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Some aspects of Leibniz algebra theory

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To Professor N. N. Semko on the occasion of his 60th birthday

ABSTRACT. One of the key tendencies in the development of Leibniz algebra theory is the search for analogues of the basic results of Lie algebra theory. In this survey, we consider the reverse situation. Here the main attention is paid to the results reflecting the difference of the Leibniz algebras from the Lie algebras.

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if it satisfies the (left) Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a,b,c \in L$.

We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras appeared first in the papers of A.M. Bloh [13–15], in which he called them the *D-algebras*. However, in that time these works were not in demand, and they have not been properly developed. Only after two decades, a real interest to Leibniz algebras rose. It happened

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thanks to the work of J.-L. Loday [33] (see also [34, Section 10.6]), who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Gottfried Wilhelm Leibniz who discovered and proved the *Leibniz rule* for differentiation of functions. Later, some authors used to call these algebras by Loday algebras though J.-L. Loday himself, sometimes under the nom-de-plume Guillaume William Zinbiel (here Zinbiel is the inverse of Leibniz), in survey [48] noted that this term does not fit.

An algebra R over a field F is called *right Leibniz algebra* if it satisfies the (right) Leibniz identity

$$[a,[b,c]] = [[a,b],c] - [[a,c],b]$$

for all $a,b,c \in R$.

Note at once that the classes of left Leibniz algebras and right Leibniz algebras are different. The following simple example justifies it.

Example 1. Let F be an arbitrary field and L be a vector space over F having a basis $\{a,b\}$. Define the operation $[\cdot,\cdot]$ on L by the following rule:

$$[a,a] = [a,b] = b, [b,a] = [b,b] = 0.$$

It is not hard to check that L becomes a left Leibniz algebra. But

$$0 = [[a,a],a] \neq [[a,a],a] + [a,[a,a]] = [a,b] = b.$$

Let R be a right Leibniz algebra, then put $\subset a,b \supset = [b,a]$. Then we have

$$\begin{aligned} \subset\subset a,b\supset, c\supset &= [c,[b,a]] = [[c,b],a] - [[c,a],b] \\ &= \subset a, \subset b,c\supset\supset - \subset b, \subset a,c\supset\supset . \end{aligned}$$

Thus, this substitution leads us to a left Leibniz algebra. Similarly, we can make a transfer from a left Leibniz algebra to a right Leibniz algebra.

An algebra L over a field F is called a *symmetric Leibniz algebra* if it is both a left and right Leibniz algebra.

We prefer to work with left Leibniz algebras even though many authors prefer to consider right Leibniz algebras. The choice of left Leibniz algebras is more suitable for us because they have more visible relationships with the differentiation of products (in which the differential operator is written to the right of a differentiable object).

Thus, in this article, the term a Leibniz algebra stands for a left Leibniz algebra.

The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, non-commutative geometry, and so on. They found some applications in physics (see, for example, [16, 23, 24]). The theory of Leibniz algebras has been developing quite intensively but un-even. On one hand, some analogues of important results from the theory of Lie algebras were proven. On the other hand, natural questions about the structure of Leibniz algebras are not considered. For example, until very recently, the cyclic subalgebras of Leibniz algebras were not fully described. In this survey, we want to gather and to systematize the main results that clarify to some extent the structure of Leibniz algebras. We will not touch issues related to the study of homological problems, we will not focus on the connections of Leibniz algebras, as well as issues related to the applications of Leibniz algebras. Note, that most of the results obtained to date relate to finite dimensional algebras. We will try to focus on the overall results, i.e. the results that hold for both finite dimensional and infinite dimensional algebras. Our goal is to see which parts of the picture involving the general structure of Leibniz algebras have already been drawn, and this will allow us to see which parts of this picture should be drawn further. Many results on of Lie algebras are practically unchanged carried over to Leibniz algebras. But we would like to draw attention not to results of this kind, but to results showing the differences between Leibniz algebras and Lie algebras.

Note at once that if L is a Lie algebra, then

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0.$$

It follows that

$$[[a,b],c]=-[[b,c],a]-[[c,a],b]=[a,[b,c]]+[b,[c,a]]=[a,[b,c]]-[b,[a,c]],$$

which shows that every Lie algebra is a Leibniz algebra.

Conversely, suppose that $[a,a]=0$ for each element $a\in L$. Then for arbitrary elements $a,b\in L$ we have

$$0=[a+b,a+b]=[a,a]+[a,b]+[b,a]+[b,b]=[a,b]+[b,a].$$

It follows that $[a,b]=-[b,a]$. Then we obtain

$$\begin{aligned}0&=[[a,b],c]-[a,[b,c]]+[b,[a,c]]\\&=[[a,b],c]+[[b,c],a]-[[a,c],b]\\&=[[a,b],c]+[[b,c],a]+[[c,a],b]\end{aligned}$$

for all $a, b, c \in L$. Thus, Lie algebras can be characterized as Leibniz algebras in which $[a, a] = 0$ for every element $a \in L$.

Like Lie algebras, Leibniz algebras are also associated with associative algebras, but this connection is a little more complicated.

Let A be an associative algebra over a field F and let $f : A \to A$ be an endomorphism of A such that $f^2 = f$. Define the binary operation $[\cdot, \cdot]$ on A by the following rule:

$$[a, b] = f(a)b - bf(a)$$

for all elements $a, b \in A$. We have

$$\begin{aligned}
[[a, b], c] &= [f(a)b - bf(a), c] \\
&= f(f(a)b - bf(a))c - cf(f(a)b - bf(a)) \\
&= f(f(a)b)c - f(bf(a))c - cf(f(a)b) + cf(bf(a)) \\
&= f(a)f(b)c - f(b)f(a)c - cf(a)f(b) + cf(b)f(a); \\
[a, [b, c]] &= [a, f(b)c - cf(b)] \\
&= f(a)(f(b)c - cf(b)) - (f(b)c - cf(b))f(a) \\
&= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a); \\
[b, [a, c]] &= [b, f(a)c - cf(a)] \\
&= f(b)(f(a)c - cf(a)) - (f(a)c - cf(a))f(b) \\
&= f(b)f(a)c - f(b)cf(a) - f(a)cf(b) + cf(a)f(b).
\end{aligned}$$

Then

$$\begin{aligned}
[a, [b, c]] - [b, [a, c]] &= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a) \\
&\quad - (f(b)f(a)c - f(b)cf(a) - f(a)cf(b) + cf(a)f(b)) \\
&= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a) \\
&\quad - f(b)f(a)c + f(b)cf(a) + f(a)cf(b) - cf(a)f(b)) \\
&= f(a)f(b)c + cf(b)f(a) - f(b)f(a)c - cf(a)f(b) \\
&= [[a, b], c].
\end{aligned}$$

Thus, with respect to the operations $+$ and $[\cdot, \cdot]$ A becomes a Leibniz algebra.

Note that if f is the identity permutation of A, then we obtain a standard transition from associative algebras to Lie algebras.

Note the following useful property of the elements of Leibniz algebras. We have

$$\begin{aligned}
[a, [b, c]] &= [[a, b], c] + [b, [a, c]], \\
[b, [a, c]] &= [[b, a], c] + [a, [b, c]]
\end{aligned}$$

or

$$[a,[b,c]] = [b,[a,c]] - [[b,a],c].$$

It follows that

$$[[a,b],c] + [b,[a,c]] = [b,[a,c]] - [[b,a],c],$$

and hence

$$[[a,b],c] = -[[b,a],c].$$

A Leibniz algebra L is called *abelian* (or *trivial*) if $[a,b] = 0$ for every elements $a, b \in L$. In particular, an abelian Leibniz algebra is a Lie algebra.

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$. As usual, a subspace A of L is called a *subalgebra* of L, if $[x, y] \in A$ for every $x, y \in A$. It follows that $[A, A] \leqslant A$.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L, then $\langle M \rangle$ denote the subalgebra of L generated by M.

A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[y, x] \in A$ (respectively $[x, y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$).

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, that is $[y, x], [x, y] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider a *factor-algebra* L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

Denote by $\mathbf{Leib}(L)$ the subspace, generated by the elements $[a, a]$, $a \in L$. We note that $\mathbf{Leib}(L)$ is an ideal of L. Indeed, for arbitrary elements $a, x \in L$ we have

$$[a,[a,x]] = [[a,a],x] + [a,[a,x]],$$

so $[[a, a], x] = 0$. Furthermore,

$$\begin{aligned}[x+[a,a], x+[a,a]] &= [x,x] + [x,[a,a]] + [[a,a],x] + [[a,a],[a,a]] \\ &= [x,x] + [x,[a,a]].\end{aligned}$$

It follows that $[x, [a, a]] = [x + [a, a], x + [a, a]] - [x, x] \in \mathbf{Leib}(L)$.

Put $K = \mathbf{Leib}(L)$. Then in factor-algebra L/K we have

$$[a+K, a+K] = [a,a] + K = K$$

for each element $a \in L$. By mentioned above we obtain that L/K is a Lie algebra. Conversely, suppose that H is an ideal of L such that L/H is a Lie algebra. Then

$$H = [a + H, a + H] = [a, a] + H,$$

which implies that $[a, a] \in H$ for every element $a \in L$. Then $\mathbf{Leib}(L) \leqslant H$.

The ideal $\mathbf{Leib}(L)$ is called the *Leibniz kernel* of algebra L.

We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0.$$

This property shows that $\mathbf{Leib}(L)$ is an abelian subalgebra of L.

Let L be a Leibniz algebra. Define the *lower central series*

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

of L by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap\limits_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots] \ldots]]$. Note the following useful properties of subalgebras and ideals.

