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Uniformly 2-absorbing primary ideals of commutative rings

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Abstract. In this study, we introduce the concept of "uniformly 2-absorbing primary ideals" of commutative rings, which imposes a certain boundedness condition on the usual notion of 2-absorbing primary ideals of commutative rings. Then we investigate some properties of uniformly 2-absorbing primary ideals of commutative rings with examples. Also, we investigate a specific kind of uniformly 2-absorbing primary ideals by the name of "special 2-absorbing primary ideals".

Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is a *proper ideal* if $I \neq R$. Then $Z_I(R) = \{r \in R \mid rs \in I$ for some $s \in R \backslash I\}$ for a proper ideal I of R. Additively, if I is an ideal of R, then *the radical of* I is given by $\sqrt{I} = \{r \in R \mid r^n \in I$ for some positive integer $n\}$. Let I, J be two ideals of R. We will denote by $(I :_R J)$, the set of all $r \in R$ such that $rJ \subseteq I$.

Cox and Hetzel have introduced uniformly primary ideals of a commutative ring with nonzero identity in [6]. They said that a proper ideal Q of a commutative ring R is *uniformly primary* if there exists a positive integer n such that whenever $r, s \in R$ satisfy $rs \in Q$ and $r \notin Q$, then

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$s^n \in Q$. A uniformly primary ideal Q has order N and write $\text{ord}_R(Q) = N$, or simply $\text{ord}(Q) = N$ if the ring R is understood, if N is the smallest positive integer for which the aforementioned property holds.

Badawi [3] said that a proper ideal I of R is a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, I_3 are ideals of R with $I_1 I_2 I_3 \subseteq I$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. Anderson and Badawi [1] generalized the notion of 2-absorbing ideals to n-absorbing ideals. A proper ideal I of R is called an *n-absorbing* (resp. *a strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. Badawi et. al. [4] defined a proper ideal I of R to be a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Let I be a 2-absorbing primary ideal of R. Then $P = \sqrt{I}$ is a 2-absorbing ideal of R by [4, Theorem 2.2]. We say that I is a P-2-absorbing primary ideal of R. For more studies concerning 2-absorbing (submodules) ideals we refer to [5, 9, 10, 15, 16]. These concepts motivate us to introduce a generalization of uniformly primary ideals. A proper ideal Q of R is said to be a *uniformly 2-absorbing primary ideal of* R if there exists a positive integer n such that whenever $a, b, c \in R$ satisfy $abc \in Q$, $ab \notin Q$ and $ac \notin \sqrt{Q}$, then $(bc)^n \in Q$. In particular, if for $n = 1$ the above property holds, then we say that Q is a *special 2-absorbing primary ideal of* R.

In section 2, we introduce the concepts of uniformly 2-absorbing primary ideals and Noether strongly 2-absorbing primary ideals. Then we investigate the relationship between uniformly 2-absorbing primary ideals, Noether strongly 2-absorbing primary ideals and 2-absorbing primary ideals. After that, in Theorem 2 we characterize uniformly 2-absorbing primary ideals. We show that if $Q_1,\ Q_2$ are uniformly primary ideals of a ring R, then $Q_1 \cap Q_2$ and $Q_1 Q_2$ are uniformly 2-absorbing primary ideals of R, Theorem 4. Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with $1 \neq 0$. It is shown (Theorem 5) that a proper ideal Q of R is a uniformly 2-absorbing primary ideal of R if and only if either $Q = Q_1 \times R_2$ for some uniformly 2-absorbing primary ideal Q_1 of R_1 or $Q = R_1 \times Q_2$ for some uniformly 2-absorbing primary ideal Q_2 of R_2 or $Q = Q_1 \times Q_2$ for some uniformly primary ideal Q_1 of R_1 and some uniformly primary ideal Q_2 of R_2.

In section 3, we give some properties of special 2-absorbing primary ideals. For example, in Theorem 7 we show that Q is a special 2-absorbing primary ideal of R if and only if for every ideals I, J, K of R, $IJK \subseteq Q$ implies that either $IJ \subseteq \sqrt{Q}$ or $IK \subseteq Q$ or $JK \subseteq Q$. We prove that if Q is a special 2-absorbing primary ideal of R and $x \in R\backslash\sqrt{Q}$, then $(Q :_R x)$ is a special 2-absorbing primary ideal of R, Theorem 8. It is proved (Theorem 9) that an irreducible ideal Q of R is special 2-absorbing primary if and only if $(Q :_R x) = (Q :_R x^2)$ for every $x \in R\backslash\sqrt{Q}$. Let R be a Prüfer domain and I be an ideal of R. In Corollary 10 we show that Q is a special 2-absorbing primary ideal of R if and only if $Q[X]$ is a special 2-absorbing primary ideal of $R[X]$.

1. Uniformly 2-absorbing primary ideals

Let Q be a P-primary ideal of R. We recall from [6] that Q is a *Noether strongly primary ideal of* R if $P^n \subseteq Q$ for some positive integer n. We say that N is the exponent of Q if N is the smallest positive integer for which the above property holds and it is denoted by $\mathfrak{e}(Q) = N$.

Definition 1. Let Q be a proper ideal of a ring R.

1) Q is a *uniformly 2-absorbing primary ideal of* R if there exists a positive integer n such that whenever $a, b, c \in R$ satisfy $abc \in Q$, $ab \notin Q$ and $ac \notin \sqrt{Q}$, then $(bc)^n \in Q$. We call that N is order of Q if N is the smallest positive integer for which the above property holds and it is denoted by $2\text{-}\operatorname{ord}_R(Q) = N$ or $2\text{-}\operatorname{ord}(Q) = N$.
2) P-2-absorbing primary ideal Q is a *Noether strongly 2-absorbing primary ideal of* R if $P^n \subseteq Q$ for some positive integer n. We say that N is the exponent of Q if N is the smallest positive integer for which the above property holds and it is denoted by $2\text{-}\mathfrak{e}(Q) = N$.

A *valuation ring* is an integral domain V such that for every element x of its field of fractions K, at least one of x or x^{-1} belongs to K.

Proposition 1. *Let V be a valuation ring with the quotient field K and let Q be a proper ideal of V. The following conditions are equivalent:*

1) *Q is a uniformly 2-absorbing primary ideal of V;*
2) *There exists a positive integer n such that for every $x, y, z \in K$ whenever $xyz \in Q$ and $xy \notin Q$, then $xz \in \sqrt{Q}$ or $(yz)^n \in Q$.*

Proof. (1)⇒(2) Assume that Q is a uniformly 2-absorbing primary ideal of V. Let $xyz \in Q$ for some $x, y, z \in K$ such that $xy \notin Q$. If $z \notin V$, then $z^{-1} \in V$, since V is valuation. So $xyzz^{-1} = xy \in Q$, a contradiction. Hence $z \in V$. If $x, y \in V$, then there is nothing to prove. If $y \notin V$, then $xz \in Q \subseteq \sqrt{Q}$, and if $x \notin V$, then $yz \in Q$. Consequently we have the claim.

