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Some combinatorial characteristics of closure operations

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Abstract. The aim of this paper investigates some combinatorial characteristics of minimal key and antikey of closure operations. We also give effective algorithms finding minimal keys and antikeys of closure operations. We estimate these algorithms. Some remarks on the closeness of closure operations class under the union and direct product operations are also studied in this paper.

Introduction

Functional dependencies (FDs) play an important role the relational database theory. The equivalence of the family of FDs is one of the hottest topics that get a lot of attention and interest currently. There are many equivalent descriptions of the family of FDs. Based on the equivalent descriptions, we can obtain many important properties of the family of FDs. The closure operation is an equivalent description of family of FDs ([1]). A closure operation here is a map between the elements of a partial ordered set that verifies three axioms: extension, order-preservation and idempotence. In recent years, the closure operations have been widely studied (e.g. see [2, 7–9]). Closed set, minimal key and antikey of closure operations are the interestring concepts and significant. Such as the family

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of closed sets of a closure operation forms a closure system (or meet-semlattice). Recently the closure operations have also been applied in data mining (e.g. see [5, 6]).

This paper investigates some characteristics of minimal key and antikey of closure operations as well as the closeness of closure operations class under some basic operations. The paper is organized as follows. After an introduction section, in Section 2, we introduce the notions of closure operation, minimal key and antikey of closure operation. Section 3 we present some characteristics of minimal key and antikey of closure operation. The algorithms for finding all minimal key and antikey of closure operations are studied in Section 4 and 5. The closeness of closure operations class under the union and direct product operations is studied in Section 6.

1. Definitions and preliminaries

This section introduces the notions of Sperner system, closure operation, closure system, closed set, minimal key and antikey of closure operation. The notions and results in this section can be found in [3, 4, 8, 9].

Let U be a finite set, and denote $\mathcal{P}(U)$ its power set. A family $\mathcal{S} \subseteq \mathcal{P}(U)$ is called a *Sperner system* on U if for any $X, Y \in \mathcal{S}$ implies $X \not\subseteq Y$.

The mapping $f : \mathcal{P}(U) \to \mathcal{P}(U)$ is called a *closure operation* on U if it satisfies the following conditions

(C1) (*Extensivity*) $X \subseteq f(X)$

(C2) (*Monotonicity*) $X \subseteq Y$ implies $f(X) \subseteq f(Y)$

(C3) (*Idempotency*) $f(f(X)) = f(X)$

for every $X, Y \subseteq U$.

We denote by $\mathrm{Cl}(U)$ the set of all closure operations on U.

Let $f \in \mathrm{Cl}(U)$ and $X \subseteq U$. Set X is called *closed* of f if $f(X) = X$. The family of closed sets is denoted $\mathrm{Closed}(f)$. Therefore, $\mathrm{Closed}(f) = \{X \subseteq U : f(X) = X\}$. It is easy to see that $U \in \mathrm{Closed}(f)$ and $X, Y \in \mathrm{Closed}(f) \Rightarrow X \cap Y \in \mathrm{Closed}(f)$. Then we also can rewrite $\mathrm{Closed}(f) = \{f(X) : X \subseteq U\}$.

A family $\mathcal{S}$ of subsets of U is called a *closure system* (or *Moore family*, *meet-semilattice*) on U if it satisfies the following conditions

(S1) $U \in \mathcal{S}$;

(S2) $\forall \mathcal{A} \subseteq \mathcal{P}(U), \varnothing \neq \mathcal{A} \subseteq \mathcal{S} \Rightarrow \bigcap \mathcal{A} \in \mathcal{S}$.

It can be seen that, if $\mathcal{S}$ is a closure system, and we define $f_{\mathcal{S}}(X)$ as

$$f_{\mathcal{S}}(X) = \bigcap \{Y \in \mathcal{S} : X \subseteq Y\}$$

then $f_{\mathcal{S}} \in \mathrm{Cl}(U)$. Conversely, if $f \in \mathrm{Cl}(U)$, then there is exactly one closure system $\mathcal{S}$ on U so that $f = f_{\mathcal{S}}$, where

$$\mathcal{S} = \{X \subseteq U : f(X) = X\}.$$

Thus $\mathrm{Closed}(f)$ is a closure system. This means that there is a 1-1 correspondence between closure operations and closure systems.

Example 1. The following mappings are basic closure operations:

(1) *A maximal mapping* $m : \mathcal{P}(U) \to \mathcal{P}(U)$ is determined by $m(X) = U$ for every $X \subseteq U$. Then $\mathrm{Closed}(m) = \{U\}$.

