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RESEARCH ARTICLE

Free ultra-groups, generators and relations

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Abstract. In this paper, we intend to define an ultra-group by its presentation. The attitude of the presentation for a group was the key for us to investigate in this area. Instead of writing whole elements of an ultra-group, we denote it by its generators and the relations among those generators. A general computational approach for finitely presented ultra-groups by quotient ultra-groups and subultra-groups is described and some examples are presented. It is the way that can clarify the structure of an ultra-group quicker than having just a list of elements.

Introduction

In universal algebra, an algebra is a set together with a collection of operations on it. The need for such a definition was noted by several mathematicians such as Whitehead in 1898, and later by Noether, the credit for realizing this goal goes to Birkhoff in 1933. S. Burris, H. P. Sankapanavar developed the most general and fundamental notions of universal algebra. Moreover, Free algebras are discussed in great detail by them [1].

In [3] the new concept of an ultra-group was presented. It is an algebraic structure which was introduced vastly in Definition 2.

In the group theory, one method of defining a group is by its presentation. We specify a set X of generators such that every element of the group can be written as a product of powers of some of these generators,

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and a set R of relations among those generators. We say the group G has the finitely presented if there is a finite presentation $\langle X|R\rangle$ with $G \cong F/N$ where F is the free group on X and N is the normal closure of R in F (see [2, Chapter I, Section 9.]).

In this paper, our aim is to provide such a strong capability in the category of ultra-groups. More precisely, we intend to write a presentation for an ultra-groups instead of their elements and Cayley operation tables. The reference [3] include the results about an ultra-group, such as subultra-group, homomorphism and isomorphism theorems. Let us recall some of the basic definitions.

Let H be a subgroup of the group G and M a subset of G. If $|M \cap Hg| = 1$ for all $g \in G$, then $G = HM$. For the group G which satisfies the above conditions, we have $MH \subseteq G = HM$. Therefore, for every element $mh \in MH$ there exists $h' \in H$ and $m' \in M$ such that $mh = h'm'$.

Definition 1. Let H be a subgroup of a multiplicative group G. A subset M of G is called (right unitary) complementary set with respect to subgroup H, if for any elements $m \in M$ and $h \in H$ there exist the unique elements $h^{'} \in H$ and $m^{'} \in M$ such that $mh = h^{'}m^{'}$ and $e \in M$. We denote h' and m' by ${}^{m}h$ and m^h, respectively.

Similarly for any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and ${}^{(m_1,m_2)}h \in H$ such that $m_1m_2 = {}^{(m_1,m_2)}h[m_1, m_2]$. For every element $a \in M$, there exists a^{-1} belonging to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$.

Definition 2. A (right) ultra-group ${}_HM$ is a complementary set of subgroup H over group G with a binary operation $\alpha : {}_HM \times {}_HM \to {}_HM$ and unary operation $\beta_h : {}_HM \to {}_HM$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.

A (left) ultra-group M_H is defined similarly via (left unitary) complementary set. In this text we concentrate on the right ultra-group.

Throughout this paper, we denote a right ultra-group over the subgroup H of the group G by ${}_HM$ and consider its binary operation and unary operation by α and β_h, respectively.

Although we have an associative property for the groups, but this property is not valid for the binary operation α of the ultra-groups. Therefore, we convent $\alpha(a, b, c) = \alpha(\alpha(a, b), c)$, where a, b, c are the elements of the ultra-group ${}_HM$ and α is its first binary operation over it. For a positive integer n, $\underbrace{[\dots[x_1, x_2], x_3], \dots, x_n]}_{n-1 \text{ times } \alpha}$ is $n - 1$ times iteration of the binary operation α for n elements of ultra-group. In this paper we denote it by $\alpha^n(x_1, x_2, \ldots, x_n)$, where $x_i \in {}_HM$. The structural and categorical properties of the ultra-groups discussed in [3, 4]. Here, we have recalled just some necessary notions which are useful in this research, for more details one can see those references.

By mimicking the techniques of obtaining a presentation for a group, we require the free ultra-group which is constructed on the set of generators of an ultra-group. By the free ultra-group we mean the free object in the concrete category of ultra-groups. In [4], we proved the existence of the free object in the ultra-groups category and its structure has been described. In this research, we generalized the Van Dyck 's Theorem for the ultra-groups and consequently the presentation for an ultra-group is defined.

1. Preliminaries

Let ${}_HM$ be an ultra-group of the subgroup H over the group G. The non-empty subset Y of elements of ${}_HM$ is called a generator set of ${}_HM$ provided that $m = \alpha^s(y_1, y_2, \ldots, y_s)$, for $y_i \in Y$ and the positive integer s. It is clear that every ultra-group has a generator set. We can consider the underlying set of an ultra-group as its generator set, in the worst conditions.