Proposition 1. *Let L be a Leibniz algebra over a field F.*

(i) *If H is an ideal of L, then $[H, H]$ is an ideal of L.*
(ii) *If H is an ideal of L, then $[L, H]$ is a subalgebra of L.*
(iii) *If H is an ideal of L, then $[H, L]$ is a subalgebra of L.*
(iv) *If H is an ideal of L, then $[L, H] + [H, L]$ is an ideal of L.*
(v) *If H is an ideal of L, then $[\gamma_j(H), \gamma_k(H)] \leqslant \gamma_{j+k}(H)$ for every positive integers j, k.*
(vi) *If H is an ideal of L, then $\gamma_j(H)$ is an ideal of L for each positive integer j. In particular, $\gamma_j(L)$ is an ideal of L for each positive integer j.*
(vii) *If H is an ideal of L, then $\gamma_j(\gamma_k(H)) \leqslant \gamma_{jk}(H)$ for every positive integers j, k.*

We remark that if A, B are ideals of a Leibniz algebra L, then, in general, $[A, B]$ needs not be an ideal. The following example justifies it (see [11]).

Example 2. Let L be a vector space over a field F with the basis $\{e_1, e_2, e_3, e_4, e_5\}$. Define the operation on basis vectors by the following rule

$[\cdot,\cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	0	e_2	0	e_4	e_5
e_2	$-e_2$	0	e_4	0	0
e_3	0	e_5	0	0	0
e_4	e_5	0	0	0	0
e_5	$-e_5$	0	0	0	0

Let $A = Fe_2 + Fe_4 + Fe_5$ and $B = Fe_3 + Fe_4 + Fe_5$. It is not hard to check that A, B are ideals of L. However, $[A, B] = Fe_4$ is not an ideal.

As usual, we say that a Leibniz algebra L is *finite dimensional*, if the dimension L as a vector space over F is finite. The condition to be finite dimensional is very strong. That is why the majority of results on Leibniz algebras were obtained for finite dimensional Leibniz algebras.

If $\mathbf{dim}_F(L) = 1$, then $L = Fa$ for some element $a \in L$. Then $[a, a] = \alpha a$ where $\alpha \in F$. We have

$$0 = [[a, a], a] = [\alpha a, a] = \alpha[a, a] = \alpha^2 a.$$

It follows that $\alpha = 0$, that is $[a, a] = 0$ and L is abelian.

Suppose now that $\mathbf{dim}_F(L) = 2$ and L is not a Lie algebra. It follows that $K = \mathbf{Leib}(L)$ is non-zero. Since K is abelian, $K \neq L$. Hence there exists an element a such that $b = [a, a] \neq 0$. By this choice, $a \notin K$. Then $L = Fa + Fb$, and we have $[b, a] = 0$. The fact that K is an ideal of L implies that $[a, b] = \beta b$ for some $\beta \in F$. Suppose that $\beta \neq 0$ and put $c = \beta^{-1}a$. Then $[c, b] = \beta^{-1}[a, b] = \beta^{-1}\beta b = b$. We have

$$[c, c] = \beta^{-2}[a, a] = \beta^{-2}b = d,$$

and

$$[c, d] = [c, \beta^{-2}b] = \beta^{-2}[c, b] = \beta^{-2}b = d.$$

By this choice, $\{c, d\}$ is a basis of L. Thus, we obtain the following two non-isomorphic algebras:

$$L_1 = Fa + Fb, [a, a] = b, [b, a] = [a, b] = [b, b] = 0,$$

and

$$L_2 = Fc + Fd, [c, c] = [c, d] = d, [d, c] = [d, d] = 0.$$

The structure of 3-dimensional Leibniz algebras is more complicated. Investigation of Leibniz algebras, having dimensions 3 and 4 has been conducted in the papers [1, 2, 4, 5, 17, 18, 20–22, 39, 41, 46].

One of the first questions that naturally arises in the study of any algebraic structure is the question of the structure of its cyclic substructures (that is, substructures generated by one element). In particular, for a Leibniz algebra, the question of the structure of its cyclic subalgebras naturally arises. Unlike Lie algebras, associative algebras, groups, etc., cyclic Leibniz algebras is no necessarily abelian. We now give some concepts that will be needed farther and not only for this description.

The *left* (respectively *right*) *center* $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{x \in L|\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{right}(L) = \{x \in L|\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\mathbf{Leib}(L) \leqslant \zeta^{left}(L)$, so that $L/\zeta^{left}(L)$ is a Lie algebra. The right center is a subalgebra of L, and in general, the left and right centers are different; they even may have different dimensions. We will construct now the following examples [30].

Example 3. Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus Fe_4$ and define an operation $[\cdot, \cdot]$ by the following rule:

$$\begin{array}{llll} [e_1, e_1] = e_2, & [e_1, e_2] = -e_2 - e_3, & [e_1, e_3] = e_2 + e_3, & [e_1, e_4] = 0, \\ [e_2, e_1] = 0, & [e_3, e_1] = 0, & [e_4, e_1] = e_2 + e_3, & [e_j, e_k] = 0 \end{array}$$

for all $j, k \in \{2, 3, 4\}$. It is possible to check that this operation defines a Leibniz algebra. We can see that $\zeta^{right}(L) = Fe_4$ and $\zeta^{right}(L)$ is not an ideal. Furthermore, $\zeta^{left}(L) = Fe_2 \oplus Fe_3$, so that

$$\zeta^{right}(L) \cap \zeta^{left}(L) = \langle 0 \rangle.$$

Moreover, $\mathbf{dim}_F(\zeta^{right}(L)) = 1$, $\mathbf{dim}_F(\zeta^{left}(L)) = 2$. Note also that $[L, L] = \mathbf{Leib}(L) = \zeta^{left}(L)$.

The *center* $\zeta(L)$ of L is the intersection of the left and right centers, that is

$$\zeta(L) = \{x \in L|\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Example 4. Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Z$ where a subspace Z has a countable basis $\{z_n |\ n \in \mathbb{N}\}$. Put $[z_n, x] = 0$ for every $x \in L$ and

$$[e_1, e_1] = [e_2, e_2] = [e_1, e_2] = [e_2, e_1] = z_1, [e_1, z_1] = [e_2, z_1] = 0.$$

By such definitions, we have

$$0 = [[e_j, e_k], e_m] \text{ and } [e_j, [e_k, e_m]] - [e_k, [e_j, e_m]] = 0 - 0 = 0$$

for all $j, k, m \in \{1, 2\}$. Take into account the equalities

$$0 = [[e_1, e_2], z] = [e_1, [e_2, z]] - [e_2, [e_1, z]],$$
$$0 = [[e_2, e_1], z] = [e_2, [e_1, z]] - [e_1, [e_2, z]],$$

we obtain $[e_2, [e_1, z]] - [e_1, [e_2, z]] = 0$ for every $z \in Z$. Now we put

$$[e_1, z_j] = z_j, [e_2, z_j] = z_{j+1}$$

for all $j > 1$. By this definition, we have

$$0 = [[e_j, z], e_k] \text{ and } [e_j, [z, e_k]] - [z, [e_j, e_k]] = [e_j, 0] - 0 = 0,$$
$$0 = [[z, e_j], e_k] \text{ and } [z, [e_j, e_k]] - [e_j, [z, e_k]] = 0 - [e_j, 0] = 0$$

for all $j, k \in \{1, 2\}$ and $z \in Z$. As we have seen above

$$[[e_j, e_k], z] = [e_j, [e_k, z]] - [e_k, [e_j, z]]$$

for all $j, k \in \{1, 2\}$ and $z \in Z$. Hence, L is a Leibniz algebra. By it construction Z is a left center of L, the right center coincides with the center of L and coincides with Fz_1, so that, the left center has finite codimension (and therefore, infinite dimension) and the right center and the center have finite dimension. By the construction, $[L, L] = Z$. Furthermore

$$[e_1 + z_1, e_1 + z_1] = [e_1, e_1] + [z_1, e_1] + [e_1, z_1] + [z_1, z_1] = z_1,$$
$$[e_1 + z_j, e_1 + z_j] = [e_1, e_1] + [z_j, e_1] + [e_1, z_j] + [z_j, z_j] = z_1 + z_j$$

for $j > 1$. It follows that $\mathbf{Leib}(L) = Z$.

Clearly, the center $\zeta(L)$ is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

Now we define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \ldots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup\limits_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter. Denote by $\mathbf{zl}(L)$ the length of upper central series of L.

The introduced here concepts of the upper and lower central series for Leibniz algebras are an analogous of others similar concepts, which became standard in several algebraic structures. They play an important role, for example, in Lie algebras and groups. Following this analogy, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\mathbf{ncl}(L)$.

It is a well-known fact for Lie algebras and groups that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length.

Consider the factors $\gamma_k(L)/\gamma_{k+1}(L)$, $k \in \mathbb{N}$. By definition $[L, \gamma_k(L)] = \gamma_{k+1}(L)$. By Proposition 1, $[\gamma_k(L), L] = [\gamma_k(L), \gamma_1(L)] \leqslant \gamma_{k+1}(L)$.

Let

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots C_\alpha \leqslant C_{\alpha+1} \leqslant \ldots C_\gamma = L$$

be an *ascending series* of ideals of Leibniz algebra L. This series is called *central* if $C_{\alpha+1}/C_\alpha \leqslant \zeta(L/C_\alpha)$ for each ordinal $\alpha < \gamma$ and $C_\lambda = \bigcup\limits_{\mu<\lambda} C_\mu$ for the limit ordinals λ. In other words, $[C_{\alpha+1}, L], [L, C_{\alpha+1}] \leqslant C_\alpha$ for each ordinal $\alpha < \gamma$.