(2) *An identity mapping* $i : \mathcal{P}(U) \to \mathcal{P}(U)$ is determined by $i(X) = X$ for every $X \subseteq U$. Then $\mathrm{Closed}(i) = \mathcal{P}(U)$.

(3) *A translation mapping* $t_M : \mathcal{P}(U) \to \mathcal{P}(U)$ is determined by $t_M(X) = M \cup X$, where M is a given subset of U and for every $X \subseteq U$. Then $\mathrm{Closed}(t_M) = \{M \cup X : X \subseteq U\}$.

It can be seen that if $M = U$ then $t_M = m$. The case if $M = \varnothing$ then $t_M = i$.

Now let $f \in \mathrm{Cl}(U)$. A subset $K \subseteq U$ is called a *minimal key* of f if it satisfies the following conditions

(K1) $f(K) = U$

(K2) $\forall a \in K : f(K \setminus \{a\}) \neq U$.

Denote $\mathrm{Key}(f)$ the set of all minimal keys of f. It is easy to see that U is the unique minimal key of f if and only if $f(X) = X$ for every $X \subseteq U$, i.e. $f = i$.

A subset $K^{-1} \subseteq U$ is called a *antikey* of f if it satisfies the following conditions

(AK1) $f(K^{-1}) \neq U$

(AK2) $\forall a \in U \setminus K^{-1} : f(K^{-1} \cup \{a\}) = U$.

Denote $\mathrm{Antikey}(f)$ the set of all antikeys of f. It it clear that $\mathrm{Key}(f)$ and $\mathrm{Antikey}(f)$ are Sperner systems on U. It is easy to see that $K^{-1} \neq U$ and $\mathrm{Antikey}(f)$ can describe by $\mathrm{Key}(f)$ as follows:

$$\mathrm{Antikey}(f) = \{K^{-1} \subset U : (K \in \mathrm{Key}(f) \Rightarrow K \not\subseteq K^{-1}) \text{ and } \\ ((K^{-1} \subset Y) \Rightarrow (\exists K \in \mathrm{Key}(f))(K \subseteq Y))\}.$$

Obviously, $\mathrm{Key}(f)$ and $\mathrm{Antikey}(f)$ are uniquely determined by one another.

Example 2. Using the definition of minimal key and antikey, we can easily imply the minimal keys and antikeys of the basic closure operations as follows:

(1) $\text{Key}(m) = \{\varnothing\}, \text{Antikey}(m) = \varnothing$;
(2) $\text{Key}(i) = \{U\}, \text{Antikey}(i) = \{U \setminus \{a\} : a \in U\}$;
(3) $\text{Key}(t_M) = \{U \setminus M\}, \text{Antikey}(t_M) = \{U \setminus \{a\} : a \in U \setminus M\}$.

2. Closed set, minimal key and antikey

Now we denote by $\text{MAX}(\mathcal{S})$ the family of maximal elements of family $\mathcal{S} \subseteq \mathcal{P}(U)$. Then the antikey of closure operations have the following basic characteristic.

Theorem 1. *Let* $f \in \text{Cl}(U)$*. Then*

$$\text{Antikey}(f) = \text{MAX}(\text{Closed}(f) \setminus \{U\}).$$

Proof. Suppose that $K^{-1} \in \text{Antikey}(f)$ and $K^{-1} \subset f(K^{-1})$. According to the definition of antikey we have $K^{-1} \neq U$ and $U = f(f(K^{-1})) = f(K^{-1})$. Thus K^{-1} is a key of f. This contradicts the fact K^{-1} is an antikey of f. Hence $f(K^{-1}) = K^{-1}$, or $K^{-1} \in \text{Closed}(f) \setminus \{U\}$. On the other hand, if there is a $Y \in \text{Closed}(f) \setminus \{U\}$ such that $Y \supset K^{-1}$, then $f(Y) = U \neq Y$. This contracdicts the fact that Y is a closed set. Consequently, $K^{-1} \in \text{MAX}(\text{Closed}(f) \setminus \{U\})$.

The case if $K^{-1} \in \text{MAX}(\text{Closed}(f) \setminus \{U\})$ and there is a $K \in \text{Key}(f)$ such that $K \subset K^{-1}$, then $f(K^{-1}) = U$. Therefore $K^{-1} = U$. This contracdicts the suppose that $K^{-1} \neq U$. Moreover, it can be seen that if there exists $Y \subseteq U$ such that $K^{-1} \subset Y$, then $f(Y) = U$. Consequently, $K^{-1} \in \text{Antikey}(f)$. □

So relying on $\text{Closed}(f)$ we also can find effectively the set of antikeys of closure operation f.