(2)⇒(1) It is clear. □

Proposition 2. *Let $Q_1,\ Q_2$ be two Noether strongly primary ideals of a ring R. Then $Q_1 \cap Q_2$ and Q_1Q_2 are Noether strongly 2-absorbing primary ideals of R such that 2-$\mathfrak{e}(Q_1 \cap Q_2) \leqslant max\{\mathfrak{e}(Q_1), \mathfrak{e}(Q_2)\}$ and 2-$\mathfrak{e}(Q_1Q_2) \leqslant \mathfrak{e}(Q_1) + \mathfrak{e}(Q_2)$.*

Proof. Since $Q_1,\ Q_2$ are primary ideals of R, then $Q_1 \cap Q_2$ and Q_1Q_2 are 2-absorbing primary ideals of R, by [4, Theorem 2.4]. □

Proposition 3. *If Q is a uniformly 2-absorbing primary ideal of R, then Q is a 2-absorbing primary ideal of R.*

Proof. Straightforward. □

Proposition 4. *Let R be a ring and Q be a proper ideal of R.*

1) *If Q is a 2-absorbing ideal of R, then*
   (a) *Q is a Noether strongly 2-absorbing primary ideal with 2-$\mathfrak{e}(Q) \leqslant 2$.*
   (b) *Q is a uniformly 2-absorbing primary ideal with 2-$\mathrm{ord}(Q) = 1$.*
2) *If Q is a uniformly primary ideal of R, then it is a uniformly 2-absorbing primary ideal with 2-$\mathrm{ord}(Q) = 1$.*

Proof. (1) (a) If Q is a 2-absorbing ideal, then it is a 2-absorbing primary ideal and $(\sqrt{Q})^2 \subseteq Q$, by [3, Theorem 2.4].

(b) It is evident.

(2) Let Q be a uniformly primary ideal of R and let $abc \in Q$ for some $a, b, c \in R$ such that $ac \notin \sqrt{Q}$. Since Q is uniformly primary, $abc \in Q$ and $ac \notin \sqrt{Q}$, then $b \in Q$. Therefore $ab \in Q$ or $bc \in Q$. Consequently Q is a uniformly 2-absorbing primary ideal with 2-ord$(Q) = 1$. □

Example 1. Let $R = K[X, Y]$ where K is a field. Then $Q = (X^2, XY, Y^2)R$ is a Noether strongly $(X, Y)R$-primary ideal of R and so it is a Noether strongly 2-absorbing primary ideal of R.

Proposition 5. *If Q is a Noether strongly 2-absorbing primary ideal of R, then Q is a uniformly 2-absorbing primary ideal of R and 2-$\mathrm{ord}(Q) \leqslant$2-$\mathfrak{e}(Q)$.*

Proof. Let Q be a Noether strongly 2-absorbing primary ideal of R. Now, let $a, b, c \in R$ such that $abc \in Q$, $ab \notin Q$, $ac \notin \sqrt{Q}$. Then $bc \in \sqrt{Q}$ since Q is a 2-absorbing primary ideal of R. Thus $(bc)^{2\text{-}\mathfrak{e}(Q)} \in (\sqrt{Q})^{2\text{-}\mathfrak{e}(Q)} \subseteq Q$. Therefore, Q is a uniformly 2-absorbing primary ideal and also 2-$\mathrm{ord}(Q) \leqslant$ 2-$\mathfrak{e}(Q)$. □

In the following example, we show that the converse of Proposition 5 is not true. We make use of [6, Example 6 and Example 7]

Example 2. Let R be a ring of characteristic 2 and $T = R[X]$ where $X = \{X_1, X_2, X_3, \dots\}$ is a set of indeterminates over R. Let $Q = (\{X_i^2\}_{i=1}^{\infty})T$. By [6, Example 7] Q is a uniformly P-primary ideal of T with $\mathrm{ord}_T(Q) = 1$ where $P = (X)T$. Then Q is a uniformly 2-absorbing primary ideal of T with 2-$\mathrm{ord}_T(Q) = 1$, by Proposition 4(2). But Q is not a Noether strongly 2-absorbing primary ideal since for every positive integer n, $P^n \nsubseteq Q$.

Remark 1. Every 2-absorbing ideal of a ring R is a uniformly 2-absorbing primary ideal, but the converse does not necessarily hold. For example, let p, q be two distinct prime numbers. Then $p^2q\mathbb{Z}$ is a 2-absorbing primary ideal of $\mathbb{Z}$, [4, Corollary 2.12]. On the other hand $(\sqrt{p^2q\mathbb{Z}})^2 = p^2q^2\mathbb{Z} \subseteq p^2q\mathbb{Z}$, and so $p^2q\mathbb{Z}$ is a Noether strongly 2-absorbing primary ideal of $\mathbb{Z}$. Hence Proposition 5 implies that $p^2q\mathbb{Z}$ is a uniformly 2-absorbing primary ideal. But, notice that $p^2q \in p^2q\mathbb{Z}$ and neither $p^2 \in p^2q\mathbb{Z}$ nor $pq \in p^2q\mathbb{Z}$ which shows that $p^2q\mathbb{Z}$ is not a 2-absorbing ideal of $\mathbb{Z}$. Also, it is easy to see that $p^2q\mathbb{Z}$ is not primary and so it is not a uniformly primary ideal of $\mathbb{Z}$. Consequently the two concepts of uniformly primary ideals and of uniformly 2-absorbing primary ideals are different in general.

Proposition 6. *Let R be a ring and Q be a proper ideal of R. If Q is a uniformly 2-absorbing primary ideal of R, then one of the following conditions must hold:*

1) $\sqrt{Q} = \mathfrak{p}$ *is a prime ideal.*
2) $\sqrt{Q} = \mathfrak{p} \cap \mathfrak{q}$, *where $\mathfrak{p}$ and $\mathfrak{q}$ are the only distinct prime ideals of R that are minimal over Q.*

Proof. Use [4, Theorem 2.3]. □

Let R be a ring and I be an ideal of R. We denote by $I^{[n]}$ the ideal of R generated by the n-th powers of all elements of I. If $n!$ is a unit in R, then $I^{[n]} = I^n$, see [2].

Theorem 1. *Let Q be a proper ideal of R. Then the following conditions are equivalent:*

1) *Q is uniformly primary;*
2) *There exists a positive integer n such that for every ideals I, J of R, $IJ \subseteq Q$ implies that either $I \subseteq Q$ or $J^{[n]} \subseteq Q$;*
3) *There exists a positive integer n such that for every $a \in R$ either $a \in Q$ or $(Q :_R a)^{[n]} \subseteq Q$;*

4) *There exists a positive integer* n *such that for every* $a \in R$ *either* $a^n \in Q$ *or* $(Q :_R a) = Q$.

Proof. (1)⇒(2) Suppose that Q is uniformly primary with $\operatorname{ord}(Q) = n$. Let $IJ \subseteq Q$ for some ideals $I,\ J$ of R. Assume that neither $I \subseteq Q$ nor $J^{[n]} \subseteq Q$. Then there exist elements $a \in I \backslash Q$ and $b^n \in J^{[n]} \backslash Q$, where $b \in J$. Since $ab \in IJ \subseteq Q$, then either $a \in Q$ or $b^n \in Q$, which is a contradiction. Therefore either $I \subseteq Q$ or $J^{[n]} \subseteq Q$.

(2)⇒(3) Note that $a(Q :_R a) \subseteq Q$ for every $a \in R$.

(3)⇒(1) and (1)⇔(4) have easy verifications. □

Corollary 1. *Let* R *be a ring. Suppose that* $n!$ *is a unit in* R *for every positive integer* n*, and* Q *is a proper ideal of* R*. The following conditions are equivalent:*

1) Q *is uniformly primary;*
2) *There exists a positive integer* n *such that for every ideals* I, J *of* R, $IJ \subseteq Q$ *implies that either* $I \subseteq Q$ *or* $J^n \subseteq Q$;
3) *There exists a positive integer* n *such that for every* $a \in R$ *either* $a \in Q$ *or* $(Q :_R a)^n \subseteq Q$;
4) *There exists a positive integer* n *such that for every* $a \in R$ *either* $a^n \in Q$ *or* $(Q :_R a) = Q$.

In the following theorem we characterize uniformly 2-absorbing primary ideals.