Proposition 2 ([30])**.** *Let L be an Leibniz algebra over a field F, and*

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = L$$

be a finite central series of L. Then

(i) $\gamma_j(L) \leqslant C_{n-j+1}$ *for every* $1 \leqslant j \leqslant n+1$, *so that* $\gamma_{n+1}(L) = \langle 0 \rangle$.

(ii) $C_j \leqslant \zeta_j(L)$ *for every* $0 \leqslant j \leqslant n$, *so that* $\zeta_n(L) = L$.

(iii) *If j, k are positive integer such that $k \geqslant j$, then* $[\gamma_j(L), \zeta_k(L)]$, $[\zeta_k(L), \gamma_j(L)] \leqslant \zeta_{k-j}(L)$.

Corollary 1 ([30])**.** *Let L be an Leibniz algebra over a field F and suppose that L has a finite central series*

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = L.$$

Then L is nilpotent and $\mathbf{ncl}(L) \leqslant n$. Furthermore, the upper central series of L is finite, $\zeta_\infty(L) = L$, $\mathbf{zl}(L) \leqslant n$. Moreover, $\mathbf{ncl}(L) = \mathbf{zl}(L)$.

This Corollary shows that a Leibniz algebra L is nilpotent if and only if there is a positive integer k such that $L = \zeta_k(L)$. The least positive integer having this property coincides with nilpotency class of L. So, as in the cases of Lie algebras and groups, the definition of nilpotency can be given here using the notion of the upper central series.

Here it will be appropriate to note the fact that the Leibniz algebra L can be associative. Indeed, if $[L, L] = \gamma_2(L) \leqslant \zeta(L)$, then $0 = [[x, y], z] = [x, [y, z]]$ for all $x, y, z \in L$. Conversely, suppose that L is associative. Then, taking into account the equality $[[x, y], z] = [x, [y, z]]$, from $[[x, y], z] = [x, [y, z] - [y, [x, z]]$ we derive that $[y, [x, z]] = 0$. Since it is true for all $x, y, z \in L$, $[L, L] \leqslant \zeta^{right}(L)$. Furthermore, $0 = [y, [x, z]] = [[y, x], z]$, which shows that $[L, L] \leqslant \zeta^{left}(L)$. So we obtain

Proposition 3. *Let L be a Leibniz algebra over a field F. Then L is associative if and only if $[L, L] \leqslant \zeta(L)$.*

Let L be a Leibniz algebra over a field F and d be an element of L. Put

$$\mathbf{ln}_1(d) = d, \quad \mathbf{ln}_2(d) = [d, d], \quad \mathbf{ln}_{k+1}(d) = [d, \mathbf{ln}_k(d)], \qquad k \in \mathbb{N}.$$

Lemma 1 ([19]). *Let L be a Leibniz algebra over a field F, $a \in L$. Then every non-zero product of k copies of an element a with any bracketing is coincides with $\mathbf{ln}_k(a)$. Hence a cyclic subalgebra $\langle a \rangle$ is generated as a subspace by the elements $\mathbf{ln}_k(a)$, $k \in \mathbb{N}$.*

The following two natural cases appear here.

The elements $d_j = \mathbf{ln}_j(d)$, $j \in \mathbb{N}$ are linearly independent. In this case, a subalgebra $D = \langle d \rangle$ has the lower central series

$$D = \gamma_1(D) \geqslant \gamma_2(D) \geqslant \ldots \geqslant \gamma_j(D) \geqslant \gamma_{j+1}(D) \geqslant \gamma_\omega(D) = \langle 0 \rangle$$

of the length ω, and $\gamma_j(D) = \bigoplus_{t \geqslant j} F d_t$, $j \in \mathbb{N}$. In this case, we will say that an element d has *infinite depth*.

Consider the second possibility, when elements $d_j = \mathbf{ln}_j(d)$, $j \in \mathbb{N}$, are not linearly independent. In this case, we have

Lemma 2 ([19]). *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. If there exists a positive integer k such that $\mathbf{ln}_{k+1}(a) \in F\mathbf{ln}_1(a) + \ldots + F\mathbf{ln}_k(a)$, then $D = F\mathbf{ln}_1(a) + \ldots + F\mathbf{ln}_k(a)$.*

In particular, in this case, the subalgebra $D = \langle d \rangle$ has finite dimension over F and we will say that an element d has *finite depth*. Let k be the least positive integer such that $\mathbf{ln}_1(d), \dots, \mathbf{ln}_k(d)$ are linearly independent, but $\mathbf{ln}_1(d), \dots, \mathbf{ln}_k(d), \mathbf{ln}_{k+1}(d)$ are not linearly independent. Then the subset $\{\mathbf{ln}_1(d), \dots, \mathbf{ln}_k(d)\}$ is a basis of D and $\mathbf{dim}_F(D) = k$. In this case, we can say that an *element d has depth k*.

The case when an element d has finite depth turned out to be much more diverse. The following theorem has described this case.

Theorem 1 ([19]). *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Suppose that an element a has finite depth. Then D is an algebra of one of the following types:*

(i) *$D = Fa$ is abelian, $[a, a] = 0$.*

(ii) *There exists a positive integer k such that $\mathbf{ln}_k(a) \neq 0$, $\mathbf{ln}_{k+1}(a) = 0$, that is D is a nilpotent cyclic algebra.*

(iii) *$D = V \oplus U$ where V is an abelian ideal, $V \leqslant \zeta^{left}(D)$, U is a nilpotent cyclic subalgebra, $[D, D] = V \oplus [U, U]$ is an abelian ideal.*

(iv) *$D = \zeta^{left}(D) \oplus \zeta^{right}(D)$ where $[D, D] = \zeta^{left}(D) = F\mathbf{ln}_2(a) + \dots + F\mathbf{ln}_k(a)$, $\zeta^{right}(D) = Fc$ for some element $c \in D$ and $[c, y] = [a, y]$ for each element $y \in \zeta^{left}(D)$.*

For the case when $F = \mathbb{C}$ is a field of complex number, a description of cyclic finite dimensional Leibniz algebras were obtained in the paper [40]. Unlike Theorem 1, it does not show the structure of cyclic Leibniz algebras and based on the following. Let an element a has a depth k. Then $\mathbf{ln}_{k+1}(d) = \alpha_2\mathbf{ln}_2(a) + \dots + \alpha_k\mathbf{ln}_k(a)$, for some $\alpha_j \in \mathbb{C}$, $2 \leqslant j \leqslant k$. In the paper [40] a characterization for the set of coefficients $(\alpha_2, \dots, \alpha_k)$ was obtained.

As we already noted above, Lie algebras are a partial type of Leibniz algebras. In this regard, it is interesting to see how the Leibniz algebras, which are the minimal non Lie algebras with all proper subalgebras of which are Lie algebras are organized. A description of such algebras was obtained in [19].

Theorem 2 ([19]). *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is a Lie algebra. Then L is an algebra of one of the following types:*

(i) *L is a Lie algebra.*

(ii) *There exists a positive integer k such that $\mathbf{ln}_k(a) \neq 0$, $\mathbf{ln}_{k+1}(a) = 0$, that is L is nilpotent.*

(iii) *$L = V \oplus U$ where V is an abelian ideal, $V \leqslant \zeta^{left}(D)$, $U = Fu$ and $[u,u] = 0$, $V = Fv + Fv_1$ and $[u,v] = v_1$, $[u,v_1] = 0$.*

Since every abelian Leibniz algebra is a Lie algebra, we obtain

Corollary 2 ([19]). *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is abelian. Then L is an algebra of one of the following types:*

(i) *L is a Lie algebra whose proper subalgebras are abelian.*

(ii) *There exists a positive integer k such that $\mathbf{ln}_k(a) \neq 0$, $\mathbf{ln}_{k+1}(a) = 0$, that is L is nilpotent.*

(iii) *$L = V \oplus U$ where V is an abelian ideal, $V \leqslant \zeta^{left}(L)$, $U = Fu$ and $[u,u] = 0$, $V = Fv + Fv_1$ and $[u,v] = v_1$, $[u,v_1] = 0$.*

This result implies that a description of Leibniz algebras, whose proper subalgebras are abelian, can be deduced to the case of Lie algebras, whose proper subalgebras are abelian. Such Lie algebras are either simple, or soluble. Soluble minimal non-abelian Lie algebras (even soluble minimal non-nilpotent Lie algebras) were described in [27], [44] and [45]. Simple minimal non-abelian Lie algebras were studied in [25] and [26], but their complete description remains to be an open problem.

Another natural question concerns the relationship of the subalgebras and ideals. In particular, what is a structure of Leibniz algebras, all of whose subalgebras are ideals? It is not hard to prove that a Lie algebra, all of whose subalgebras are ideals, is abelian. For groups the situation is different: there exists non-abelian groups, all of whose subgroups are normal. Such groups have been described in [6]. In the case of associative algebras, the situation is much more complicated. For Leibniz algebras the situation is quite diverse. At once it is possible to specify a simple example of non-abelian Leibniz algebra, all of whose subalgebras are ideals.

Example 5. Let L be a vector space over a field F, having dimension 2, $\{a,b\}$ be a basis of L. Define the operation $[\cdot,\cdot]$ by the following rule:

$$[a,a] = b, [b,b] = [b,a] = [a,b] = 0.$$

A direct check justifies that L becomes a Leibniz algebra. If $\lambda a + \mu b$ is an arbitrary element of L and $\lambda \neq 0$, then $[\lambda a + \mu b, \lambda a + \mu b] = \lambda^2 b$. Since $\lambda^2 \neq 0$, we obtain that the subalgebra generated by $\lambda a + \mu b$ includes Fb. Since L/Fb is abelian, $\langle \lambda a + \mu b \rangle$ is an ideal. Hence, every cyclic subalgebra of L is an ideal. It follows that every subalgebra of L is an ideal.