Example 3. Let us consider the mapping $f : \mathcal{P}(U) \to \mathcal{P}(U)$, with $U = \{a, b, c, d\}$, as follows:

X	$f(X)$	X	$f(X)$	X	$f(X)$	X	$f(X)$
$\varnothing$	$\varnothing$	$\{d\}$	$\{d\}$	$\{b,c\}$	U	$\{a,b,d\}$	U
$\{a\}$	$\{a\}$	$\{a,b\}$	U	$\{b,d\}$	$\{b,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
$\{b\}$	$\{b\}$	$\{a,c\}$	$\{a,c\}$	$\{c,d\}$	$\{c,d\}$	$\{b,c,d\}$	U
$\{c\}$	$\{c\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b,c\}$	U	U	U

It is easy to see that $f \in \text{Cl}(U)$. Then $\text{Key}(f) = \{\{a,b\}, \{b,c\}\}$ and $\text{Closed}(f) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,c,d\}, U\}$.

By Theorem 1, we obtain $\text{Antikey}(f) = \{\{b,d\}, \{a,c,d\}\}$.

The minimal key and antikey of closure operations have the following correlation.

Proposition 1. *Let $f \in \mathrm{Cl}(U)$. Then*

$$\bigcup \mathrm{Key}(f) = U \setminus \bigcap \mathrm{Antikey}(f).$$

Proof. It is clear that if $a \in \bigcup \mathrm{Key}(f)$, then there exists a $K \in \mathrm{Key}(f)$ such that $a \in K$. Let $M = K \setminus \{a\}$. It can be seen that M does not contain any minimal keys of f. Hence, there exists an antikey $K^{-1} \in \mathrm{Antikey}(f)$ such that $M \subseteq K^{-1}$. It is easy to see that $a \notin K^{-1}$. Consequently, we obtain that $a \in U \setminus K^{-1}$, or $a \in U \setminus \bigcap \mathrm{Antikey}(f)$.

Now assume that $a \notin \bigcup \mathrm{Key}(f)$ and let $K^{-1} \in \mathrm{Antikey}(f)$. Obviously, if $a \notin K^{-1}$ then $K^{-1} \cup \{a\}$ contains a minimal key $K \in \mathrm{Key}(f)$. Thus, $K \subseteq K^{-1}$. This contracdicts the fact that K^{-1} is a antikey of f. Consequently, we have $a \in K^{-1}$. □

3. Finding the set of all antikeys of closure operations

In this section, we present the algorithm for finding all antikeys of closure operations.

Algorithm 1. (Finding all antikeys)

Input: $f \in \mathrm{Cl}(U)$ with $\mathrm{Key}(f) = \{K_1, K_2, \ldots, K_m\}$.

Output: $\mathrm{Antikey}(f)$.

Step 1: From K_1 we construct a family $\mathcal{T}_1 = \{U \setminus \{a\} : a \in K_1\}$. It is obvious that $\mathcal{T}_1 = \mathrm{Antikey}(g_1)$ such that $\mathrm{Key}(g_1) = \{K_1\}$, where $g_1 \in \mathrm{Cl}(U)$.

Step $j+1$ ($j = 1, 2, \ldots, m-1$): Suppose that $\mathcal{T}_j = \mathcal{F}_j \cup \{X_1, \ldots, X_{t_j}\}$, where $X_1, \ldots, X_{t_j}$ are elements of $\mathcal{T}_j$ containing K_{j+1} and $\mathcal{F}_j = \{Y \in \mathcal{T}_j : K_{j+1} \not\subseteq Y\}$. For all i ($i = 1, 2, \ldots, t_j$) we construct $\mathrm{Antikey}(g_{j+1})$ such that $\mathrm{Key}(g_{j+1}) = \{K_{j+1}\}$, where $g_{j+1} \in \mathrm{Cl}(X_i)$, in an analogous way as $\mathcal{T}_1$ in Step 1, which are the maximal subsets of X_i not containing K_{j+1}. We denote them by $Y_1^i, \ldots, Y_{r_i}^i$ ($i = 1, 2, \ldots, t_j$). Let

$$\mathcal{T}_{j+1} = \mathcal{F}_j \cup \{Y_p^i : Y \in \mathcal{F}_j \Rightarrow Y_p^i \not\subseteq Y, 1 \leqslant i \leqslant t_j, 1 \leqslant p \leqslant r_i\}.$$

Step $m+1$: Let $\mathrm{Antikey}(f) = \mathcal{T}_m$.

