Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **23** (2017). Number 2, pp. 216–222

On unicyclic graphs of metric dimension 2

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Communicated by A. P. Petravchuk

Abstract. A metric basis S of a graph G is the subset of vertices of minimum cardinality such that all other vertices are uniquely determined by their distances to the vertices in S. The metric dimension of a graph G is the cardinality of the subset S. A unicyclic graph is a graph containing exactly one cycle. The construction of a knitting unicyclic graph is introduced. Using this construction all unicyclic graphs with two main vertices and metric dimensions 2 are characterized.

Introduction

The notion of a metric basis was introduced by L. Blumenthal in [1] for semimetric spaces. F. Harary and R. Melter considered the concepts of a metric basis and metric dimension for simple, connected graphs in [2]. In general case the problem to find a metric basis of a graph is NP-hard [3].

There are three main ways to study metric dimension of graphs. The first way is a characterization of metric dimension of some families of graphs. For example, in [4] metric dimension of trees was considered, in [5] metric dimension of wheels was determined, in [6] for metric dimension of unicyclic graphs some bounds were given. The second is a characterization of metric dimension of constructions of graphs (e.g. [7], [8]). The third way is a description of graphs having a fixed value of metric dimensions. For instance, in [9] it was proved that a graph G has metric dimension 1

2010 MSC: 05C12.
Key words and phrases: distance, metric dimensions, unicyclic graph.

if and only if G is a chain, the metric dimension of G equals to $n-1$ if and only if G is the complete graph on n vertices and all graphs on n vertices with metric dimension $n-2$ were characterized.

In this paper we characterize all unicyclic graphs with two main vertices such that their metric dimension equals 2.

1. Metric dimension of unicyclic graphs

We consider only simple, finite, undirected, connected and nontrivial graphs. Let $G=(V,E)$ be a graph with set of vertices V and set of edges E. The *degree* $deg_G(v)$ of a vertex v in G is the number of edges that incident to v in G. The *path* between v_1 and v_2 in graph G is a sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_n$, such that any edge e_i is incident to vertices v_i and v_i+1, $1 \leqslant i \leqslant n-1$. A *unicyclic graph* is a graph containing exactly one cycle.

The *distance* between two vertices v_1 and v_2 is denoted by $d_G(v_1,v_2)$ and it equals to the length of the shortest path between v_1 and v_2. We denote by C_n and L_n the cycle and the path on n vertices correspondingly. For unspecified notions in graph theory we refer to [10].

A vertex u of a graph G is said *to resolve* two vertices v_1 and v_2 of graph G if the following inequality holds:

$$d_G(u,v_1) \neq d_G(u,v_2).$$

An ordered vertex set S of G is a *resolving* set of G if every two distinct vertices of G are resolved by some vertex of S. A resolving set also is called a *metric generator*. A *metric basis* of G is a resolving set of minimum cardinality. The *metric dimension* of G is the cardinality of its basis. We denote metric dimension of G by the symbol $dimG$.

Let $\widehat{G}=(\widehat{V},\widehat{E})$ be a subgraph of the unicyclic graph $G=(V,E)$, which is a simple cycle. In other words, $\widehat{G}$ is isomorphic to C_m for some positive integer m.

Proposition 1. *Let $G=(V,E)$ be a unicyclic graph. If metric dimension of G equals 2 then for any $v \in V \setminus \widehat{V}$ the inequality $deg_G(v) \leqslant 3$ holds.*

Proof. Assume that w is a vertex of G such that $deg_G(w) \geqslant 4$. Then there are 4 vertices u_1, u_2, u_3, u_4 such that the distance from any of these vertices to w equals 1 (see Figure 1). Then all pairs of vertices u_i, u_j,

$1 \leqslant i < j \leqslant 4$ are resolved by some set S of vertices that consists of more than three vertices. Therefore, $\dim(G) > 2$. □

In this paper we consider only graphs $G = (V, E)$ such that the inequality $deg_G(v) \leqslant 3$ holds for all $v \in V$.