As we shall see later, any non-abelian Leibniz algebra, whose subalgebras are ideals, is constructed from such algebras as from bricks. Here are more details.

A Leibniz algebra L is called an *extraspecial* algebra if it satisfies the following condition:

(i) $\zeta(L)$ is non-trivial and has dimension 1;

(ii) $L/\zeta(L)$ is abelian.

It is important to observe that there are extraspecial Leibniz algebras in which not every subalgebra is an ideal. The following example of an extraspecial Leibniz algebra from the paper [31] shows this. Moreover, the existence of subalgebras that are not ideals depends on the choice of the field.

Example 6. Let F be a field, put $L = Fa \oplus Fb \oplus Fc$. Define on L an operation $[\cdot,\cdot]$ by the following rule:

$$c = [a, a] = [b, b] = [a, b], [c, c] = [c, a] = [c, b] = [a, c] = [b, c] = [b, a] = 0.$$

From this definition it follows that $[L, L] \leqslant Fc$, $c \in \zeta(L)$, $\langle c\rangle = Fc$. The equality

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

occurs automatically, because $[x, y], [y, z], [x, z] \in \zeta(L)$. Thus L is a Leibniz algebra. Let x be an arbitrary element of L, then $x = \lambda a + \mu b + \nu c$ for some $\lambda, \mu, \nu \in F$. We have

$$\begin{aligned}
[x, x] &= [\lambda a + \mu b + \nu c, \lambda a + \mu b + \nu c] \\
&= \lambda^2[a, a] + \lambda\mu[a, b] + \lambda\nu[a, c] + \lambda\mu[b, a] + \mu^2[b, b] \\
&\quad + \mu\nu[b, c] + \lambda\nu[c, a] + \mu\nu[c, b] + \nu^2[c, c] \\
&= \lambda^2 c + \lambda\mu c + \mu^2 c = (\lambda^2 + \lambda\mu + \mu^2)c.
\end{aligned}$$

Let $F = \mathbb{F}_2$. If $(\lambda, \mu) \neq (0, 0)$, then $\lambda^2 + \lambda\mu + \mu^2 = 1$, that is, $[x, x] = c$ whenever $x \notin Fc$. It follows that $\zeta(L) = Fc$ and $\langle x\rangle = Fx \oplus Fc$. It follows that $\langle x\rangle$ is an ideal of L. Since Fc is an ideal, we obtain that every subalgebra of L is an ideal.

Let $F = \mathbb{F}_5$. Suppose that $\lambda^2 + \lambda\mu + \mu^2 = 0$. It follows that $(\lambda + \frac{1}{2}\mu)^2 = \mu^2(\frac{1}{4} - 1)$. In field $\mathbb{F}_5$ a solution of an equation $4x = 1$ is 4, so that $\frac{1}{4} - 1 = 3$. But the equation $x^2 = 3$ has no solutions in $\mathbb{F}_5$. This shows that the equality $\lambda^2 + \lambda\mu + \mu^2 = 0$ is true only when $\lambda = \mu = 0$. Thus if $(\lambda, \mu) \neq (0, 0)$, then $[x, x] \neq 0$ and $[x, x] \in Fc$. Hence, in this case, every subalgebra of L is an ideal.

If $F = \mathbb{Q}$, then using the similar arguments we obtain again that every subalgebra of L is an ideal and the center of L is Fc.

Consider now the case when $F = \mathbb{F}_3$. For element $x = a + b$ we have $[a + b, a + b] = 3c = 0$. It follows that $\langle x\rangle = Fx$. But $[x, a] = [a + b, a] = c \notin Fx$, which shows that a cyclic subalgebra $\langle x\rangle$ is not an ideal.

The following theorem concerned with Leibniz algebras whose every subalgebra is an ideal.

Theorem 3 ([31])**.** *Let L be a Leibniz algebra over a field F, all of whose subalgebras are ideals. If L is non-abelian, then $L = E \oplus Z$ where $Z \leqslant \zeta(L)$, and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.*

With an extraspecial algebra we can connect a bilinear form in the following way. Let $Z = \zeta(L)$, $V = L/Z$, and c be a fixed non-zero element of Z. Define the mapping $\Phi : V \times V \to F$ by the following rule: if $x, y \in L$, then $[x, y] \in Z$, so that $[x, y] = \alpha c$ for some element $\alpha \in F$. Put $\Phi(x + Z, y + Z) = \alpha$. This mapping is correct. Indeed, let x_1, y_1 be elements of L such that $x_1 + Z = x + Z$, $y_1 + Z = y + Z$. Then $x_1 = x + c_1$, $y_1 = y + c_2$ for some elements $c_1, c_2 \in Z$. Then

$$[x_1, y_1] = [x + c_1, y + c_2] = [x, y] + [x, c_2] + [c_1, y] + [c_1, c_2] = [x, y].$$

The mapping Φ is bilinear. In fact, let $x, y, u \notin Z$, $[x, u] = \lambda c$, $[y, u] = \mu c$. Then $[x + y, u] = [x, u] + [y, u] = \lambda c + \mu c = (\lambda + \mu)c$, so that

$$\begin{aligned}\Phi(x + Z + y + Z, u + Z) &= \Phi(x + y + Z, u + Z) = \lambda + \mu \\ &= \Phi(x + Z, u + Z) + \Phi(y + Z, u + Z).\end{aligned}$$

Similarly, we can show that

$$\Phi(x + Z, y + Z + u + Z) = \Phi(x + Z, y + Z) + \Phi(x + Z, u + Z).$$

Let $\beta \in F$, then $[\beta x, y] = \beta[x, y] = \beta(\alpha c) = (\beta\alpha)c$. It follows that

$$\Phi(\beta(x + Z), y + Z) = \Phi(\beta x + Z, y + Z) = \beta\alpha = \beta\Phi(x + Z, y + Z).$$

Likewise we can show that

$$\Phi(x + Z, \beta(y + Z)) = \beta\Phi(x + Z, y + Z).$$

By the definition of an extraspecial algebra we obtain that a bilinear form Φ is non-degenerate. Moreover, Theorem 3 shows that $\Phi(x, x) \neq 0$ for every non-zero element $x \in V$.

Conversely, let V be a vector space over a field F and Φ be a bilinear form on V such that $\Phi(x, x) \neq 0$ for every non-zero element $x \in V$. Put $L = V \oplus F$. Define the operation $[\cdot, \cdot]$ on L by the following rule: if $a, b \in V$, $\alpha, \beta \in F$, then $[(a, \alpha), (b, \beta)] = (0, \Phi(a, b))$. Put $C = \{(0, \alpha)|\ \alpha \in F\}$. Then $\mathbf{dim}_F(C) = 1$. By this definition, $[L, L] = [L, C] = [C, L] = [C, C] = \langle 0\rangle$. It follows from here that the constructed algebra a Leibniz algebra. Furthermore, $C \leqslant \zeta(L)$. Moreover, $C = \zeta(L)$. Indeed, let $(z, \gamma) \in \zeta(L)$ and suppose that $z \neq 0$. Then $[(z, \gamma), (a, \alpha)] = [(a, \alpha), (z, \gamma)] = (0, 0)$, in particular, $[(z, \gamma), (z, \gamma)] = (0, 0)$. But $[(z, \gamma), (z, \gamma)] = (0, \Phi(z, z))$. Since $z \neq 0$, $\Phi(z, z) \neq 0$, and we obtain a contradiction. This contradiction proves the equality $C = \zeta(L)$.

Let V be a vector space over a field F, U a subspace of V, and Φ be a bilinear form on V. Put

$$
\begin{aligned}
{}^{\perp}U &= \{x \in V|\ \Phi(x, u) = 0 \text{ for all elements } u \in U\},\\
U^{\perp} &= \{x \in V|\ \Phi(u, x) = 0 \text{ for all elements } u \in U\}.
\end{aligned}
$$

Clearly ${}^{\perp}U$ and $U^{\perp}$ are subspaces of V. ${}^{\perp}U$ is called a *left orthogonal complement* of U in V, $U^{\perp}$ is called a *right orthogonal complement* of U in V.

Using standard linear algebra tools one can prove the following statement.

Proposition 4. *Let V be a finite dimensional vector space over a field F, U a subspace of V, and Φ be a non-degenerate bilinear form on V. If the restriction of Φ on U is non-degenerate, then*

$$\mathbf{dim}_F({}^{\perp}U) = \mathbf{dim}_F(U^{\perp}) = \mathbf{dim}_F(V) - \mathbf{dim}_F(U).$$

Let V be a vector space over a field F having countable dimension, and Φ be a bilinear form on V. A basis $\{a_j|\ j \in \mathbb{N}\}$ is called *left orthogonal*, if $\Phi(a_j, a_k) = 0$ whenever $j > k$.

Corollary 3. *Let V be a finite dimensional vector space over a field F and Φ be a bilinear form on V. If $\Phi(a, a) \neq 0$ for each element $0 \neq a \in V$, then V has a left orthogonal basis.*

Indeed, let U be an arbitrary non-zero subspace of V. If we suppose that the restriction of Φ on U is degenerate, then ${}^{\perp}U \cap U \neq \langle 0\rangle$. Let $0 \neq a \in {}^{\perp}U \cap U$, then $\Phi(a, u) = 0$ for all elements $u \in U$. In particular, $\Phi(a, a) = 0$, and we obtain a contradiction. Hence the restriction of Φ on every non-zero subspace is non-degenerate, and we can apply Proposition 4.

We note that if V is a finite dimensional vector space and Φ be a bilinear form on V such that V has a left orthogonal basis, then the matrix of the form Φ in this basis is triangular.