Theorem 2. *Let* Q *be a proper ideal of* R*. Then the following conditions are equivalent:*

1) Q *is uniformly 2-absorbing primary;*
2) *There exists a positive integer* n *such that for every* $a, b \in R$ *either* $(ab)^n \in Q$ *or* $(Q :_R ab) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R b)$;
3) *There exists a positive integer* n *such that for every* $a, b \in R$ *either* $(ab)^n \in Q$ *or* $(Q :_R ab) = (Q :_R a)$ *or* $(Q :_R ab) \subseteq (\sqrt{Q} :_R b)$;
4) *There exists a positive integer* n *such that for every* $a, b \in R$ *and every ideal* I *of* R, $abI \subseteq Q$ *implies that either* $aI \subseteq Q$ *or* $bI \subseteq \sqrt{Q}$ *or* $(ab)^n \in Q$;
5) *There exists a positive integer* n *such that for every* $a, b \in R$ *either* $ab \in Q$ *or* $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$;
6) *There exists a positive integer* n *such that for every* $a, b \in R$ *either* $ab \in Q$ *or* $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a)$ *or* $(Q :_R ab)^{[n]} \subseteq (Q :_R b^n)$.

Proof. (1)⇒(2) Suppose that Q is uniformly 2-absorbing primary with $2\text{-}\operatorname{ord}(Q) = n$. Assume that $a, b \in R$ such that $(ab)^n \notin Q$. Let $x \in (Q:_R ab)$.

Thus $xab \in Q$, and so either $xa \in Q$ or $xb \in \sqrt{Q}$. Hence $x \in (Q :_R a)$ or $x \in (\sqrt{Q} :_R b)$ which shows that $(Q :_R ab) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R b)$.

(2)⇒(3) By the fact that if an ideal is a subset of the union of two ideals, then it is a subset of one of them.

(3)⇒(4) Suppose that n is a positive number which exists by part (3). Let $a, b \in R$ and I be an ideal of R such that $abI \subseteq Q$ and $(ab)^n \notin Q$. Then $I \subseteq (Q :_R ab)$, and so $I \subseteq (Q :_R a)$ or $I \subseteq (\sqrt{Q} :_R b)$, by (3). Consequently $aI \subseteq Q$ or $bI \subseteq \sqrt{Q}$.

(4)⇒(1) Is easy.

(1)⇒(5) Suppose that Q is uniformly 2-absorbing primary with $2\text{-}\operatorname{ord}(Q) = n$. Assume that $a, b \in R$ such that $ab \notin Q$. Let $x \in (Q :_R ab)$. Then $abx \in Q$. So $ax \in \sqrt{Q}$ or $(bx)^n \in Q$. Hence $x^n \in (\sqrt{Q} :_R a)$ or $x^n \in (Q :_R b^n)$. Consequently $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$.

(5)⇒(6) Is similar to the proof of (2)⇒(3).

(6)⇒(1) Assume (6). Let $abc \in Q$ for some $a, b, c \in R$ such that $ab \notin Q$. Then $c \in (Q :_R ab)$ and thus $c^n \in (Q :_R ab)^{[n]}$. So, by part (6) we have that $c^n \in (\sqrt{Q} :_R a)$ or $c^n \in (Q :_R b^n)$. Therefore $ac \in \sqrt{Q}$ or $(bc)^n \in Q$, and so Q is uniformly 2-absorbing primary. □

Corollary 2. *Let R be a ring. Suppose that $n!$ is a unit in R for every positive integer n, and Q is a proper ideal of R. The following conditions are equivalent:*

1) *Q is uniformly 2-absorbing primary;*
2) *There exists a positive integer n such that for every $a, b \in R$ either $ab \in Q$ or $(Q :_R ab)^n \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$;*
3) *There exists a positive integer n such that for every $a, b \in R$ either $ab \in Q$ or $(Q :_R ab)^n \subseteq (\sqrt{Q} :_R a)$ or $(Q :_R ab)^n \subseteq (Q :_R b^n)$.*

Proposition 7. *Let Q be a uniformly 2-absorbing primary ideal of R and $x \in R \backslash Q$ be idempotent. The following conditions hold:*

1) $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$.
2) *$(Q :_R x)$ is a uniformly 2-absorbing primary ideal of R with 2-$\operatorname{ord}((Q :_R x)) \leqslant$ 2-$\operatorname{ord}(Q)$.*

Proof. (1) Is easy.

(2) Suppose that 2-$\operatorname{ord}(Q) = n$. Let $abc \in (Q :_R x)$ for some $a, b, c \in R$. Then $a(bc)x \in Q$ and so either $abc \in Q$ or $ax \in \sqrt{Q}$ or $(bc)^n x \in Q$. If $abc \in Q$, then either $ab \in Q \subseteq (Q :_R x)$ or $ac \in \sqrt{Q} \subseteq \sqrt{(Q :_R x)}$ or $(bc)^n \in Q \subseteq (Q :_R x)$. If $ax \in \sqrt{Q}$, then $ac \in (\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$ by part (1). In the third case we have $(bc)^n \in (Q :_R x)$. Hence $(Q :_R x)$ is a uniformly 2-absorbing primary ideal of R with 2-$\operatorname{ord}((Q :_R x)) \leqslant n$. □

Proposition 8. *Let I be a proper ideal of a ring R.*

1) *$\sqrt{I}$ is a 2-absorbing ideal of R.*
2) *For every $a, b, c \in R$, $abc \in I$ implies that $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$;*
3) *$\sqrt{I}$ is a 2-absorbing primary ideal of R;*
4) *$\sqrt{I}$ is a Noether 2-absorbing primary ideal of R ($2\text{-}\mathfrak{e}(\sqrt{I}) = 1$);*
5) *$\sqrt{I}$ is a uniformly 2-absorbing primary ideal of R.*

Proof. (1)⇒(2) It is trivial.

(2)⇒(1) Let $xyz \in \sqrt{I}$ for some $x, y, z \in R$. Then there exists a positive integer m such that $x^m y^m z^m \in I$. So, the hypothesis in (2) implies that $x^m y^m \in \sqrt{I}$ or $x^m z^m \in \sqrt{I}$ or $y^m z^m \in \sqrt{I}$. Hence $xy \in \sqrt{I}$ or $xz \in \sqrt{I}$ or $yz \in \sqrt{I}$ which shows that $\sqrt{I}$ is a 2-absorbing ideal.

(1)⇔(3) and (3)⇒(4) are clear.

(4)⇒(5) By Proposition 5.

(5)⇒(3) It is easy. □

Proposition 9. *If Q_1 is a uniformly P-primary ideal of R and Q_2 is a uniformly P-2-absorbing primary ideal of R such that $Q_1 \subseteq Q_2$, then $2\text{-}\operatorname{ord}(Q_2) \leqslant \operatorname{ord}(Q_1)$.*

Proof. Let $\operatorname{ord}(Q_1) = m$ and $2\text{-}\operatorname{ord}(Q_2) = n$. Then there are $a, b, c \in R$ such that $abc \in Q_2$, $ab \notin Q_2$, $ac \notin \sqrt{Q_2}$ and $(bc)^n \in Q_2$ but $(bc)^{n-1} \notin Q_2$. Thus $bc \in \sqrt{Q_2} = \sqrt{Q_1}$. Hence $(bc)^m \in Q_1 \subseteq Q_2$ by [6, Proposition 8]. Therefore, $n > m - 1$ and so $n \geqslant m$. □

Theorem 3. *Let R be a ring and $\{Q_i\}_{i \in I}$ be a chain of uniformly P-2-absorbing primary ideals such that $\max_{i \in I}\{2\text{-}\operatorname{ord}(Q_i)\} = n$, where n is a positive integer. Then $Q = \bigcap_{i \in I} Q_i$ is a uniformly P-2-absorbing primary ideal of R with $2\text{-}\operatorname{ord}(Q) \leqslant n$.*

Proof. It is clear that $\sqrt{Q} = \bigcap_{i \in I} \sqrt{Q_i} = P$. Let $a, b, c \in R$ such that $abc \in Q$, $ab \notin Q$ and $(bc)^n \notin Q$. Since $\{Q_i\}_{i \in I}$ is a chain, there exists some $k \in I$ such that $ab \notin Q_k$ and $(bc)^n \notin Q_k$. On the other hand Q_k is uniformly 2-absorbing primary with $2\text{-}\operatorname{ord}(Q_k) \leqslant n$, thus $ac \in \sqrt{Q_i} = \sqrt{Q}$, and so Q is a uniformly 2-absorbing primary ideal of R with $2\text{-}\operatorname{ord}(Q) \leqslant n$. □

In the following remark, we show that if Q_1 and Q_2 are uniformly 2-absorbing primary ideals of R, then $Q_1 \cap Q_2$ need not be a uniformly 2-absorbing primary ideal of R.