The normal subultra-group was discussed vastly in [3, Definition 2.8]. The normal subultra-group generated by a set $T \subseteq {}_HM$ is the intersection of all normal subultra-groups of ${}_HM$ that contains T.

If S is a normal subultra-group of ${}_HM$, then the quotient ultra-group ${}_HM/S$ is an ultra-group over the subgroup $H_1 = \{h[S, e] : h \in H\}$ of the group $G_1 = \{h[S, a] : h \in H, a \in {}_HM\} = H[S,_H M]$, where $[\ ,\ ]$ denotes the binary operation of ${}_HM$, (see [3] for the notations). One can verify that G_1 is a group with a binary operation $*$, which is defined by $h_1[S, a_1] * h_2[S, a_2] = h_1h_2[[S, a_1], [S, a_2]] = h_1h_2[S, [a_1, a_2]]$, $h_i \in H$ and $a_i \in {}_HM$, $i = 1, 2$. Note that, the normality of S in ${}_HM$ implies the second equality (see [3, Lemma 2.5 (ii)]). With the same notations here, we conclude the following result.

Lemma 1. *The map π: ${}_HM \to {}_HM/S$ is an ultra-group epimorphism.*

Proof. The map π with the rule $a \mapsto [S, a]$, satisfies the definition of ultra-groups homomorphism (see [3, Definition 2.5]). By the normal subultra-groups property $\pi([a_1, a_2]) = [S, [a_1, a_2]] = [[S, a_1], [S, a_2]] =$

$[\pi(a_1), \pi(a_2)]$. Moreover,

$$\pi(a^h) = [S, a^h] = [S, a]^{hS} = [S, a]^{\varphi(h)} = \pi(a)^{\varphi(h)},$$

where $\varphi : H \to H_1$. Hence the assertion is clear. □

We are going to prepare the tools to generalized the Van Dyck's Theorem in the group theory for the ultra-groups.

Theorem 1. *Let $f\colon {}_{H_1}M_1 \to {}_{H_2}M_2$ be an ultra-group homomorphism and S_i is a normal subultra-group of ${}_{H_i}M_i$, $i = 1, 2$ such that $f(S_1)$ be the proper subultra-group of S_2. Then f induce the ultra-group homomorphism $\overline{f}\colon {}_{H_1}M_1/S_1 \to {}_{H_2}M_2/S_2$.*

Proof. Consider the composition of ultra-group homomorphisms

$${}_{H_1}M_1 \to {}_{H_2}M_2 \to {}_{H_2}M_2/S_2.$$

Clearly, $S_1 \subset \mathrm{Ker}(\pi f) \subseteq f^{-1}(S_2)$, where $\pi\colon {}_{H_2}M_2 \to {}_{H_2}M_2/S_2$. Define the map $\overline{f}\colon {}_{H_1}M_1/S_1 \to {}_{H_2}M_2/S_2$. By the rule $[S_1, a] \mapsto [S_2, f(a)]$, where $a \in {}_{H_1}M_1$. Assume $a, a_1, a_2 \in {}_{H_1}M_1$ and $h \in H_1$. The map $\overline{f}$ is an ultra-group homomorphism, because

$$\begin{aligned}\overline{f}([[S_1, a_1], [S_1, a_2]]) &= \overline{f}([[S_1, [a_1, a_2]]) = [S_2, f([a_1, a_2])]\\ &= [S_2, [f(a_1), f(a_2)]] = [\overline{f}([S_1, a_1]), \overline{f}([S_1, a_2])],\end{aligned}$$

and also,

$$\begin{aligned}\overline{f}([S_1, a]^{hS_1}) &= \overline{f}([S_1, a^h] = [S_2, f(a^h)] = [S_2, (f(a))^{\psi(h)}]\\ &= [S_2, f(a)]^{\psi(h)S_2} = \overline{f}([S_1, a])^{\varphi(hS_1)},\end{aligned}$$

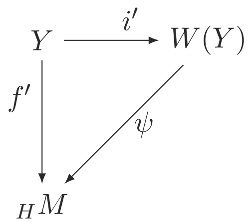
where φ is the group homomorphism between the two subgroups of which two ultra-groups ${}_{H_1}M_1/S_1$ and ${}_{H_2}M_2/S_2$ are constructed (see the argument before Lemma 1) and $\psi : H_1 \to H_2$ is the group homomorphism which is extracted from the ultra-group homomorphism f. □

