1 is a regular element of R. If a_0u+b_0v-1 is a regular element, by $d(a_0u+b_0v-1)=0$ we have $d=0$, i.e. $a=b=0$ but this is impossible. Let $a_0u+b_0v=r$ be a regular element of R.

Let $a_0R+b_0R=\delta R$. If $\delta\notin U(R)$ we have $a_0x+b_0y=\delta$, $a_0=\delta a_1$, $b_0=\delta b_1$ for some elements $a_1,b_1,x,y\in R$. This implies that $\delta(a_1u+b_1v)=a_0u+b_0v=r$. Since $r\in\mathfrak{R}(R)$, we have $\delta\in\mathfrak{R}(R)$.

This implies that $\delta(a_1x+b_1y-1)=0$ and, since $\delta\neq 0$, we have $a_1x+b_1y-1=0$ i.e. $a_1R+b_1R=R$. Thus, we have $a=d\delta a_1$, $b=d\delta b_1$, $a_1R+b_1R=R$. By [2], R is an Hermite ring and, by Theorem 1, we obtain that R is a ring of stable range 2. □

In the class of rings of regular range 1 we allocate a class of ring of idempotent regular range 1.

Proposition 8. *A ring R is said to be a ring of idempotent regular range 1 if for any such elements $a,b\in R$ that $aR+bR=R$ there exists such an idempotent $e\in\mathfrak{B}(R)$ and a regular element $r\in\mathfrak{R}(R)$ that $a+be=r$.*

An obvious example of a ring of idempotent regular range 1 is a ring of idempotent stable range 1, i.e. a commutative clean ring.

Proposition 9. *A commutative regular local ring is a ring of idempotent regular range 1.*

Proof. Let R be a regular local ring and $aR+bR=R$. If a is a regular element, then we have a representation $a+b0=a$. If a is not a regular element, so since $aR+(a+b)R=R$, the element $a+b1$ is regular. □

Theorem 4. *A commutative semihereditary ring is a ring of idempotent regular range 1.*

Proof. Let R be a commutative semihereditary ring and $aR+bR=R$. By [5] and Proposition 3, we have $a=er$ where e is an idempotent and r is a regular element. Note that if $e=1$, we have that a is a regular element and $a+b\cdot 0$ is a necessary representation. If $e\neq 1$, let $s=a+b(1-e)$. We need to show that s is a regular element of R. Let $sx=0$, then $ax=-b(1-e)x$. Since $a=er$, we have that

$$erx=(1-e)(-b)x.$$

Thus, we obtained that $e\cdot erx=e(1-e)(-b)=0$. Since $erx=exr=0$ and r is a regular nonzero element, we have $ex=0$ and $b(1-e)x=0$, therefore $bx = bex = 0$. Hence we have $ax = 0$ and $bx = 0$. Since $aR + bR = R$ we have $au + bv = 1$ for some elements $u, v \in R$. Then $x = axu + bxv = 0$ and $s = a + b(1 - e)$ is a regular element. Thus, we have that R is a ring of idempotent regular range 1. $\square$

Consequently, we obtain a result.

Proposition 10. *A commutative ring of idempotent regular range 1 is an almost clean ring.*

Proof. Let R be a ring of idempotent regular range 1 and let $a \in R$ be any nonzero element $a \in R$. Then $aR + (-1)R = R$ and $a - e = r$, where e is an idempotent and r is a regular element of R. $\square$

Open question: Is every commutative almost clean ring a ring of idempotent regular range 1?

Proposition 11. *For a commutative ring R the following conditions are equivalent:*

1) *R is an indecomposable almost clean ring;*
2) *R is a regular local ring.*

Proof. Let R be an indecomposable almost clean ring. Since 0 and 1 are all idempotents of R, we have for any a that either a or $1 - a$ is a regular element of R.

Let R be a regular local ring. Since for each idempotent $e \in R$ we have, that both e and $1 - e$ are idempotents, then we have that R is indecomposable ring. By Proposition 9, we have that R is a ring of idempotent regular range 1 and by Proposition 10, R is an almost clean ring. $\square$

By Theorem 1 and Proposition 7 we have the following result.