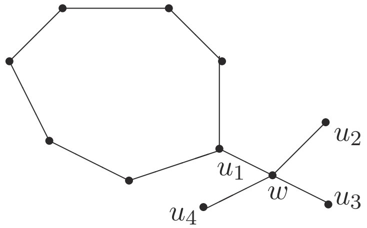

FIGURE 1. Graph G.

A vertex $u \in V \setminus \widehat{V}$ of the graph G is said to be *projected* in the vertex $w \in \widehat{V}$ if for any vertex $q \in \widehat{V}$ the inequality

$$d_G(u, w) < d_G(u, q)$$

holds. A vertex with degree 3 from $\widehat{G}$, in which the vertices that have degree 3 and are located outside the cycle are projected, is called a *main vertex*. For example, the graph G on Figure 1 has a unique main vertex w.

We need the following lemma from [11].

Lemma 1 ([11])**.** *Let $G = (V, E)$ be a unicyclic graph and* $\dim(G) = 2$*. Then there exist at most two main vertices in the graph G.*

A vertex of degree at least 3 in a graph G will be called an exterior vertex of G. Any end-vertex u of G is said to be a terminal vertex of an exterior vertex v of the graph G if for every other exterior vertex w of G the inequality $d_G(u, v) < d_G(u, w)$ holds. An exterior vertex v will be called *two-leaf* if there exist two different terminal vertices of the vertex v. An exterior vertex v will be called *one-leaf* if there exist exactly one terminal vertex of the vertex v. For example, the vertex z_1 is two-leaf and the vertex w is one-leaf on Figure 3.

Lemma 2. *Let $G = (V, E)$ be a unicyclic graph and* $\dim(G) = 2$*. A vertex $v \in \widehat{V}$ with degree* 3 *is a main vertex of the graph G if an only if v is not a one-leaf vertex.*

Proof. Let $v \in \widehat{V}$ with degree 3 be a main vertex. Then there is an exterior vertex $w \in V \setminus \widehat{V}$ that is projected in v. Moreover, v is a vertex of a cycle G_1. Hence, v is not a one-leaf vertex. Note, that v is not a two-leaf vertex also.

Assume now, that $u \in \widehat{V}$ is an one-leaf vertex. Then there is one terminal end-vertex $z \in V \setminus \widehat{V}$ of the vertex v projected in v. Hence there is no vertices with degree 3 projected in u. Therefore, the vertex u is not a main vertex of graph G. □

Lemma 3. *Let $G = (V, E)$ be a unicyclic graph and* $\dim(G) = 2$*. Then for any main vertex v of G there exists exactly one two-leaf vertex that projected in v.*

The proof of this lemma directly follows from the proof of Lemma 1 in [11].

2. Metric dimension of knitting of unicyclic graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Fix vertices $v_1 \in V_1$ and $v_2 \in V_2$. A graph G is built from G_1 and G_2 by *gluing* along the vertices v_1 and v_2 if $G = (V, E)$ has the set of vertices $V = V_1 \cup (V_2 \setminus v_2)$ and the set of edges $E = E_1 \cup E_2$ (a vertex v_2 is replaced by v_1 for all edges of G_2). Roughly speaking, we identify vertices v_1 and v_2 of graphs G_1 and G_2 (see Figure 2).

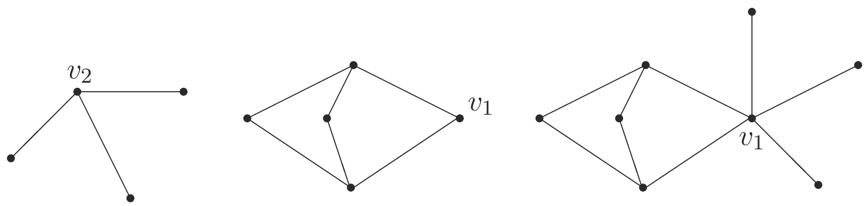

Figure 2. Gluing of two graphs.