Because $\mathrm{Key}(f)$ and $\mathrm{Antikey}(f)$ are uniquely determined by one another, the determination of $\mathrm{Antikey}(f)$ based on Algorithm 1 does not depend on the order of $K_1, K_2, \ldots, K_m$.

Lemma 1. $\mathcal{T}_m = \text{Antikey}(f)$.

Proof. (Proof by induction) It is clear that $\mathcal{T}_1 = \text{Antikey}(g_1)$ such that $\text{Key}(g_1) = \{K_1\}$, where $g_1 \in \text{Cl}(U)$. Now we assume $\mathcal{T}_l = \text{Antikey}(g_l)$ such that $\text{Key}(g_l) = \{K_1, \ldots, K_l\}$, where $l \geqslant 1$. We have to prove that $\mathcal{T}_{l+1} = \text{Antikey}(g_{l+1})$ such that $\text{Key}(g_{l+1}) = \{K_1, \ldots, K_{l+1}\}$.

Firstly, we show that if $Y \in \mathcal{T}_{l+1}$, then Y is the subset of U not containing K_t $(t = 1, 2, \ldots, l+1)$ and being maximal for this property, i.e. $Y \in \text{Antikey}(g_{l+1})$. Indeed, suppose that $Y \in \mathcal{T}_{l+1}$. If $Y \in \mathcal{F}_l$, then Y does not contain the elements K_t $(t = 1, 2, \ldots, l)$ and Y is maximal for this property and at the same $K_{l+1} \not\subseteq Y$. Therefore, Y is a maximal subset of U not containing Y_t $(t = 1, 2, \ldots, l+1)$. Clearly, if $Y \in \mathcal{T}_{l+1} \setminus \mathcal{F}_l$, then there is a Y_p^i $(1 \leqslant i \leqslant t_j, 1 \leqslant p \leqslant r_i)$ such that $Y = Y_p^i$. Our construction shows that $K_t \not\subseteq Y_p^i$ for all t $(t = 1, 2, \ldots, l+1)$. On the other hand $Y_p^i = X_i \setminus \{b\}$ for some $b \in K_{l+1}$. It is obvious that $K_{l+1} \subseteq Y_p^i \cup \{b\}$. If $a \in U \setminus X_i$ then, by the inductive hypothesis, for $X_i \cup \{a\}$ there exists K_s $(s = 1, 2, \ldots, l)$ such that $K_s \subseteq X_i \cup \{a\}$. Note that $Y_p^i \cup \{a, b\} = X_i \cup \{a\}$ and X_i does not contain $K_1, \ldots, K_l$. Thus, $a \in K_s$. Then, if $K_s \setminus \{a\} \subseteq Y_p^i$ then $K_s \subseteq Y_p^i \cup \{a\}$. Case, for every K_s $(s = 1, 2 \ldots, l)$ with $K_s \subseteq X_i \cup \{a\}$ and $K_s \not\subseteq Y_p^i$, we have $b \in K_s$. Therefore, $K_s \setminus \{a, b\} \subseteq Y_p^i$. Consequently, there exists a $Y' \in \mathcal{F}_l$ such that $Y_p^i \subseteq Y'$. This contradicts $Y \in \mathcal{T}_{l+1} \setminus \mathcal{F}_l$. So there is a K_s $(1 \leqslant s \leqslant l)$ such that $K_s \subseteq Y_p^i \cup \{a\}$.

Next we show that every $Y \subseteq U$ not containing the elements K_t $(t = 1, 2, \ldots, l+1)$ and being maximal for this property is an element of $\mathcal{T}_{l+1}$. Assume that Y is the maximal subset of U not containing Y_t $(t = 1, 2, \ldots, l+1)$. By the inductive hypothesis, there is a $Z \in \mathcal{T}_l$ such that $Y \subseteq Z$. The first case, if $K_{l+1} \not\subseteq Z$ then Z does not contain $K_1, \ldots, K_{l+1}$. Because Y is the maximal subset of U not containing K_t $(t = 1, 2, \ldots, l+1)$, we obtain $Y = Z$. This implies $Y \in \mathcal{F}_l$. Consequently, we have $Y \in \mathcal{T}_{l+1}$. The second case, if $K_{l+1} \subseteq Z$ then $Z = X_i$ holds for some $i \in \{1, 2, \ldots, t_j\}$ and $Y \subseteq Y_p^i$ holds for some $p \in \{1, 2, \ldots, r_i\}$. Then, if there exists a $Y' \in \mathcal{F}_l$ such that $Y_p^i \subset Y'$, then we also have $Y \subset Y'$. This contradicts the definition of Y. Thus, $Y_p^i \in \mathcal{T}_{l+1}$. Furthermore Y_p^i does not contain $K_1, \ldots, K_{l+1}$. Therefore, $Y = Y_p^i$. This means that $\mathcal{T}_{l+1} = \text{Antikey}(g_{l+1})$. □