Now we can get a more detailed description of such bilinear forms. For clarity, we confine ourselves to the case when a vector space V has finite dimension n. Let Φ be a bilinear form on V such that $\Phi(x,x) \neq 0$ for each non-zero element $x \in V$. Choose a non-zero element $v_1 \in V$ and put $V_1 = Fv_1$, $U_1 = {}^{\perp}V_1$. By Proposition 4 $\mathbf{dim}_F(U_1) = n-1$. Suppose that U_1 has an element v_2 such that $\Phi(v_1, v_2) \neq 0$. Choose in the subspace U_1 a left orthogonal complement U_2 to a subspace Fv_2, and let $\{v_3, \ldots, v_n\}$ be a basis of U_2. Put $\Phi(v_j, v_k) = \gamma_{jk}$, $1 \leqslant j,k \leqslant n$, then $\gamma_{j1} = 0$ whenever $j > 1$, $\gamma_{j2} = 0$ for $j > 2$. Consider the elements γ_{1k} where $k \geqslant 3$. Suppose that not these elements are zeros. Without loss of generality we may assume that $\gamma_{13} \neq 0$. Put $a_1 = v_1$, $a_2 = v_2$, $a_3 = v_3$, $a_k = v_k - \gamma_{1k}\gamma_{13}^{-1}v_3$ if $k > 3$. Then clearly, $\{a_1, \ldots, a_n\}$ is the basis of V such that $\Phi(a_k, a_1) = 0$ for $k > 1$, $\Phi(a_k, a_2) = 0$ for $k > 2$ and $\Phi(a_1, a_k) = 0$ for $k > 3$. If $\gamma_{1k} = 0$ for all $k \geqslant 3$, then we will not change the basis.

Suppose that $\Phi(a_2, a_k) \neq 0$ for some $k > 2$. Without loss of generality we can assume that $\Phi(a_2, a_3) \neq 0$. Put $U_3 = Fa_3 + \ldots + Fa_n$. Let $\Phi(a_j, a_k) = \alpha_{jk}$, $1 \leqslant j,k \leqslant n$, and consider the elements α_{2k} where $k \geqslant 4$. Suppose that not these elements are zeros. Without loss of generality we may assume that $\alpha_{24} \neq 0$. Put $b_1 = a_1$, $b_2 = a_2$, $a_3 = b_3$, $a_4 = b_4$, $b_k = a_k - \alpha_{2k}\alpha_{24}^{-1}a_4$ if $k > 4$. Then clearly, $\{b_1, \ldots, b_n\}$ is the basis of V such that $\Phi(b_k, b_1) = 0$ for $k > 1$, $\Phi(b_k, b_2) = 0$ for $k > 2$, $\Phi(b_1, b_k) = 0$ for $k > 3$ and $\Phi(b_2, b_k) = 0$ for $k > 4$. If $\alpha_{2k} = 0$ for all $k \geqslant 4$, we will remain to use the previous basis.

Repeating these arguments, we come to a basis $\{e_1, \ldots, e_n\}$ of V such that $\Phi(e_j, e_k) = 0$ whenever $j > k$, and $\Phi(e_j, e_k) = 0$ whenever $k > j+3$. It is also easy to see that this description can be extended to the case of a vector space having countable dimension, so that we have

Theorem 4. *Let V be a vector space over a field F having countable dimension, and Φ be a bilinear form on V. If $\Phi(a,a) \neq 0$ for each element $0 \neq a \in V$, then V has a basis $\{e_1, \ldots, e_n\}$ such that $\Phi(e_j, e_k) = 0$ whenever $j > k$, and $\Phi(e_j, e_k) = 0$ whenever $k > j+3$, $j,k \in \mathbb{N}$.*

Corollary 4. *Let L be an extraspecial Leibniz algebra over a field F, having countable dimension. If $[a,a] \neq 0$ for every element $a \notin \zeta(L)$, then L has a basis $\{c, e_n | \ n \in \mathbb{N}\}$ such that $[c, e_n] = [e_n, c] = 0$, $0 \neq [e_n, e_n] \in Fc$ for all $n \in \mathbb{N}$, $[e_j, e_k] = 0$ whenever $j > k$ and $[e_j, e_k] = 0$ whenever $k > j+3$, $j,k \in \mathbb{N}$.*

It will be useful to consider the properties of nilpotent Leibniz algebras. Let L be a Leibniz algebra. As for Lie algebras, a linear transformation f of L is called a *derivation*, if $f([a,b]) = [f(a),b] + [a,f(b)]$ for all $a,b \in L$. Denote by $\mathbf{End}_F(L)$ the set of all linear transformations of L, then L is an associative algebra by the operation $+$ and $\circ$. As usual, $\mathbf{End}_F(L)$ is a Lie algebra by the operations $+$ and $[\cdot,\cdot]$, where $[f,g] = f\circ g - g\circ f$ for all $f,g \in \mathbf{End}_F(L)$. Let $\mathbf{Der}(L)$ be the subset of all derivations of L. If $f,g \in \mathbf{Der}(L)$, then

$$\begin{aligned}(f-g)([a,b]) &= f([a,b]) - g([a,b])\\ &= [f(a),b] + [a,f(b)] - [g(a),b] - [a,g(b)]\\ &= [f(a)-g(a),b] + [a,f(b)-g(b)]\\ &= [(f-g)(a),b] + [a,(f-g)(b)];\end{aligned}$$

$$\begin{aligned}[f,g]([a,b]) &= (f\circ g - g\circ f)([a,b]) = (f\circ g)([a,b]) - (g\circ f)([a,b])\\ &= f(g([a,b])) - g(f([a,b]))\\ &= f([g(a),b] + [a,g(b)]) - g([f(a),b] + [a,f(b)])\\ &= f([g(a),b]) + f([a,g(b)]) - g([f(a),b]) - g([a,f(b)])\\ &= [f(g(a)),b] + [g(a),f(b)] + [f(a),g(b)] + [a,f(g(b))]\\ &\quad - [g(f(a)),b] - [f(a),g(b)] - [g(a),f(b)] - [a,g(f(b))]\\ &= [f(g(a)),b] + [a,f(g(b))] - [g(f(a)),b] - [a,g(f(b))]\\ &= [f(g(a)) - g(f(a)),b] + [a,f(g(b)) - g(f(b))]\\ &= [[f,g](a),b] + [a,[f,g](b)].\end{aligned}$$

This shows that $\mathbf{Der}(L)$ is a subalgebra of a Lie algebra $\mathbf{End}_F(L)$. $\mathbf{Der}(L)$ is called the *algebra of derivations* of a Leibniz algebra L.

Consider the mapping $\mathfrak{l}_a : L \to L$, defined by the rule $\mathfrak{l}_a(x) = [a,x]$. For every $x,y \in L$ and $\alpha \in F$ we have

$$\mathfrak{l}_a(x+y) = [a,x+y] = [a,x] + [a,y] = \mathfrak{l}_a(x) + \mathfrak{l}_a(y),$$
$$\mathfrak{l}_a(\alpha x) = [a,\alpha x] = \alpha[a,x] = \alpha\mathfrak{l}_a(x),$$

and

$$\mathfrak{l}_a([x,y]) = [a,[x,y]] = [[a,x],y] + [x,[a,y]] = [\mathfrak{l}_a(x),y] + [x,\mathfrak{l}_a(y)].$$

These equalities show that $\mathfrak{l}_a$ is a derivation of L. Consider some properties of the mappings $\mathfrak{l}_a$. If $a,b \in L$ and $\beta \in F$, then

$$\beta\mathfrak{l}_a(x) = \beta[a,x] = [\beta a,x] = \mathfrak{l}_{\beta a}(x)$$

for every $x \in L$, which implies that $\beta\mathfrak{l}_a = \mathfrak{l}_{\beta}a$. Further,

$$(\mathfrak{l}_a + \mathfrak{l}_b)(x) = \mathfrak{l}_a(x) + \mathfrak{l}_b(x) = [a, x] + [b, x] = [a + b, x] = \mathfrak{l}_{a+b}(x),$$

which follows that $\mathfrak{l}_a + \mathfrak{l}_b = \mathfrak{l}_{a+b}$. And finally,

$$\begin{aligned}[\mathfrak{l}_a, \mathfrak{l}_b](x) &= (\mathfrak{l}_a \circ \mathfrak{l}_b - \mathfrak{l}_b \circ \mathfrak{l}_a)(x) = (\mathfrak{l}_a \circ \mathfrak{l}_b)(x) - (\mathfrak{l}_b \circ \mathfrak{l}_a)(x)\\ &= \mathfrak{l}_a(\mathfrak{l}_b(x)) - \mathfrak{l}_b(\mathfrak{l}_a(x)) = \mathfrak{l}_a([b, x]) - \mathfrak{l}_b([a, x])\\ &= [a, [b, x]] - [b, [a, x]] = [[a, b], x] = \mathfrak{l}_{[a,b]}(x),\end{aligned}$$

which follows that $[\mathfrak{l}_a, \mathfrak{l}_b] = \mathfrak{l}_{[a,b]}$. This shows that the set $\{\mathfrak{l}_a |\ a \in L\}$ is a subalgebra of $\mathbf{Der}(L)$.

Similarly, consider the mapping $\mathfrak{r}_a : L \to L$, defined by the rule $\mathfrak{r}_a(x) = [x, a]$. For every $x, y \in L$ and $\alpha \in F$ we have $\mathfrak{r}_a(x + y) = \mathfrak{r}_a(x) + \mathfrak{r}_a(y)$, $\mathfrak{r}_a(\alpha x) = \alpha\mathfrak{r}_a(x)$ and

$$\mathfrak{r}_a([x, y]) = [[x, y], a] = [x, [y, a]] - [y, [x, a]] = [x, \mathfrak{r}_a(y)] - [y, \mathfrak{r}_a(x)].$$

Also we have $\beta\mathfrak{r}_a = \mathfrak{r}_{\beta a}$ and $\mathfrak{r}_a + \mathfrak{r}_b = \mathfrak{r}_{a+b}$ for all $a, b \in L$ and $\beta \in F$.