Definition 1. A unicyclic graph G is said to be a *basic graph* if following conditions hold:

(A) for any vertex v from G $deg_G(v) \leqslant 3$;

(B) for any main vertex v of G there exists exactly one two-leaf vertex projected in v;

(C) G has exactly two main vertices;
(D) $\widehat{G}$ has only main vertices with degree more than two.

Definition 2. Let now G_1 be a basic graph. Denote the main vertices of G_1 by u and v. A unicyclic graph G is called a *knitting* of the graph G_1 by chains $L_1, \ldots, L_r$ if G is obtained from the graph G_1 by gluing vertices with degree 2 of its cycle and beginings of the chains $L_1, \ldots, L_r$ such that each vertex with degree 2 of the cycle is glued to the end of exactly one chain and for any one-leaf vertex w and any adjacent to w vertex a the following inequality holds (see Figure 3):

$$d_G(u,v) + d_G(v,w) + 1 \neq d_G(u,a). \tag{1}$$

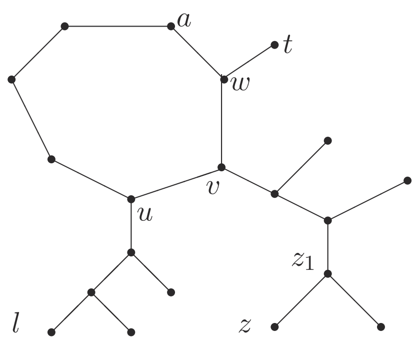

FIGURE 3. Knitting condition does not hold.

A unicyclic graph G is *even*, if $|\widehat{V}| = 2k$ for some positive integer k. A unicyclic graph G is *odd*, if $|\widehat{V}| = 2k+1$ for some positive integer k.

Proposition 2. *Let $G = (V,E)$ be an even unicyclic graph with two main vertices u and v, $|\widehat{V}| = 2k$. If $dimG = 2$, then $d(u,v) \neq k$.*

Proof. Assume that $dimG = 2$ and $d(u,v) = k$. Since u and v are main vertices of graph G, a resolving set of G contains two end-vertices z and l projected in u and v respectively. Let a and b be vertices from the cycle $\widehat{G}$ adjacent to u. Hence, $d_G(a,u) = d_G(b,u) = 1$ and $d_G(a,v) = d_G(b,v) = k-1$. Therefore, the set $\{z,l\}$ is not a resolving set and $dimG > 2$. □

Theorem 1. *An odd unicyclic graph $G = (V,E)$ with two main vertices has metric dimension 2 if and only if one of the following conditions hold:*

1) *G is a basic graph;*

2) *G is a knitting of some basic graph G_1 .*

An even unicyclic graph $G = (V, E)$ with two main vertices u and v, $|\widehat{V}| = 2k$, has metric dimension 2 if and only if one of the conditions 1) or 2) holds and $d(u, v) \neq k$.

Proof. **1.** Assume, that $G = (V, E)$ is an odd basic graph. Then if some vertex u from $\widehat{G}$ has degree 3 then u is a main vertex of G. Let u and v be main vertices of the graph G, z and l be end-vertices projected in u and v respectively. It is not hard to verify that the set $\{z, l\}$ is a resolving set of the graph G. Since vertices of a cycle are resolved of two vertices the set $\{z, l\}$ is a metric basic and then $dimG = 2$. If G is a knitting of some basic graph G_1 then from the construction of knitting it follows that a metric basis of G_1 is a metric basis of G.

2. Let now $G = (V, E)$ be an odd unicyclic graph with two main vertices and metric dimension 2. It follows from Proposition 1, Lemma 3 and Lemma 1 that for the graph G conditions $(A) - (C)$ of Definition 1 hold. If $\widehat{G}$ has only main vertices with degree more than two then G is a basic graph. Assume that $\widehat{G}$ has not only main vertices with degree 3. From Lemma 2 it follows that all of these vertices are one-leaf ones. Then the graph G can be considered as a knitting of some basic graph. □

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Received by the editors: 08.06.2017
and in final form 10.06.2017.