2. The presentation of an ultra-group

Let F be a free group on the non-empty set X (see [2] for more details). We know every subgroup of a free group is itself a free group. Choose K one of the subgroups of F. Constructing all the ultra-groups of a subgroup over a group has been vastly discussed in [3]. Suppose $(W(X), \alpha, \beta_k)$ is the ultra-group of the subgroup K over the free group F, where α and β_k

are binary and unary operations, for all $k \in K$. Let $w_1, w_2 \in W(X)$. Since $W(X) \subseteq F$ elements of $W(X)$ are all reduced words. The binary operation on the free group F is just juxtaposition of two reduced words. Therefore, since $w_1w_2 \in F$ and $F = KW(X)$ we deduce $w_1w_2 = {}^{(w_1,w_2)}k\ [w_1, w_2]$, where ${}^{(w_1,w_2)}k \in K$ and $[w_1, w_2] \in W(X)$. It is not hard to see that $\alpha(w_1, w_2) = [w_1, w_2]$ by an ultra-group definition. Furthermore, since $W(X)K \subseteq F = KW(X)$ we have $wk = {}^{w}kw^k$. Thus $\beta_k(w) = w^k$, for $w \in W(X)$ and all $k \in K$. We call $W(X)$ the free ultra-group on the non-empty set $Y \subseteq X$, where Y is the set of all one letter word such that the words of $W(X)$ is obtained. We observed that $W(X)$ is a free object in the category of ultra-groups (see [4] for more details). In the following we denote the free ultra-group by $W(Y)$.

By considering the ultra-group ${}_HM = \langle Y \rangle$ over the subgroup H of the group G, there is a generating set X for the group G such that $Y \subseteq X$. Moreover, note that $G = HM$ and $F = KW(Y)$, where F is the free group on X, K is the free subgroup of F on the set $X - Y$ and $W(Y)$ is free ultra-group on Y. Since F is the free group on the set X, the group G is homomorphic image of F. Thus there exists a unique group epimorphism $\varphi : F \to G$ such that $\varphi i = f$, where $i : X \to F$ and $f : X \to G$ are inclusion maps. Now restrict i, f on Y and φ on $W(Y)$. Let $i|_Y = i'$, $f|_Y = f'$ and $\varphi|_{W(Y)} = \psi$. We have the following diagram.



In the following, we use the same notations of the above argument.

Lemma 2. *For every $w \in F \cap W(Y)$, $\psi(w) = [[\ldots[w_1, w_2], w_3], \ldots, w_n]$, where $w = w_1w_2 \ldots w_n$.*

Proof. By definition of the binary operation of the free ultra-group, we deduce that $\varphi(ab) = {}^{(\varphi(a),\varphi(b))}k\ [\varphi(a), \varphi(b)]$, where $a, b \in F$ and $k \in K$. Since $\psi = \varphi|_{W(Y)}$ the assertion is clear. □

The above discussion deduce the following theorem.

Theorem 2. *Every ultra-group is homomorphic image of a free-ultra group.*

Proof. Let ${}_HM$ be an ultra-group with the generating set Y and $W(Y)$ a free ultra-group which is constructed on Y. Since $W(Y)$ is a free object in the category of ultra-groups, there exists an ultra-groups homomorphism $\psi : W(Y) \to {}_HM$ such that

$$\begin{aligned}\psi(y_1y_2\dots y_m) &= \varphi(y_1y_2\dots y_m)\\ &= \varphi(y_1)\varphi(y_2)\dots\varphi(y_m)\\ &= \varphi(i(y_1))\varphi(i(y_2))\dots\varphi(i(y_m))\\ &= [[\dots[f(y_1), f(y_2)], f(y_3)], \dots, f(y_m)]\\ &= [[\dots[y_1, y_2], y_3], \dots, y_s] = \alpha^m(y_1y_2\dots y_m)\end{aligned}$$

and $\varphi : F \to G$ is the group homomorphism, where F is the free group which is constructed on the generating set of G, (See [4, Theorem 3.1]). Suppose $x \in {}_HM$. Clearly x can be written of the form $[[\dots[y_1, y_2], y_3], \dots, y_s]$, for $y_i \in Y$. Now, consider the word $y_1y_2\dots y_s \in W(Y)$ which maps to x by the ultra-group homomorphism ψ. This shows ψ is an ultra-group epimorphism. □