Theorem 5 ([9])**.** *(Engel's theorem for Leibniz algebras). Let L be a finite dimensional left Leibniz algebra over a field F of characteristic 0. If the mappings $\mathfrak{l}_a$ are nilpotent for each $a \in L$, then the algebra L is itself nilpotent. In particular, all operators $\mathfrak{l}_a$ possess the common eigenvector with zero eigenvalue. Moreover, there exists a basis of L such that the matrix of $\mathfrak{l}_a$ in this basis is upper zero-triangular for every $a \in L$.*

First proof of this statement (for right Leibniz algebras) was given in [4]. In the paper [37] the following result has been obtained.

Theorem 6. *Let L be a finite dimensional left Leibniz algebra over a field F of characteristic 0. If the mappings $\mathfrak{l}_a$ are nilpotent for each $a \in L$, then all operators $\mathfrak{r}_a$ are nilpotent for each $a \in L$. Moreover, there exists a basis of L such that the matrices of $\mathfrak{l}_a$ and $\mathfrak{r}_a$ in this basis are upper zero-triangular for every $a \in L$.*

Let L be a Leibniz algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$\mathbf{Ann}_H^{left}(M) = \{a \in H|\ [a, M] = \langle 0\rangle\},$$
$$\mathbf{Ann}_H^{right}(M) = \{a \in H|\ [M, a] = \langle 0\rangle\}.$$

The subset $\mathbf{Ann}_H^{left}(M)$ is called the *left annihilator* or the *left centralizer* of M in subalgebra H. The subset $\mathbf{Ann}_H^{right}(M)$ is called the *right annihilator* or the *right centralizer* of M in subalgebra H. The intersection

$$\begin{aligned}\mathbf{Ann}_H(M) &= \mathbf{Ann}_H^{left}(M) \cap \mathbf{Ann}_H^{right}(M)\\ &= \{a \in H \mid [a, M] = \langle 0\rangle = [M, a]\}\end{aligned}$$

is called the *annihilator* or the *centralizer* of M in subalgebra H.

It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathbf{Ann}_L^{left}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathbf{Ann}_L^{left}(M)$, $b \in M$. Then

$$\begin{aligned}&[[a, x], b] = [a, [x, b]] - [x, [a, b]] = 0 - [x, 0] = 0, \text{ and}\\ &[[x, a], b] = [x, [a, b]] - [a, [x, b]] = [x, 0] - 0 = 0.\end{aligned}$$

If M is an ideal of L, then $\mathbf{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathbf{Ann}_L(M)$, $b \in M$. Using the above arguments, we obtain that $[[a, x], b] = [[x, a], b] = 0$. Further,

$$\begin{aligned}&[b, [a, x]] = [[b, a], x] + [a, [b, x]] = [0, x] + 0 = 0, \text{ and}\\ &[b, [x, a]] = [[b, x], a] + [x, [b, a]] = 0 + [x, 0] = 0.\end{aligned}$$

Let H be a left ideal of a Leibniz algebra L, $a \in L$. Consider the mapping $\mathfrak{l}_a : H \to H$ defined by the rule $\mathfrak{l}_a(x) = [a, x]$. As above, we can show that $\mathfrak{l}_a$ is a derivation of H for every $a \in L$ and the set $\{\mathfrak{l}_a \mid a \in L\}$ is a subalgebra of $\mathbf{Der}(H)$.

Consider now the mapping $\delta : L \to \mathbf{Der}(H)$, defined by the rule $\delta(a) = \mathfrak{l}_a$. By proved above,

$$\begin{aligned}\delta(\beta a) &= \mathfrak{l}_{\beta a} = \beta\mathfrak{l}_a = \beta\delta(a);\\ \delta(a + b) &= \mathfrak{l}_{a+b} = \mathfrak{l}_a + \mathfrak{l}_b = \delta(a) + \delta(b);\\ \delta([a, b]) &= \mathfrak{l}_{[a,b]} = [\mathfrak{l}_a, \mathfrak{l}_b] = [\delta(a), \delta(b)].\end{aligned}$$

These equations shows that the mapping δ is a homomorphism of the Leibniz algebra L in the Lie algebra $\mathbf{Der}(H)$. Then $\mathbf{Im}(\delta)$ is a subalgebra of $\mathbf{Der}(H)$, and $\mathbf{Im}(\delta) \cong L/\mathbf{Ker}(\delta)$. Finally,

$$\mathbf{Ker}(\delta) = \{a \in L \mid \mathfrak{l}_a = \delta(a) = 0\}.$$

In turn out, $\mathfrak{l}_a = 0$ means that $0 = \mathfrak{l}_a(x) = [a, x]$ for every element $x \in H$. In other words,

$$\mathbf{Ker}(\delta) \leqslant \mathbf{Ann}_L^{left}(H) = \{a \in L \mid [a, H] = \langle 0\rangle\}$$

the left annihilator of H in L. The converse inclusion is obvious, so that $\mathbf{Ker}(\delta) = \mathbf{Ann}_L^{left}(H)$. As we remarked above, $\mathbf{Ann}_L^{left}(H)$ is a two-side ideal of L, so we obtain that $L/\mathbf{Ann}_L^{left}(H)$ is isomorphic to some subalgebra of $\mathbf{Der}(H)$.

Let H be a subalgebra of L. The *left idealizer* or the *left normalizer* of H in L is defined by the following:

$$\mathbf{I}_L^{left}(H) = \{x \in L |\ [x, h] \in H \text{ for all } h \in H\}.$$

Clearly, that the term the *left normalizer* arose from group theory analogous.

Similarly, the *right idealizer* of H in L is defined by the following:

$$\mathbf{I}_L^{right}(H) = \{x \in L |\ [h, x] \in H \text{ for all } h \in H\}.$$

The *idealizer* of H in L is defined by the following:

$$\begin{aligned}\mathbf{I}_L(H) &= \{x \in L |\ [h, x], [x, h] \in H \text{ for all } h \in H\}\\ &= \mathbf{I}_L^{left}(H) \cap \mathbf{I}_L^{right}(H).\end{aligned}$$

The left idealizer of H is a subalgebra of L. Indeed, let $x, y \in \mathbf{I}_L^{left}(H)$, $h \in H$, $\alpha \in F$, then

$$\begin{aligned}[x - y, h] &= [x, h] - [y, h] \in H;\\ [\alpha x, h] &= \alpha[x, h] \in H; \text{ and}\\ [[x, y], h] &= [x, [y, h]] - [y, [x, h]] \in H.\end{aligned}$$

The idealizer of H is also a subalgebra of L. Indeed, let $x, y \in \mathbf{I}_L(H)$, $h \in H$, $\alpha \in F$. As above we can show that $x - y$, αx, $[x, y] \in \mathbf{I}_L(H)$. Further,

$$\begin{aligned}[x - y, h] &= [x, h] - [y, h] \in H;\\ [\alpha x, h] &= \alpha[x, h] \in H; \text{ and}\\ [h, [x, y]] &= [[h, x], y] + [x, [h, y]] \in H.\end{aligned}$$

However the right idealizer need not be a subalgebra. This is shown in the following example from [8].

Example 7. Let L be a vector space over F, and $\{a, b, c, d\}$ be a basis of L. Define the operation $[\cdot, \cdot]$ by the rule

$$\begin{aligned}&[a, b] = a, [b, a] = -a + c, [b, b] = d, [a, d] = c,\\ &[a, a] = [a, c] = 0, [b, c] = -c, [d, [d, d]] = 0,\end{aligned}$$

and

$$[c, x] = [[d, d], x] = 0 \text{ for all } x \in L.$$

It is not hard to prove that L is a Leibniz algebra. Let $H = \langle a \rangle$. Since $[a, a] = 0$, $\langle a \rangle = Fa$. Clearly, $\mathbf{I}_L^{right}(H) = Fb + Fc$. However $[c, c] = d \notin \mathbf{I}_L^{right}(H)$, which shows that $\mathbf{I}_L^{right}(H)$ is not a subalgebra of L.

However, if H is a left ideal of L, then its right idealizer is a subalgebra. Indeed, let $x, y \in \mathbf{I}_L^{right}(H)$, $h \in H$, then $[h, [x, y]] = [[h, x], y] + [x, [h, y]]$. By definition, $[h, x], [h, y] \in H$, and $[[h, x], y] \in H$. Since H is a left ideal, $[x, [h, y]] \in H$, which implies that $[x, y] \in \mathbf{I}_L^{right}(H)$.

Let L be a hypercentral Leibniz algebra and let

$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots Z_\alpha \leqslant Z_{\alpha+1} \leqslant \ldots Z_\gamma = L$$

be the upper central series of L. Let H be a proper subalgebra of L. Then there exists an ordinal α such that $Z_\alpha \leqslant H$ but H does not include $Z_{\alpha+1}$. Choose an element $x \in Z_{\alpha+1} \setminus H$. For every element $h \in H$ we have $[x, h], [h, x] \in Z_\alpha$. The inclusion $Z_\alpha \leqslant H$ implies that $[x, h], [h, x] \in H$. This shows that $\mathbf{I}_L(H) \neq H$, in particular $\mathbf{I}_L^{right}(H) \neq H \neq \mathbf{I}_L^{left}(H)$, so we obtain

Proposition 5. *Let L be a Leibniz algebra over a field F. If L is hypercentral, then $\mathbf{I}_L(H) \neq H$ for every proper subalgebra H of L.*

Corollary 5. *Let L be a nilpotent Leibniz algebra over a field F. Then $\mathbf{I}_L(H) \neq H$ for every proper subalgebra H of L.*

Corollary 6. *Let L be a nilpotent Leibniz algebra over a field F. Then every maximal subalgebra of L is an ideal of L.*

For finitely dimensional Leibniz algebras the just mentioned properties can help to characterize nilpotent Leibniz algebras.