Remark 2. Let p, q, r be distinct prime numbers. Then $p^2q\mathbb{Z}$ and $r\mathbb{Z}$ are uniformly 2-absorbing primary ideals of $\mathbb{Z}$. Notice that $p^2qr \in p^2q\mathbb{Z} \cap r\mathbb{Z}$ and neither $p^2q \in p^2q\mathbb{Z} \cap r\mathbb{Z}$ nor $p^2r \in \sqrt{p^2q\mathbb{Z} \cap r\mathbb{Z}} = p\mathbb{Z} \cap q\mathbb{Z} \cap r\mathbb{Z}$ nor $qr \in \sqrt{p^2q\mathbb{Z} \cap r\mathbb{Z}} = p\mathbb{Z} \cap q\mathbb{Z} \cap r\mathbb{Z}$. Hence $p^2q\mathbb{Z} \cap r\mathbb{Z}$ is not a 2-absorbing primary ideal of $\mathbb{Z}$ which shows that it is not a uniformly 2-absorbing primary ideal of $\mathbb{Z}$.

Theorem 4. *Let $Q_1,\ Q_2$ be uniformly primary ideals of a ring R.*

1) *$Q_1 \cap Q_2$ is a uniformly 2-absorbing primary ideal of R with 2-$\mathrm{ord}(Q_1 \cap Q_2) \leqslant max\{\mathrm{ord}(Q_1), \mathrm{ord}(Q_2)\}$.*
2) *Q_1Q_2 is a uniformly 2-absorbing primary ideal of R with 2-$\mathrm{ord}(Q_1Q_2) \leqslant \mathrm{ord}(Q_1) + \mathrm{ord}(Q_2)$.*

Proof. (1) Let $Q_1,\ Q_2$ be uniformly primary. Set $n = \max\{\mathrm{ord}(Q_1), \mathrm{ord}(Q_2)\}$. Assume that for some $a, b, c \in R$, $abc \in Q_1 \cap Q_2$, $ab \notin Q_1 \cap Q_2$ and $ac \notin \sqrt{Q_1 \cap Q_2}$. Since Q_1 and Q_2 are primary ideals of R, then $Q_1 \cap Q_2$ is 2-absorbing primary by [4, Theorem 2.4]. Therefore $bc \in \sqrt{Q_1 \cap Q_2} = \sqrt{Q_1} \cap \sqrt{Q_2}$. By [6, Proposition 8] we have that $(bc)^{\mathrm{ord}(Q_1)} \in Q_1$ and $(bc)^{\mathrm{ord}(Q_2)} \in Q_2$. Hence $(bc)^n \in Q_1 \cap Q_2$ which shows that $Q_1 \cap Q_2$ is uniformly 2-absorbing primary and 2-$\mathrm{ord}(Q_1 \cap Q_2) \leqslant n$.

(2) Similar to the proof in (1). □

We recall from [7], if R is an integral domain and P is a prime ideal of R that can be generated by a regular sequence of R, then, for each positive integer n, the ideal P^n is a P-primary ideal of R.

Lemma 1. *([6, Corollary 4]) Let R be a ring and P be a prime ideal of R. If P^n is a P-primary ideal of R for some positive integer n, then P^n is a uniformly primary ideal of R with $\mathrm{ord}(P^n) \leqslant n$.*

Corollary 3. *Let R be a ring and $P_1,\ P_2$ be prime ideals of R. If P_1^n is a P_1-primary ideal of R for some positive integer n and P_2^m is a P_2-primary ideal of R for some positive integer m, then $P_1^nP_2^m$ and $P_1^n \cap P_2^m$ are uniformly 2-absorbing primary ideals of R with 2-$\mathrm{ord}(P_1^nP_2^m) \leqslant n + m$ and 2-$\mathrm{ord}(P_1^n \cap P_2^m) \leqslant max\{n, m\}$.*

Proof. By Theorem 4 and Lemma 1. □

Proposition 10. *Let $f : R \longrightarrow R'$ be a homomorphism of commutative rings. Then the following statements hold:*

1) *If Q' is a uniformly 2-absorbing primary ideal of R', then $f^{-1}(Q')$ is a uniformly 2-absorbing primary ideal of R with 2-$\mathrm{ord}_R(f^{-1}(Q')) \leqslant$ 2-$\mathrm{ord}_{R'}(Q')$.*

2) *If f is an epimorphism and Q is a uniformly 2-absorbing primary ideal of R containing* $\ker(f)$*, then $f(Q)$ is a uniformly 2-absorbing primary ideal of R' with* $2\text{-}\operatorname{ord}_{R'}(f(Q)) \leqslant 2\text{-}\operatorname{ord}_R(Q)$.

Proof. (1) Set $N = 2\text{-}\operatorname{ord}_{R'}(Q')$. Let $a, b, c \in R$ such that $abc \in f^{-1}(Q')$, $ab \notin f^{-1}(Q')$ and $ac \notin \sqrt{f^{-1}(Q')} = f^{-1}(\sqrt{Q'})$. Then $f(abc) = f(a)f(b)f(c) \in Q'$, $f(ab) = f(a)f(b) \notin Q'$ and $f(ac) = f(a)f(c) \notin \sqrt{Q'}$. Since Q' is a uniformly 2-absorbing primary ideal of R', then $f^N(bc) \in Q'$. Then $f((bc)^N) \in Q'$ and so $(bc)^N \in f^{-1}(Q')$. Thus $f^{-1}(Q')$ is a uniformly 2-absorbing primary ideal of R with $2\text{-}\operatorname{ord}_R(f^{-1}(Q')) \leqslant N = 2\text{-}\operatorname{ord}_{R'}(Q')$.

(2) Set $N = 2\text{-}\operatorname{ord}_R(Q)$. Let $a, b, c \in R'$ such that $abc \in f(Q)$, $ab \notin f(Q)$ and $ac \notin \sqrt{f(Q)}$. Since f is an epimorphism, then there exist $x, y, z \in R$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$. Then $f(xyz) = abc \in f(Q)$, $f(xy) = ab \notin f(Q)$ and $f(xz) = ac \notin \sqrt{f(Q)}$. Since $\ker(f) \subseteq Q$, then $xyz \in Q$. Also $xy \notin Q$, and $xz \notin \sqrt{Q}$, since $f(\sqrt{Q}) \subseteq \sqrt{f(Q)}$. Then $(yz)^N \in Q$ since Q is a uniformly 2-absorbing primary ideal of R. Thus $f((yz)^N) = (f(y)f(z))^N = (bc)^N \in f(Q)$. Therefore, $f(Q)$ is a uniformly 2-absorbing primary ideal of R'. Moreover $2\text{-}\operatorname{ord}_{R'}(f(Q)) \leqslant N = 2\text{-}\operatorname{ord}_R(Q)$. □

As an immediate consequence of Proposition 10 we have the following result:

Corollary 4. *Let R be a ring and Q be an ideal of R.*

1) *If R' is a subring of R and Q is a uniformly 2-absorbing primary ideal of R, then $Q \cap R'$ is a uniformly 2-absorbing primary ideal of R' with* $2\text{-}\operatorname{ord}_{R'}(Q \cap R') \leqslant 2\text{-}\operatorname{ord}_R(Q)$.
2) *Let I be an ideal of R with $I \subseteq Q$. Then Q is a uniformly 2-absorbing primary ideal of R if and only if Q/I is a uniformly 2-absorbing primary ideal of R/I.*

Corollary 5. *Let Q be an ideal of a ring R. Then $\langle Q, X\rangle$ is a uniformly 2-absorbing primary ideal of $R[X]$ if and only if Q is a uniformly 2-absorbing primary ideal of R.*

Proof. By Corollary 4(2) and regarding the isomorphism $\langle Q, X\rangle/\langle X\rangle \simeq Q$ in $R[X]/\langle X\rangle \simeq R$ we have the result. □

Corollary 6. *Let R be a ring, Q a proper ideal of R and $X = \{X_i\}_{i\in I}$ a collection of indeterminates over R. If $QR[X]$ is a uniformly 2-absorbing primary ideal of $R[X]$, then Q is a uniformly 2-absorbing primary ideal of R with* $2\text{-}\operatorname{ord}_R(Q) \leqslant 2\text{-}\operatorname{ord}_{R[X]}(QR[X])$.