We continue with the same notations as in the proof of Theorem 2 and the argument before that. By the first isomorphism theorem of ultra-groups, $W(Y)/\mathrm{Ker}(\psi)$ is an ultra-group isomorphic to ${}_HM$, where $\mathrm{Ker}(\psi)$ is the set,

$$\{y_ky_l\dots y_t \in W(Y) : \psi(y_ky_l\dots y_t) = [[\dots[y_k, y_l], y_m], \dots, y_t] = e\},$$

and e is the identity element of the group of which ${}_HM$ is constructed on (see [3, Definition 2.6, Theorem 2.3] for more details). Therefore, in order to describe ${}_HM$ up to isomorphism we need only specify Y, $W(Y)$ and $\mathrm{Ker}(\psi)$. By [2, Theorem 7.8] $W(Y)$ is determined up to ultra-group isomorphism by Y and $\mathrm{Ker}(\psi)$ is determined by any subset that generates it as a subgroup of $W(Y)$. If $y_1y_2\dots y_s \in W(Y)$ is a generator of $\mathrm{Ker}(\psi)$, then under the ultra-group epimorphism $\psi : W(Y) \to {}_HM$, $y_1y_2\dots y_s \mapsto [[\dots[y_1, y_2], y_3], \dots, y_s] = e \in {}_HM$. The equation

$$[[\dots[y_1, y_2], y_3], \dots, y_s] = e$$

in ${}_HM$ is called a relation on the generators y_i. Conversely, suppose we are given a set Y and a set T of reduced words on the elements of Y (see [4, Section 3.]). Does there exists an ultra-group $({}_HM, \alpha, \beta_h)$ such that is generated by Y and all the relations $[[\dots[y_1, y_2], y_3], \dots, y_s] = e$, where $y_1y_2\dots y_s \in T$ and $[[\dots[y_1, y_2], y_3], \dots, y_s] = \alpha^s(y_1, y_2, \dots, y_s)$?

The answer is positive. We construct such an ultra-group as follows. Let $W(Y)$ be the free ultra-group on Y and N the normal subultra-group of $W(Y)$ generated by T. Let ${}_HM$ be the quotient ultra-group $W(Y)/N$ and identifying Y with its image under the map $Y \subset W(Y) \xrightarrow{\pi} W(Y)/N$. Every coset $[N, w] \in W(Y)/N$ correspondence to the element $m = \alpha^s(y_1, y_2, y_3, \ldots, y_s)$, where $w = y_1y_2 \ldots y_s$ is a reduced word in $W(Y)$. Thus we recognize ${}_HM$ by this method, which implies ${}_HM$ is generated by Y. If $w = y_{l_1} y_{l_2} \ldots y_{l_s} \in T$, then $y_{l_1} y_{l_2} \ldots y_{l_s} \in N$, since N is generated by T. This fact deduce that $[N, y_{l_1} y_{l_2} \ldots y_{l_s}] = N$ as N is a normal subultra-group and so $\alpha^s(y_{l_1} y_{l_2} \ldots y_{l_s}) = e$ by considering the map π. Hence, ${}_HM$ is the ultra-group which asked in the above question.

Similar to the definition of the presentation for the groups [2, Definition 9.4], and according to the above discussion we have the following significant definition.

Definition 3. If X be a finite set and Y a finite set of (reduced) words on X, then an ultra-group ${}_HM$ is said to be the ultra-group defined by the generators $x \in X$ and relations $w = e$ $(w \in Y)$ provided ${}_HM \cong W(X)/N$, where $W(X)$ is the free ultra-group on X and N the normal subultra-group of $W(X)$ generated by Y. One says that $\langle X|Y\rangle$ is a finite presentation of ${}_HM$.

We are ready to present the Van Dyck's Theorem for ultra-groups. The proof follows by Theorems 1, 2 and is very similar to the proof of Theorem 9.5 in [2].

Theorem 3. *Let X be a set, Y a set of reduced words on X and ${}_{H_1}M_1$ the ultra-group defined by generators $x \in X$ and relations $w = e$ $(w \in Y)$. If ${}_{H_2}M_2$ is any ultra-group such that ${}_{H_2}M_2 = \langle X\rangle$ and satisfies all relations $w = e$ $(w \in Y)$, then there is an epimorphism ${}_{H_1}M_1 \to {}_{H_2}M_2$.*

Proof. If $W(X)$ is the free ultra-group on X, then the inclusion map $X \to {}_{H_2}M_2$ induces the ultra-group epimorphism $\psi : W(X) \to {}_{H_2}M_2$ by Theorem 2. Since ${}_{H_2}M_2$ satisfies the relations $w = e$ $(w \in Y)$, $Y \subset \operatorname{Ker}\psi$. Consequently, the normal subgroup N generated by Y in $W(X)$ is contained in $\operatorname{Ker}\psi$. By Theorem 1 follows an epimorphism ${}_{H_1}M_1 \cong W(X)/N \to_{H_2} M_2/\{e\} \cong {}_{H_2}M_2$. □

We can associate to a given group different ultra-groups. In the following, we are going to find the presentations for the distinct ultra-groups which are assign to the dihedral group D_n.