Theorem 7. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then the following statements are equivalent:*

(i) *L is nilpotent.*
(ii) *Every proper subalgebra of L does not coincide with its idealizer.*
(iii) *Every proper subalgebra of L does not coincide with its right idealizer.*
(iv) *Every maximal subalgebra of L is an ideal of L.*
(v) *Every maximal subalgebra of L is a right ideal of L.*

The most significant of these characteristics were proved in [8].

In [38] the following properties of finite dimensional Leibniz algebras have been obtained.

Theorem 8. *Let L be a finite dimensional Leibniz algebra over a field F and H be a nilpotent ideal of L. Then L is nilpotent if and only if $L/[H,H]$ is nilpotent. Moreover, if $\mathbf{ncl}(H) = c$ and $\mathbf{ncl}(L/[H,H]) = d+1$, then $\mathbf{ncl}(L) \leqslant \binom{c+1}{2}d - \binom{c}{2}$.*

Theorem 9. *Let L be a finite dimensional Leibniz algebra over a field F. If L is nilpotent and H is a subalgebra of L such that $H + [L,L] = L$, then $H = L$. Conversely, if for every subalgebra H of L such that $H+[L,L] = L$ we have $H = L$, then L is nilpotent.*

Let L be a Leibniz algebra. The intersection of the maximal subalgebras of L is called the *Frattini subalgebra* of L and denoted by $\mathbf{Frat}(L)$. If L does not include maximal subalgebras, then put $L = \mathbf{Frat}(L)$.

Theorem 10 ([11])**.** *Let L be a Leibniz algebra over a field F of characteristic 0. Then $\mathbf{Frat}(L)$ is an ideal of L.*

Note that if $\mathbf{char}(F)$ is prime, it is not true even for soluble Lie algebras [12].

Combining Theorem 10 with Corollary 5.6 of the paper [8], we obtain

Theorem 11. *Let L be a Leibniz algebra over a field F of characteristic 0. If $\mathbf{dim}_F(L)$ is finite, then $\mathbf{Frat}(L)$ is nilpotent.*

Note the following important property of Frattini subalgebras.

Proposition 6. *Let L be a finite dimensional Leibniz algebra over a field F. If M is a subset of L such that $\langle M, \mathbf{Frat}(L)\rangle = L$, then $\langle M\rangle = L$.*

Indeed, suppose the contrary. Let $\langle M\rangle$ is a proper subalgebra of L. Since $\mathbf{dim}_F(L)$ is finite, there is a maximal subalgebra H such that $\langle M\rangle \leqslant H$. Being maximal, H includes $\mathbf{Frat}(L)$, so that $\langle M, \mathbf{Frat}(L)\rangle \leqslant H \neq L$. This contradiction proves that $\langle M\rangle = L$.

Using the Frattini subalgebra, we can obtain the following characterization of nilpotent Leibniz algebras. But first we give a slightly more general statement.

Proposition 7. *Let L be a Leibniz algebra over a field F. Then every maximal subalgebra of L is an ideal if and only if $[L,L] = \mathbf{Frat}(L)$.*

Indeed, suppose that each maximal subalgebra of L is an ideal. Let K be an arbitrary maximal subalgebra of L. Then $\langle K, x\rangle = L$ for each element $x \notin K$. Since K is an ideal, L/K is a cyclic algebra. If we suppose that $\mathbf{Leib}(L/K)$ is non-zero, then $\mathbf{Leib}(L/K)$ is a proper subalgebra of L/K, which is impossible. Hence $\mathbf{Leib}(L/K) = \langle 0\rangle$, so that L/K is a cyclic Lie algebra. In particular, it is abelian, which follows that $[L, L] \leqslant K$. It is valid for each maximal subalgebra, therefore their intersection $\mathbf{Frat}(L)$ includes $[L, L]$. On the other hand, factor-algebra $L/[L, L]$ is abelian, so that every its subspace is a subalgebra. Since the intersection of all maximal subspaces of $L/[L, L]$ is zero, then $\mathbf{Frat}(L) = [L, L]$.

Conversely, if $[L, L] = \mathbf{Frat}(L)$, then $\mathbf{Frat}(L)$ is an ideal and the factor-algebra $L/\mathbf{Frat}(L)$ is abelian. It follows that every subalgebra including $\mathbf{Frat}(L)$ is an ideal of L, in particular, every maximal subalgebra of L is an ideal.

Using this result and Theorem 7 we obtain

Corollary 7. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then L is nilpotent if and only if $[L, L] = \mathbf{Frat}(L)$.*

In [11] the following properties of nilpotent ideals of Leibniz algebras have been obtained.

Theorem 12. *Let L be a Leibniz algebra over a field F and K_1, K_2 are ideals of L. Suppose that K_1, K_2 are nilpotent and $\mathbf{ncl}(K_1) = c_1$, $\mathbf{ncl}(K_2) = c_2$. Then the ideal $K_1 + K_2$ is nilpotent, and $\mathbf{ncl}(K_1 + K_2) = c_1 + c_2$.*

Leibniz algebra L is called *locally nilpotent* if every finitely generated subalgebra of L is nilpotent.

If L is an arbitrary Leibniz algebra, then denote by $\mathbf{Nil}(L)$ the subalgebra, generated by all nilpotent ideals of L. $\mathbf{Nil}(L)$ is called the *nil-radical* of L.

If $L = \mathbf{Nil}(L)$, then L is called a *Leibniz nil-algebra.* Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. If L is finite dimensional, then Theorem 12 shows that $\mathbf{Nil}(L)$ is the greatest nilpotent ideal of L. In general case, $\mathbf{Nil}(L)$ is locally nilpotent.

Let L be a Leibniz algebra. Define the *lower derived series*

$$L = \delta_0(L) \geqslant \delta_1(L) \geqslant \dots \delta_\alpha(L) \geqslant \delta_{\alpha+1}(L) \geqslant \dots \delta_\nu(L)$$

of L by the following rule: $\delta_0(L) = L$, $\delta_1(L) = [L, L]$, and recursively $\delta_{\alpha+1}(L) = [\delta_\alpha(L), \delta_\alpha(L)]$ for all ordinals α and $\delta_\lambda(L) = \bigcap_{\mu<\lambda} \delta_\mu(L)$ for the limit ordinals λ. For the last term $\delta_\nu(L)$ we have $\delta_\nu(L) = [\delta_\nu(L), \delta_\nu(L)]$. The length ν of this series is called the *derived length* of L and denoted by $\mathbf{dl}(L)$.

If $\delta_\nu(L) = \langle 0 \rangle$ for some ordinal ν, then L is called a *hypoabelian* Leibniz algebra. If $\delta_n(L) = \langle 0 \rangle$ for some positive integer n, then we say that L is a *soluble* Leibniz algebra.

If K_1, K_2 are soluble ideals of Leibniz algebra L, then clearly their sum $K_1 + K_2$ is a soluble ideal of L. Therefore if L is a finite dimensional Leibniz algebra, then its subalgebra $\mathbf{Sol}(L)$ generated by all soluble ideals of L is called the *soluble radical* of L. By above remarked, $\mathbf{Sol}(L)$ is a soluble ideal of L, and a factor-algebra $L/\mathbf{Sol}(L)$ does not include non-zero soluble ideals.

Note some properties of the nil-radical and the soluble radical obtained in [29].

Theorem 13. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then $[L, \mathbf{Sol}(L)] \leqslant \mathbf{Nil}(L)$.*

Corollary 8. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then $[\mathbf{Sol}(L), \mathbf{Sol}(L)]$ is nilpotent.*

Corollary 9. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then L is soluble if and only if $[L, L]$ is nilpotent.*

The last two corollaries were proved earlier in [4].

For a finite dimensional Leibniz algebra the following analogue of the Levi's Theorem from Lie algebras takes place. It was proved by D. Barnes [10].

Theorem 14. *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then L includes a subalgebra S (being a semisimple Lie algebra) such that $L = \mathbf{Sol}(L) + S$ and $\mathbf{Sol}(L) \cap S = \langle 0 \rangle$.*

The examples given in [10] show that the subalgebra S is not unique.

We will not go deeper into the structure of the finite dimensional Leibniz algebras. These questions have been adequately reflected in many articles on Leibniz algebras. Many of the results obtained on this problem are analogs (not always complete) of the corresponding theorems from the theory of Lie algebras.

Let us now consider some other natural questions of the general theory of Leibniz algebras.

Note that the relation “to be a subalgebra of a Leibniz algebra” is transitive. However, the relation “to be an ideal” is not transitive even for Lie algebras.

Therefore it is natural to ask the question about the structure of Leibniz algebras, in which the relation “to be an ideal” is transitive.

In this context, the following important type of subalgebras naturally arises. A subalgebra A of a Leibniz algebra L is called a *left* (respectively *right*) *subideal* of L, if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is a left (respectively right) ideal of A_j, $1 \leqslant j \leqslant n$.

Similarly, a subalgebra A of a Leibniz algebra L is called a *subideal* of L, if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is an ideal of A_j, $1 \leqslant j \leqslant n$.

We note the following property of nilpotent Leibniz algebras.

Proposition 8. *Let L be a nilpotent Leibniz algebra over a field F. Then every subalgebra of L is a subideal of L.*

A Leibniz algebra L is called a *T-algebra*, if a relation “to be an ideal” is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that in a Leibniz T-algebra every subideal is an ideal.