Theorem 5. *A commutative indecomposable almost clean Bezout ring is a Hermite ring.*

Proposition 12. *A commutative semihereditary local ring is a ring of idempotent regular range 1.*

Proof. Let R be a commutative semihereditary local ring and $aR+bR = R$. If a is semihereditary element then we have a representation $a = er$, where e is an idempotent and r is a regular element. Then we have that $a+b(1-e)$ is a regular element by the proof of Theorem 4. If a is not a semihereditary element, then by the equality $aR+(a+b)R=R$, we have that $a+b=er$ is a semihereditary element, i.e. $e^2=e$ and $r\in\mathfrak{R}(R)$.

Since $(a+b)R+(-b)R=R$, the equalities $a+b-b(1-e)=a+be=s$ we provide a necessary representation. $\square$

Theorem 6. *Let R be a commutative Bezout ring. Then $Q_{CL}(R)$ is a (von Neumann) regular local ring if and only if R is a semihereditary local ring.*

Proof. Let $aR+bR=R$, then $\frac{a}{1}Q_{CL}(R)+\frac{b}{1}Q_{CL}(R)=Q_{CL}(R)$. Since $Q_{CL}(R)$ is (von Neumann) regular local ring, either $\frac{a}{1}$ or $\frac{b}{1}$ is a (von Neumann) regular element. If $\frac{a}{1}$ is a (von Neumann) regular element, then by Proposition 2 we have that $\frac{a}{1}=eu$, where $e^2=e\in Q_{CL}(R)$ and $u\in U(Q_{CL}(R))$. By Proposition 1, we have that $e\in R$. Then we have $a=er$, where r is a regular element of R. The case element $\frac{b}{1}$ is a (von Neumann) regular is similar.

Let R be a semihereditary local ring and

$$\frac{a}{s}Q_{CL}(R)+\frac{b}{s}Q_{CL}(R)=Q_{CL}(R),$$

either $\frac{a}{s}\neq 0$ or $\frac{b}{s}\neq 0$. Then $au+bv=t$ for some elements $u,v\in R$ and t is a regular element of R. Since R is a commutative Bezout ring, then $aR+bR=dR$. Let $a=a_0d$, $b=b_0d$ and $ax+by=d$ for some elements $a_0,b_0,x,y\in R$. By the equality $au+bv=t$, we have that $d(a_0u+b_0v)=t$. Then d is a regular element as a divisor of t. By the equality $ax+by=d$, we have $d(a_0x+b_0y-1)=0$. Since $d\neq 0$ and d is a regular element, we have $a_0x+b_0y=1$. Hence $a_0R+b_0R=R$, so a_0 or b_0 is a semihereditary element.

If a_0 is a semihereditary element, by Proposition 3, we have $a_0=er$, where $e^2=e$ and r is a regular element of R. Since $a=a_0d=e(rd)$, we have $\frac{a}{s}=e\frac{rd}{s}$. Since $e^2=e$ and $\frac{rd}{s}\in U(Q_{CL}(R))$, we get that $\frac{a}{s}$ is a (von Neumann) regular element. If b_0 is a (von Neumann) regular element, we have a similar proof. Then $Q_{CL}(R))$ is a (von Neumann) regular local ring. $\square$

Definition 6. [3] A commutative ring R is said to be additely regular if for each $a\in R$ and each regular element $b\in R$ there exists an element $u\in R$ such that $a+ub$ is regular in R.

Proposition 13. *A commutative Bezout ring of regular range 1 is additively regular.*

Proof. Let R be a commutative Bezout ring of regular range 1 and let a be any element of R and let b be any regular element of R. Since R is a commutative Bezout ring, we have $aR+bR=dR$, and where $au+bv=d$, $a=a_0d$, $b=b_0d$ for some element $u,v,a_0,b_0 \in R$. Since b is a regular element of R, we have that d is a regular element of R, since d is a divisor of b.

Since $au+bv=d$, we have $d(a_0u+b_0v-1)=0$. Hence $d \neq 0$ and we obtain that $a_0u+b_0v-1=0$ i.e. $a_0R+b_0R=R$. Thus, R is a ring of regular range 1 and we obtain that $a_0+b_0t=r$ is a regular element for some $t \in R$. Then $a+bt=rd$ is a regular ring, i.e. R is an additively regular ring. □

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