Proof. It is clear from Corollary 4(1). □

Proposition 11. *Let S be a multiplicatively closed subset of R and Q be a proper ideal of R. Then the following conditions hold:*

1) *If Q is a uniformly 2-absorbing primary ideal of R such that $Q\cap S=\varnothing$, then $S^{-1}Q$ is a uniformly 2-absorbing primary ideal of $S^{-1}R$ with $2\text{-}\operatorname{ord}(S^{-1}Q)\leqslant 2\text{-}\operatorname{ord}(Q)$.*
2) *If $S^{-1}Q$ is a uniformly 2-absorbing primary ideal of $S^{-1}R$ and $S\cap Z_Q(R)=\varnothing$, then Q is a uniformly 2-absorbing primary ideal of R with $2\text{-}\operatorname{ord}(Q)\leqslant 2\text{-}\operatorname{ord}(S^{-1}Q)$.*

Proof. (1) Set $N:=2\text{-}\operatorname{ord}(Q)$. Let $a,b,c\in R$ and $s,t,k\in S$ such that $\frac{a}{s}\frac{b}{t}\frac{c}{k}\in S^{-1}Q$, $\frac{a}{s}\frac{b}{t}\notin S^{-1}Q$, $\frac{a}{s}\frac{c}{k}\notin\sqrt{S^{-1}Q}=S^{-1}\sqrt{Q}$. Thus there is $u\in S$ such that $uabc\in Q$. By assumptions we have that $uab\notin Q$ and $uac\notin\sqrt{Q}$. Since Q is a uniformly 2-absorbing primary ideal of R, then $(bc)^N\in Q$. Hence $(\frac{b}{t}\frac{c}{k})^N\in S^{-1}Q$. Consequently, $S^{-1}Q$ is a uniformly 2-absorbing primary ideal of $S^{-1}R$ and $2\text{-}\operatorname{ord}(S^{-1}Q)\leqslant N=2\text{-}\operatorname{ord}(Q)$.

(2) Set $N:=2\text{-}\operatorname{ord}(S^{-1}Q)$. Let $a,b,c\in R$ such that $abc\in Q$, $ab\notin Q$ and $ac\notin\sqrt{Q}$. Then $\frac{abc}{1}=\frac{a}{1}\frac{b}{1}\frac{c}{1}\in S^{-1}Q$, $\frac{ab}{1}=\frac{a}{1}\frac{b}{1}\notin S^{-1}Q$ and $\frac{ac}{1}=\frac{a}{1}\frac{c}{1}\notin\sqrt{S^{-1}Q}=S^{-1}\sqrt{Q}$, because $S\cap Z_Q(R)=\varnothing$ and $S\cap Z_{\sqrt{Q}}(R)=\varnothing$. Since $S^{-1}Q$ is a uniformly 2-absorbing primary ideal of $S^{-1}R$, then $(\frac{b}{1}\frac{c}{1})^N=\frac{(bc)^N}{1}\in S^{-1}Q$. Then there exists $u\in S$ such that $u(bc)^N\in Q$. Hence $(bc)^N\in Q$ because $S\cap Z_Q(R)=\varnothing$. Thus Q is a uniformly 2-absorbing primary ideal of R and $2\text{-}\operatorname{ord}(Q)\leqslant N=2\text{-}\operatorname{ord}(S^{-1}Q)$. □

Proposition 12. *Let Q be a 2-absorbing primary ideal of a ring R and $P=\sqrt{Q}$ be a finitely generated ideal of R. Then Q is a Noether strongly 2-absorbing primary ideal of R. Thus Q is a uniformly 2-absorbing primary ideal of R.*

Proof. It is clear from [14, Lemma 8.21] and Proposition 5. □

Corollary 7. *Let R be a Noetherian ring and Q a proper ideal of R. Then the following conditions are equivalent:*

1) Q is a uniformly 2-absorbing primary ideal of R;
2) Q is a Noether strongly 2-absorbing primary ideal of R;
3) Q is a 2-absorbing primary ideal of R.

Proof. Apply Proposition 5 and Proposition 12. □

We recall from [8] the construction of idealization of a module. Let R be a ring and M be an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $\sqrt{I}(+)M = \sqrt{I(+)M}$.

Proposition 13. *Let R be a ring, Q be a proper ideal of R and M be an R-module. The following conditions are equivalent:*

1) *$Q(+)M$ is a uniformly 2-absorbing primary ideal of $R(+)M$;*
2) *Q is a uniformly 2-absorbing primary ideal of R.*

Proof. The proof is routine. □

Theorem 5. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with $1 \neq 0$. Let Q be a proper ideal of R. Then the following conditions are equivalent:*

1) *Q is a uniformly 2-absorbing primary ideal of R;*
2) *Either $Q = Q_1 \times R_2$ for some uniformly 2-absorbing primary ideal Q_1 of R_1 or $Q = R_1 \times Q_2$ for some uniformly 2-absorbing primary ideal Q_2 of R_2 or $Q = Q_1 \times Q_2$ for some uniformly primary ideal Q_1 of R_1 and some uniformly primary ideal Q_2 of R_2.*

Proof. (1)⇒(2) Assume that Q is a uniformly 2-absorbing primary ideal of R with $2\text{-ord}_R(Q) = n$. We know that Q is in the form of $Q_1 \times Q_2$ for some ideal Q_1 of R_1 and some ideal Q_2 of R_2. Suppose that $Q_2 = R_2$. Since Q is a proper ideal of R, $Q_1 \neq R_1$. Let $R' = \frac{R}{\{0\} \times R_2}$. Then $Q' = \frac{Q}{\{0\} \times R_2}$ is a uniformly 2-absorbing primary ideal of R' by Corollary 4(2). Since R' is ring-isomorphic to R_1 and $Q_1 \simeq Q'$, Q_1 is a uniformly 2-absorbing primary ideal of R_1. Suppose that $Q_1 = R_1$. Since Q is a proper ideal of R, $Q_2 \neq R_2$. By a similar argument as in the previous case, Q_2 is a uniformly 2-absorbing primary ideal of R_2. Hence assume that $Q_1 \neq R_1$ and $Q_2 \neq R_2$. We claim that Q_1 is a uniformly primary ideal of R_1. Assume that $x,\ y \in R_1$ such that $xy \in Q_1$ but $x \notin Q_1$. Notice that $(x, 1)(1, 0)(y, 1) = (xy, 0) \in Q$, but neither $(x, 1)(1, 0) = (x, 0) \in Q$ nor $(x, 1)(y, 1) = (xy, 1) \in \sqrt{Q}$. So $[(1, 0)(y, 1)]^n = (y^n, 0) \in Q$. Therefore $y^n \in Q_1$. Thus Q_1 is a uniformly primary ideal of R_1 with $\text{ord}_{R_1}(Q_1) \leqslant n$. Now, we claim that Q_2 is a uniformly primary ideal of R_2. Suppose that for some $z,\ w \in R_2$, $zw \in Q_2$ but $z \notin Q_2$. Notice that $(1, z)(0, 1)(1, w) = (0, zw) \in Q$, but neither $(1, z)(0, 1) = (0, z) \in Q$ nor $(1, z)(1, w) = (1, zw) \in \sqrt{Q}$. Therefore $[(0, 1)(1, w)]^n = (0, w^n) \in Q$, and so $w^n \in Q_2$ which shows that Q_2 is a uniformly primary ideal of R_2 with $\text{ord}_{R_2}(Q_2) \leqslant n$. Consequently when $Q_1 \neq R_1$ and $Q_2 \neq R_2$ we have that $\max\{\mathrm{ord}_{R_1}(Q_1), \mathrm{ord}_{R_2}(Q_2)\} \leqslant$ 2-$\mathrm{ord}_R(Q)$.