Lie algebras, in which a relation “to be an ideal” is transitive have been studied by I. Stewart [42] and A.G. Gein and Yu.N. Muhin [28]. In particular, soluble T-algebras and finite dimensional T-algebras over a field of characteristic 0 has been described.

As in the mentioned above cases, the situation in Leibniz algebras is much more complex and diverse than it was in Lie algebras. Here are few simple examples illustrating this point.

Example 8. Let F be an arbitrary field, L be a vector space over F with a basis $\{a, c\}$. Define the operation $[\cdot,\cdot]$ on L by the following rule:

$$[a, a] = c, [c, a] = [a, c] = [c, c] = 0.$$

Then L is a cyclic Leibniz algebra, Fc is an unique its non-zero subalgebra. Moreover, Fc is the center of L, in particular, Fc is an ideal of L. Thus every subalgebra of L is an ideal and L is a Leibniz T-algebra.

Example 9. Let now $F = \mathbb{F}_2$ and L be a constructed above Leibniz algebra. Put $A = L \oplus Fv$ and let

$$[v, v] = [v, c] = [c, v] = 0, [v, a] = [a, v] = a.$$

It is not hard to check that A is a Leibniz algebra and L is an ideal of A. Moreover, if B is a non-zero ideal of A and L does not include B, then $B = A$. As we have seen above, Fc is an unique non-zero ideal of L. But $Fc = \zeta(L)$, thus Fc is an ideal of A. Thus A is a Leibniz T-algebra.

Example 10. Let again $F = \mathbb{F}_2$ and L be a constructed above Leibniz algebra. Put $D = L \oplus Fu$. Let

$$[u, u] = [u, c] = [c, u] = 0, [u, a] = [a, u] = a + c.$$

It is not hard to check that D is a Leibniz algebra and L is an ideal of D. As above, we can check that D is a Leibniz T-algebra.

As we will see further, these examples are typical in some sense.

The subalgebra $\mathbf{Ba}(L)$ of a Leibniz algebra L generated by all nilpotent subideals of L is called the *Baer radical* of L. It is possible to show that $\mathbf{Ba}(L)$ is an ideal of L and $\mathbf{Nil}(L) \leqslant \mathbf{Ba}(L)$. If $L = \mathbf{Ba}(L)$, then L is called a Leibniz *Baer algebra*. Every nil-algebra is a Baer algebra, but converse is not true even for a Lie algebra (see, for example, [3, Theorem 6.4.5]).

The description of Leibniz T-algebras has been obtained in the paper [32]. Here are the main results of this paper.

Theorem 15. *Let L be a Leibniz T-algebra over a field F. If L is a Baer algebra, then either L is abelian, or $L = E \oplus Z$ where $Z \leqslant \zeta(L)$ and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.*

A Leibniz algebra L is called *hyperabelian* if it has an ascending series

$$\langle 0 \rangle = L_0 \leqslant L_1 \leqslant \dots L_\alpha \leqslant L_{\alpha+1} \leqslant \dots L_\gamma = L$$

of ideals whose factors $L_{\alpha+1}/L_\alpha$ are abelian for all ordinals $\alpha < \gamma$ and $L_\lambda = \bigcup_{\mu<\lambda} L_\mu$ for the limit ordinals λ. If this series is finite, then L is called a *soluble Leibniz algebra.*

The structure of Leibniz T-algebras essentially depends of the structure of its nil-radical.

Theorem 16. *Let L be a hyperabelian Leibniz T-algebra over a field F. If L is non-nilpotent and $\mathbf{Nil}(L) = D$ is abelian, then $L = D \oplus V$ where $V = Fv$, $[v, v] = 0$, $[v, d] = d = -[d, v]$ for every element $d \in \mathbf{Nil}(L)$. In particular, L is a Lie algebra.*

Theorem 17. *Let L be a hyperabelian Leibniz T-algebra over a field F. If $\mathbf{char}(F) \neq 2$, then $\mathbf{Nil}(L)$ is abelian.*

In other words, if $\mathbf{char}(F) \neq 2$, then every Leibniz T-algebra is a Lie algebra. Thus we can see that the case when $\mathbf{char}(F) = 2$ is very specific here. We will consider this case with the following additional restriction.

We say that a field F is *2-closed*, if an equation $x^2 = a$ has a solution in F for every element $a \neq 0$. We note that every locally finite (in particular, finite) field of characteristic 2 is 2-closed.

Theorem 18. *Let L be a hyperabelian Leibniz T-algebra over a field F. Suppose that L is non-nilpotent and $\mathbf{Nil}(L)$ is non-abelian. If a field F is 2-closed and $\mathbf{char}(F) = 2$, then $L = Fe \oplus Fc \oplus Fv$ where*

$$[e, e] = c, [c, e] = [e, c] = [c, v] = [v, c] = 0,$$
$$[v, v] = 0, [v, e] = e + \gamma c = [e, v], \gamma \in F.$$

As we can see in Corollary 1 the fact that $\gamma_{c+1}(L) = \langle 0 \rangle$ is equivalent to the fact that $\zeta_c(L) = L$, i.e. the lower and the upper central series in nilpotent Leibniz algebras have the same length. The next natural step is the consideration of the case, when the upper (respectively lower) central series has finite length. For this case the question about the relationships between $L/\zeta_k(L)$ and $\gamma_{k+1}(L)$ naturally appears.

If L is a Lie algebra such that $L/\zeta_k(L)$ is finitely dimensional, then $\gamma_{k+1}(L)$ is also finitely dimensional, it follows from Theorem 5.2 of the paper [43] by I. Stewart. A corresponding result for groups has been obtained early by R. Baer [7]. In the paper [30] the following analog of these theorems has been obtained.

Theorem 19. *Let L be a Leibniz algebra over a field F. Suppose that $\mathbf{codim}_F(\zeta_k(L)) = d$ is finite for some positive integer k. Then $\gamma_{k+1}(L)$ has finite dimension. Moreover $\mathbf{dim}_F(\gamma_{k+1}(L)) \leqslant 2^{k-1}d^{k+1}$.*

As a corollary we obtained a bound for a dimension of $\gamma_{k+1}(L)$ in a Lie algebra L.

Corollary 10. *Let L be a Lie algebra over a field F. Suppose that $\mathbf{codim}_F(\zeta_k(L)) = d$ is finite for some positive integer k. Then $\gamma_{k+1}(L)$ has finite dimension. Moreover $\mathbf{dim}_F(\gamma_{k+1}(L)) \leqslant \frac{1}{2}d^{k-1}(d-1)$.*

An important specific case here is the case when the center of a Leibniz algebra L has finite codimension. For Lie algebras the following result is well known (see, for example [47]).

Theorem 20. *Let L be a Lie algebra over a field F. If a factor-algebra $L/\zeta(L)$ has finite dimension d, then the derived subalgebra $[L, L]$ also has finite dimension, moreover, $\mathbf{dim}_F([L, L]) \leqslant \frac{1}{2}d(d+1)$.*

A corresponding result for groups was proved much earlier.

Theorem 21. *Let G be a group, C a subgroup of the center $\zeta(G)$ such that G/C is finite. Then the derived subgroup $[G, G]$ is finite.*

In this formulation, for the first time it appears in the paper of B.H. Neumann [36]. This theorem was obtained also by R. Baer [7].

For Leibniz algebras we obtain the following analog of these results.

Theorem 22. *Let L be a Leibniz algebra over a field F. Suppose that $\mathbf{codim}_F(\zeta^{left}(L)) = d$ and $\mathbf{codim}_F(\zeta^{right}(L)) = r$ are finite. Then $[L, L]$ has finite dimension, moreover, $\mathbf{dim}_F([L, L]) \leqslant d(d+r)$.*

In this connection, the following question appears: suppose that only $\mathbf{codim}_F(\zeta^{left}(L))$ is finite. Is $\mathbf{dim}_F([L, L])$ finite? The above constructed Example 4 gives a negative answer on this question.

Corollary 11. *Let L be a Leibniz algebra over a field F. Suppose that $\mathbf{codim}_F(\zeta(L)) = d$ is finite. Then $[L, L]$ has finite dimension. Moreover, $\mathbf{dim}_F([L, L]) \leqslant d^2$.*

Corollary 12. *Let L be a Leibniz algebra over a field F. Suppose that $\mathbf{codim}_F(\zeta(L)) = d$ is finite. Then the Leibniz kernel of L has finite dimension at most $\frac{1}{2}d(d-1)$.*

We did not talk about the links of Leibniz algebras with other algebraic structures. However, in conclusion, we would like to note one such link of Leibniz algebras with not very ordinary but interesting algebraic structures that were introduced by J.-L. Loday (see [35]).

Let D be a vector space over a field F. Then D is called a *dialgebra* if two associative binary operation $\vdash$ and $\dashv$ are defined on D and they satisfy the following conditions:

$$(D1)\ x \vdash (y \dashv z) = (x \vdash y) \dashv z,$$
$$(D2)\ x \dashv (y \vdash z) = x \dashv (y \dashv z),$$
$$(D3)\ (x \dashv y) \vdash z = (x \vdash y) \vdash z$$

for all $x, y, z \in D$.

Note that our use of $\vdash$ and $\dashv$ in this bracket is the opposite of that of Loday. This convention matches our preference for left Leibniz algebras instead of right Leibniz algebras.

For a given a dialgebra D we define the operation $[\cdot,\cdot]$ by the rule

$$[x, y] = x \vdash y - y \dashv x, \; x, y \in D.$$

One can check that D becomes a Leibniz algebra relatively the operations $+$ and $[\cdot,\cdot]$. This algebra is called a *Leibniz algebra associated with dialgebra* D. Conversely, J.-L. Loday proved that for any Leibniz algebra L there exists a dialgebra $D(L)$ such that a Leibniz algebra associated with $D(L)$ includes a subalgebra, which is isomorphic to L.

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