Let $D_n = \langle a, b : a^n = b^2 = e, a^b = a^{-1}\rangle$ be the dihedral group of order $2n$ and $H = \langle a^d\rangle$ its subgroup, where $d|n$. We know all ultra-groups over the subgroup H of the group D_n are isomorphic to

$$\{e, a, a^2, \ldots, a^{d-1}, b, ab, \ldots, a^{d-1}b\}.$$

We call it dihedral ultra-group over H and denote it by ${}_H\mathtt{Dn}$.

Recall that the order of the ultra-group ${}_HM$ is the number of the elements in its underlying set which is equal to $|G|/|H|$ and we denote it by $|{}_HM|$.

Example 1. Let ${}_HM$ be an ultra-group with generators x, y and the relations $\underbrace{[\ldots[x,x],x],\ldots,x]}_{d \text{ times } x} = e$ and $[y,y] = e$, where $[\ ,\ ]$ denotes the first binary operation of the ultra-group ${}_HM$. Since ${}_H\mathtt{Dn}$, the dihedral ultra-group of order $2d$, is generated by a, b satisfies the relations, by Theorem 3 we have the ultra-group epimorphism $\varphi : {}_HM \to {}_H\mathtt{Dn}$. Therefore, $|{}_HM| \geqslant |{}_H\mathtt{Dn}| = 2d$.

Now, consider the free ultra-group $W(X)$ on the set $X = \{x, y\}$ and its normal subultra-group S generated by $\underbrace{[\ldots[x,x],x,\ldots,x]}_{d \text{ times } x}$ and $[y,y]$.

All the elements of $W(X)/S$ are the form $\left[S, [\underbrace{[\ldots[x,x],x],\ldots],x]}_{i \text{ times } x}, y]\right]$, where $0 \leqslant i \leqslant d-1$. Since, we can show that every element in $W(X)$ by $[\underbrace{[\ldots[x,x],x],\ldots],x]}_{i \text{ times } x}, y]$ such that $0 \leqslant i \leqslant d-1$. Thus $|{}_HM| \leqslant 2d$ which implies that φ is an isomorphism and ${}_H\mathtt{Dn}$ has the presentation $\langle x, y : \underbrace{[\ldots[x,x],x,\ldots,x]}_{d \text{ times } x} = [y,y] = e\rangle$.

If we change the subgroup of D_n to $K = \langle a^d, b\rangle$, then the ultra-group over the subgroup K of D_n is ${}_K\mathtt{Dn} = \{e, ab, a^2b, \ldots, a^{d-1}b\}$, where $d|n$. The following table is its binary operation α.

Example 2. Suppose ${}_HM$ is an ultra-group with generators x and y and relations $[x,x] = [y,y] = [x,[x,[x,y]]] = [[[x,y],y],x] = e$. By the Table 1 of the binary operation α of ${}_K\mathtt{Dn}$, it is clear that ab and a^3b are generators of ${}_K\mathtt{Dn}$ and satisfy the relations of ${}_HM$. Therefore, by Theorem 3 we have the ultra-group epimorphism $\varphi : {}_HM \to {}_K\mathtt{Dn}$ and $|{}_HM| \geqslant d$. Now, consider the free ultra-group $W(X)$ on the set $X = \{x, y\}$

TABLE 1.

α	e	ab	a^2b	a^3b	$\cdots$	$a^{d-1}b$
e	e	ab	a^2b	a^3b	$\cdots$	$a^{d-2}b$
ab	ab	e	ab	a^2b	$\cdots$	$a^{d-1}b$
a^2b	a^2b	$a^{d-1}b$	e	ab	$\cdots$	$a^{d-3}b$
a^3b	a^3b	$a^{d-2}b$	$a^{d-3}b$	e	$\cdots$	$a^{d-4}b$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a^{d-1}b$	$a^{d-1}b$	a^2b	a^3b	e	$\cdots$	e

and its normal subultra-group S generated by $[x,x], [y,y], [x,[x,[x,y]]]$ and $[[[x,y],y],x]$. Thus every element of $W(X)/S$ is of the form $[S,w]$ such that $w = \underbrace{[[x,y],x],\ldots,x]}_{\text{at most } d \text{ positions}}$. Thus $|_HM| \leqslant d$ which implies that φ is an isomorphism and $_K\mathtt{Dn}$ has the presentation $\langle x, y : [x,x] = [y,y] = [x,[x,[x,y]]] = [[[x,y],y],x] = e\rangle$.

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