(2)⇒(1) If $Q = Q_1 \times R_2$ for some uniformly 2-absorbing primary ideal Q_1 of R_1, or $Q = R_1 \times Q_2$ for some uniformly 2-absorbing primary ideal Q_2 of R_2, then it is clear that Q is a uniformly 2-absorbing primary ideal of R. Hence assume that $Q = Q_1 \times Q_2$ for some uniformly primary ideal Q_1 of R_1 and some uniformly primary ideal Q_2 of R_2. Then $Q'_1 = Q_1 \times R_2$ and $Q'_2 = R_1 \times Q_2$ are uniformly primary ideals of R with $\mathrm{ord}_R(Q'_1) \leqslant \mathrm{ord}_{R_1}(Q_1)$ and $\mathrm{ord}_R(Q'_2) \leqslant \mathrm{ord}_{R_2}(Q_2)$. Hence $Q'_1 \cap Q'_2 = Q_1 \times Q_2 = Q$ is a uniformly 2-absorbing primary ideal of R with 2-$\mathrm{ord}_R(Q) \leqslant \max\{\mathrm{ord}_{R_1}(Q_1), \mathrm{ord}_{R_2}(Q_2)\}$ by Theorem 4. □

Lemma 2. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_1, R_2, \ldots, R_n$ are rings with $1 \neq 0$. A proper ideal Q of R is a uniformly primary ideal of R if and only if $Q = \times_{i=1}^n Q_i$ such that for some $k \in \{1, 2, \ldots, n\}$, Q_k is a uniformly primary ideal of R_k, and $Q_i = R_i$ for every $i \in \{1, 2, \ldots, n\} \backslash \{k\}$.*

Proof. (⇒) Let Q be a uniformly primary ideal of R with $\mathrm{ord}_R(Q) = m$. We know $Q = \times_{i=1}^n Q_i$ where for every $1 \leqslant i \leqslant n$, Q_i is an ideal of R_i, respectively. Assume that Q_r is a proper ideal of R_r and Q_s is a proper ideal of R_s for some $1 \leqslant r < s \leqslant n$. Since

$$(0, \ldots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \ldots, 0)(0, \ldots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \ldots, 0) = (0, \ldots, 0) \in Q,$$

then either $(0, \ldots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \ldots, 0) \in Q$ or $(0, \ldots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \ldots, 0)^m \in Q$, which is a contradiction. Hence exactly one of the Q_i's is proper, say Q_k. Now, we show that Q_k is a uniformly primary ideal of R_k. Let $ab \in Q_k$ for some $a, b \in R_k$ such that $a \notin Q_k$. Therefore

$$\begin{aligned}(0, \ldots, 0, \overbrace{a}^{k\text{-th}}, 0, \ldots, 0)&(0, \ldots, 0, \overbrace{b}^{k\text{-th}}, 0, \ldots, 0)\\ &= (0, \ldots, 0, \overbrace{ab}^{k\text{-th}}, 0, \ldots, 0) \in Q,\end{aligned}$$

but $(0, \ldots, 0, \overbrace{a}^{k\text{-th}}, 0, \ldots, 0) \notin Q$, and so $(0, \ldots, 0, \overbrace{b}^{k\text{-th}}, 0, \ldots, 0)^m \in Q$. Thus $b^m \in Q_k$ which implies that Q_k is a uniformly primary ideals of R_k with $\mathrm{ord}_{R_k}(Q_k) \leqslant m$.

(⇐) Is easy. □

Theorem 6. *Let* $R = R_1 \times R_2 \times \cdots \times R_n$*, where* $2 \leqslant n < \infty$*, and* $R_1, R_2, \ldots, R_n$ *are rings with* $1 \neq 0$*. For a proper ideal* Q *of* R *the following conditions are equivalent:*

1) Q *is a uniformly 2-absorbing primary ideal of* R*.*
2) *Either* $Q = \times_{t=1}^{n} Q_t$ *such that for some* $k \in \{1, 2, \ldots, n\}$*,* Q_k *is a uniformly 2-absorbing primary ideal of* R_k*, and* $Q_t = R_t$ *for every* $t \in \{1, 2, \ldots, n\} \backslash \{k\}$ *or* $Q = \times_{t=1}^{n} Q_t$ *such that for some* $k, m \in \{1, 2, \ldots, n\}$*,* Q_k *is a uniformly primary ideal of* R_k*,* Q_m *is a uniformly primary ideal of* R_m*, and* $Q_t = R_t$ *for every* $t \in \{1, 2, \ldots, n\} \backslash \{k, m\}$*.*

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 5 . Then let $3 \leqslant n < \infty$ and suppose that the result is valid when $K = R_1 \times \cdots \times R_{n-1}$. We show that the result holds when $R = K \times R_n$. By Theorem 5, Q is a uniformly 2-absorbing primary ideal of R if and only if either $Q = L \times R_n$ for some uniformly 2-absorbing primary ideal L of K or $Q = K \times L_n$ for some uniformly 2-absorbing primary ideal L_n of R_n or $Q = L \times L_n$ for some uniformly primary ideal L of K and some uniformly primary ideal L_n of R_n. Notice that by Lemma 2, a proper ideal L of K is a uniformly primary ideal of K if and only if $L = \times_{t=1}^{n-1} Q_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, Q_k is a uniformly primary ideal of R_k, and $Q_t = R_t$ for every $t \in \{1, 2, \ldots, n-1\} \backslash \{k\}$. Consequently we reach the claim. □

2. Special 2-absorbing primary ideals

Definition 2. We say that a proper ideal Q of a ring R is *special 2-absorbing primary* if it is uniformly 2-absorbing primary with 2-ord$(Q) = 1$.

Remark 3. By Proposition 4(2), every primary ideal is a special 2-absorbing primary ideal. But the converse is not true in general. For example, let $p,\ q$ be two distinct prime numbers. Then $pq\mathbb{Z}$ is a 2-absorbing ideal of $\mathbb{Z}$ and so it is a special 2-absorbing primary ideal of $\mathbb{Z}$, by Proposition 4(1). Clearly $pq\mathbb{Z}$ is not primary.

Recall that a prime ideal $\mathfrak{p}$ of R is called *divided prime* if $\mathfrak{p} \subset xR$ for every $x \in R \backslash \mathfrak{p}$.

Proposition 14. *Let* Q *be a special 2-absorbing primary ideal of* R *such that* $\sqrt{Q} = \mathfrak{p}$ *is a divided prime ideal of* R*. Then* Q *is a* $\mathfrak{p}$*-primary ideal of* R*.*

Proof. Let $xy \in Q$ for some $x, y \in R$ such that $y \notin \mathfrak{p}$. Then $x \in \mathfrak{p}$. Since $\mathfrak{p}$ is a divided prime ideal, $\mathfrak{p} \subset yR$ and so there exists $r \in R$ such that $x = ry$. Hence $xy = ry^2 \in Q$. Since Q is special 2-absorbing primary and $y \notin \mathfrak{p}$, then $x = ry \in Q$. Consequently Q is a $\mathfrak{p}$-primary ideal of R. □

Remark 4. Let p, q be distinct prime numbers. Then by [4, Theorem 2.4] we can deduce that $p\mathbb{Z} \cap q^2\mathbb{Z}$ is a 2-absorbing primary ideal of $\mathbb{Z}$. Since $pq^2 \in p\mathbb{Z} \cap q^2\mathbb{Z}$, $pq \notin p\mathbb{Z} \cap q^2\mathbb{Z}$ and $q^2 \notin p\mathbb{Z} \cap q\mathbb{Z}$, then $p\mathbb{Z} \cap q^2\mathbb{Z}$ is not a special 2-absorbing primary ideal of $\mathbb{Z}$.

Notice that for $n = 1$ we have that $I^{[n]} = I$.

Theorem 7. *Let Q be a proper ideal of R. Then the following conditions are equivalent:*

1) *Q is special 2-absorbing primary;*
2) *For every $a, b \in R$ either $ab \in Q$ or $(Q :_R ab) = (Q :_R a)$ or $(Q :_R ab) \subseteq (\sqrt{Q} :_R b)$;*
3) *For every $a, b \in R$ and every ideal I of R, $abI \subseteq Q$ implies that either $ab \in Q$ or $aI \subseteq Q$ or $bI \subseteq \sqrt{Q}$;*
4) *For every $a \in R$ and every ideal I of R either $aI \subseteq Q$ or $(Q :_R aI) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R I)$;*
5) *For every $a \in R$ and every ideal I of R either $aI \subseteq Q$ or $(Q :_R aI) = (Q :_R a)$ or $(Q :_R aI) \subseteq (\sqrt{Q} :_R I)$;*
6) *For every $a \in R$ and every ideals I, J of R, $aIJ \subseteq Q$ implies that either $aI \subseteq Q$ or $IJ \subseteq \sqrt{Q}$ or $aJ \subseteq Q$;*
7) *For every ideals I, J of R either $IJ \subseteq \sqrt{Q}$ or $(Q :_R IJ) \subseteq (Q :_R I) \cup (Q :_R J)$;*
8) *For every ideals I, J of R either $IJ \subseteq \sqrt{Q}$ or $(Q :_R IJ) = (Q :_R I)$ or $(Q :_R IJ) = (Q :_R J)$;*
9) *For every ideals I, J, K of R, $IJK \subseteq Q$ implies that either $IJ \subseteq \sqrt{Q}$ or $IK \subseteq Q$ or $JK \subseteq Q$.*

Proof. (1)⇔(2)⇔(3) By Theorem 2.

(3)⇒(4) Let $a \in R$ and I be an ideal of R such that $aI \nsubseteq Q$. Suppose that $x \in (Q :_R aI)$. Then $axI \subseteq Q$, and so by part (3) we have that $x \in (Q :_R a)$ or $x \in (\sqrt{Q} :_R I)$. Therefore $(Q :_R aI) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R I)$.

(4)⇒(5)⇒(6)⇒(7)⇒(8)⇒(9)⇒(1) Have straightforward proofs. □

Theorem 8. *Let Q be a special 2-absorbing primary ideal of R and $x \in R \backslash \sqrt{Q}$. The following conditions hold:*

1) *$(Q :_R x) = (Q :_R x^n)$ for every $n \geqslant 2$.*

2) $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$.
3) $(Q :_R x)$ *is a special 2-absorbing primary ideal of* R.

Proof. (1) Clearly $(Q :_R x) \subseteq (Q :_R x^n)$ for every $n \geqslant 2$. For the converse inclusion we use induction on n. First we get $n = 2$. Let $r \in (Q :_R x^2)$. Then $rx^2 \in Q$, and so either $rx \in Q$ or $x^2 \in \sqrt{Q}$. Notice that $x^2 \in \sqrt{Q}$ implies that $x \in \sqrt{Q}$ which is a contradiction. Therefore $rx \in Q$ and so $r \in (Q :_R x)$. Therefore $(Q :_R x) = (Q :_R x^2)$. Now, assume $n > 2$ and suppose that the claim holds for $n-1$, i.e. $(Q :_R x) = (Q :_R x^{n-1})$. Let $r \in (Q :_R x^n)$. Then $rx^n \in Q$. Since $x \notin \sqrt{Q}$, then we have either $rx^{n-1} \in Q$ or $rx \in Q$. Both two cases implies that $r \in (Q :_R x)$. Consequently $(Q :_R x) = (Q :_R x^n)$.

(2) It is easy to investigate that $\sqrt{(Q :_R x)} \subseteq (\sqrt{Q} :_R x)$. Let $r \in (\sqrt{Q} :_R x)$. Then there exists a positive integer m such that $(rx)^m \in Q$. So, by part (1) we have that $r^m \in (Q :_R x)$. Hence $r \in \sqrt{(Q :_R x)}$. Thus $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$.

(3) Let $abc \in (Q :_R x)$ for some $a, b, c \in R$. Then $ax(bc) \in Q$ and so $ax \in Q$ or $abc \in Q$ or $bcx \in \sqrt{Q}$. In the first case, we have $ab \in (Q :_R x)$. If $abc \in Q$, then either $ab \in Q \subseteq (Q :_R x)$ or $ac \in Q \subseteq (Q :_R x)$ or $bc \in \sqrt{Q} \subseteq \sqrt{(Q :_R x)}$. In the third case we have $bc \in (\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$ by part (2). Therefore $(Q :_R x)$ is a special 2-absorbing primary ideal of R. □

Theorem 9. *Let* Q *be an irreducible ideal of* R*. Then* Q *is special 2-absorbing primary if and only if* $(Q :_R x) = (Q :_R x^2)$ *for every* $x \in R \backslash \sqrt{Q}$.

Proof. (⇒) By Theorem 8.

(⇐) Let $abc \in Q$ for some $a, b, c \in R$ such that neither $ab \in Q$ nor $ac \in Q$ nor $bc \in \sqrt{Q}$. We search for a contradiction. Since $bc \notin \sqrt{Q}$, then $b \notin \sqrt{Q}$. So, by our hypothesis we have $(Q :_R b) = (Q :_R b^2)$. Let $r \in (Q + Rab) \cap (Q + Rac)$. Then there are $q_1, q_2 \in Q$ and $r_1, r_2 \in R$ such that $r = q_1 + r_1ab = q_2 + r_2ac$. Hence $q_1b + r_1ab^2 = q_2b + r_2abc \in Q$. Thus $r_1ab^2 \in Q$, i.e., $r_1a \in (Q :_R b^2) = (Q :_R b)$. Therefore $r_1ab \in Q$ and so $r = q_1 + r_1ab \in Q$. Then $Q = (Q + Rab) \cap (Q + Rac)$, which contradicts the assumption that Q is irreducible. □

A ring R is said to be a *Boolean ring* if $x = x^2$ for all $x \in R$. It is famous that every prime ideal in a Boolean ring R is maximal. Notice that every ideal of a Boolean ring R is radical. So, every (uniformly) 2-absorbing primary ideal of R is a 2-absorbing ideal of R.

Corollary 8. *Let R be a Boolean ring. Then every irreducible ideal of R is a maximal ideal.*

Proof. Let I be an irreducible ideal of R. Thus, Theorem 9 implies that I is special 2-absorbing primary. Therefore by Proposition 6, either $I = \sqrt{I}$ is a maximal ideal or is the intersection of two distinct maximal ideals. Since I is irreducible, then I cannot be in the second form. Hence I is a maximal ideal. □

Proposition 15. *Let Q be a special 2-absorbing primary ideal of R and $\mathfrak{p}$, $\mathfrak{q}$ be distinct prime ideals of R.*

1) *If $\sqrt{Q} = \mathfrak{p}$, then $\{(Q :_R x) \mid x \in R\backslash\mathfrak{p}\}$ is a totally ordered set.*
2) *If $\sqrt{Q} = \mathfrak{p} \cap \mathfrak{q}$, then $\{(Q :_R x) \mid x \in R\backslash\mathfrak{p} \cup \mathfrak{q}\}$ is a totally ordered set.*

Proof. (1) Let $x, y \in R\backslash\mathfrak{p}$. Then $xy \in R\backslash\mathfrak{p}$. It is clear that $(Q :_R x) \cup (Q :_R y) \subseteq (Q :_R xy)$. Assume that $r \in (Q :_R xy)$. Therefore $rxy \in Q$, whence $rx \in Q$ or $ry \in Q$, because $xy \notin \sqrt{Q}$. Consequently $(Q :_R xy) = (Q :_R x) \cup (Q :_R y)$. Thus, either $(Q :_R xy) = (Q :_R x)$ or $(Q :_R xy) = (Q :_R y)$, and so either $(Q :_R y) \subseteq (Q :_R x)$ or $(Q :_R x) \subseteq (Q :_R y)$.

(2) Is similar to the proof of (1). □

Corollary 9. *Let $f : R \longrightarrow R'$ be a homomorphism of commutative rings. Then the following statements hold:*

1) *If Q' is a special 2-absorbing primary ideal of R', then $f^{-1}(Q')$ is a special 2-absorbing primary ideal of R.*
2) *If f is an epimorphism and Q is a special 2-absorbing primary ideal of R containing $\ker(f)$, then $f(Q)$ is a special 2-absorbing primary ideal of R'.*

Proof. By Proposition 10. □

Let R be a ring with identity. We recall that if $f = a_0 + a_1X + \cdots + a_tX^t$ is a polynomial on the ring R, then *content* of f is defined as the ideal of R, generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \ldots, a_t)$. Let T be an R-algebra and c the function from T to the ideals of R defined by $c(f) = \cap\{I \mid I \text{ is an ideal of } R \text{ and } f \in IT\}$ known as the content of f. Note that the content function c is nothing but the generalization of the content of a polynomial $f \in R[X]$. The R-algebra T is called a *content R-algebra* if the following conditions hold:

1) For all $f \in T$, $f \in c(f)T$.

2) (Faithful flatness) For any $r \in R$ and $f \in T$, the equation $c(rf) = rc(f)$ holds and $c(1_T) = R$.
3) (Dedekind-Mertens content formula) For each $f, g \in T$, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.

For more information on content algebras and their examples we refer to [11], [12] and [13]. In [10] Nasehpour gave the definition of a Gaussian R-algebra as follows: Let T be an R-algebra such that $f \in c(f)T$ for all $f \in T$. T is said to be a Gaussian R-algebra if $c(fg) = c(f)c(g)$, for all $f, g \in T$.

Example 3. ([10]) Let T be a content R-algebra such that R is a Prüfer domain. Since every nonzero finitely generated ideal of R is a cancellation ideal of R, the Dedekind-Mertens content formula causes T to be a Gaussian R-algebra.

Theorem 10. *Let R be a Prüfer domain, T a content R-algebra and Q an ideal of R. Then Q is a special 2-absorbing primary ideal of R if and only if QT is a special 2-absorbing primary ideal of T.*

Proof. ($\Rightarrow$) Assume that Q is a special 2-absorbing primary ideal of R. Let $fgh \in QT$ for some $f, g, h \in T$. Then $c(fgh) \subseteq Q$. Since R is a Prüfer domain and T is a content R-algebra, then T is a Gaussian R-algebra. Therefore $c(fgh) = c(f)c(g)c(h) \subseteq Q$. Since Q is a special 2-absorbing primary ideal of R, Theorem 7 implies that either $c(f)c(g) = c(fg) \subseteq Q$ or $c(f)c(h) = c(fh) \subseteq Q$ or $c(g)c(h) = c(gh) \subseteq \sqrt{Q}$. So $fg \in c(fg)T \subseteq QT$ or $fh \in c(fh)T \subseteq QT$ or $gh \in c(gh)T \subseteq \sqrt{Q}T \subseteq \sqrt{QT}$. Consequently QT is a special 2-absorbing primary ideal of T.

($\Leftarrow$) Note that since T is a content R-algebra, $QT \cap R = Q$ for every ideal Q of R. Now, apply Corollary 4(1). □

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminates is an example of content algebras.

Corollary 10. *Let R be a Prüfer domain and Q be an ideal of R. Then Q is a special 2-absorbing primary ideal of R if and only if $Q[X]$ is a special 2-absorbing primary ideal of $R[X]$.*

Corollary 11. *Let S be a multiplicatively closed subset of R and Q be a proper ideal of R. Then the following conditions hold:*

1) *If Q is a special 2-absorbing primary ideal of R such that $Q \cap S = \varnothing$, then $S^{-1}Q$ is a special 2-absorbing primary ideal of $S^{-1}R$ with 2-$\operatorname{ord}(S^{-1}Q) \leqslant$ 2-$\operatorname{ord}(Q)$.*

2) *If* $S^{-1}Q$ *is a special 2-absorbing primary ideal of* $S^{-1}R$ *and* $S \cap Z_Q(R) = \varnothing$*, then* Q *is a special 2-absorbing primary ideal of* R *with* $2\text{-}\operatorname{ord}(Q) \leqslant 2\text{-}\operatorname{ord}(S^{-1}Q)$.

Proof. By Proposition 11. □

In view of Theorem 5 and its proof, we have the following result.

Corollary 12. *Let* $R = R_1 \times R_2$*, where* R_1 *and* R_2 *are rings with* $1 \neq 0$*. Let* Q *be a proper ideal of* R*. Then the following conditions are equivalent:*

1) Q *is a special 2-absorbing primary ideal of* R*;*
2) *Either* $Q = Q_1 \times R_2$ *for some special 2-absorbing primary ideal* Q_1 *of* R_1 *or* $Q = R_1 \times Q_2$ *for some special 2-absorbing primary ideal* Q_2 *of* R_2 *or* $Q = Q_1 \times Q_2$ *for some prime ideal* Q_1 *of* R_1 *and some prime ideal* Q_2 *of* R_2*.*

Corollary 13. *Let* $R = R_1 \times R_2$*, where* R_1 *and* R_2 *are rings with* $1 \neq 0$*. Suppose that* Q_1 *is a proper ideal of* R_1 *and* Q_2 *is a proper ideal of* R_2*. Then* $Q_1 \times Q_2$ *is a special 2-absorbing primary ideal of* R *if and only if it is a 2-absorbing ideal of* R*.*

Proof. See Corollary 12 and apply [1, Theorem 4.7]. □

Corollary 14. *Let* $R = R_1 \times R_2 \times \cdots \times R_n$*, where* $2 \leqslant n < \infty$*, and* $R_1, R_2, \ldots, R_n$ *are rings with* $1 \neq 0$*. For a proper ideal* Q *of* R *the following conditions are equivalent:*

1) Q *is a special 2-absorbing primary ideal of* R*.*
2) *Either* $Q = \times_{t=1}^{n} Q_t$ *such that for some* $k \in \{1, 2, \ldots, n\}$*,* Q_k *is a special 2-absorbing primary ideal of* R_k*, and* $Q_t = R_t$ *for every* $t \in \{1, 2, \ldots, n\} \backslash \{k\}$ *or* $Q = \times_{t=1}^{n} Q_t$ *such that for some* $k, m \in \{1, 2, \ldots, n\}$*,* Q_k *is a prime ideal of* R_k*,* Q_m *is a prime ideal of* R_m*, and* $Q_t = R_t$ *for every* $t \in \{1, 2, \ldots, n\} \backslash \{k, m\}$*.*

Proof. By Theorem 6. □

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