Denote $|U| = n, \mathcal{T}_j = \mathcal{F}_j \cup \{X_1, \ldots, X_{t_j}\}$ and l_j be the number of elements of $\mathcal{T}_j$. Note that if $\mathcal{F}_j = \varnothing$, then $l_j = t_j$.

Lemma 2. *The worst-case time complexity of Algorithm 1 is*

$$\mathcal{O}(n^2 \sum_{j=1}^{m-1} t_j u_j)$$

where

$$u_j = \begin{cases} l_j - t_j & \text{if } l_j > t_j, \\ 1 & \text{if } l_j = t_j. \end{cases}$$

Proof. It is easy to see that for constructing $\mathcal{K}_{j+1}$ the worst-case time complexity of Algorithm 1 is

$$\begin{cases} \mathcal{O}(n^2(l_j - t_j)t_j) & \text{if } l_j > t_j, \\ \mathcal{O}(n^2 t_j) & \text{if } l_j = t_j. \end{cases}$$

Therefore, the total time of Algorithm 1 in the worst-case is

$$\mathcal{O}(n^2 \sum_{j=1}^{m-1} t_j u_j)$$

where

$$u_j = \begin{cases} l_j - t_j & \text{if } l_j > t_j, \\ 1 & \text{if } l_j = t_j. \end{cases}$$ □

It can be seen that when closure operation f has only a few minimal keys, Algorithm 1 is very effective, it does not requires exponential time in n. In cases for which $l_j \leqslant l_m$ (for all $q = 1, 2, \ldots, m-1$), the worst-case time complexity of Algorithm 1 is not greater than $\mathcal{O}(n^2 m |\operatorname{Antikey}(f)|^2)$. Hence, in these cases Algorithm 1 finds Antikey(f) in polynomial time in n, m and $|\operatorname{Antikey}(f)|$.

Example 4. Let $U = \{a, b, c, d, e, f\}$ and $f \in \operatorname{Cl}(U)$ with $\operatorname{Key}(f) = \{\{a, c, d\}, \{b, c, d\}, \{e, f\}\}$.

According to Algorithm 1, we have

$\mathcal{T}_1 = \{\{b, c, d, e, f\}\} \cup \mathcal{F}_1$, where $\mathcal{F}_1 = \{\{a, b, d, e, f\}, \{a, b, c, e, f\}\}$;

$\mathcal{T}_2 = \{\{a, b, d, e, f\}, \{a, b, c, e, f\}, \{c, d, e, f\}\} \cup \mathcal{F}_2$, where $\mathcal{F}_2 = \varnothing$;

$\mathcal{T}_3 = \{\{a, b, d, f\}, \{a, b, d, e\}, \{a, b, c, f\}, \{a, b, c, e\}, \{c, d, f\}, \{c, d, e\}\}$.

Consequently, the set of all antikeys of f is

$$\{\{a, b, d, f\}, \{a, b, d, e\}, \{a, b, c, f\}, \{a, b, c, e\}, \{c, d, f\}, \{c, d, e\}\}.$$

4. Finding the set of all minimal keys of closure operations

In this section, we firstly construct the following algorithm for finding a minimal key of clsoure operations.

Algorithm 2 (H)**.** (Finding a minimal key)

Input: $f \in \mathrm{Cl}(U)$ with $AntiKey(f) = \{K_1^{-1}, K_2^{-1}, \ldots, K_p^{-1}\}$.

Output: $K \in \mathrm{Key}(f)$.

Step 1: We select a set $X \subseteq U$ such that there exists an antikey $K_l^{-1} \in \mathrm{Antikey}(f)$ that $X = K_l^{-1} \cup \{a\}$, where $a \notin K_l^{-1}$. Suppose that $X = \{a_1, a_2, \ldots, a_q\}$. Set $T_0 = X$.

Step $i+1$ $(i = 0, 1, \ldots, q-1)$: We compute

$$T_{i+1} = \begin{cases} T_i \setminus \{a_{i+1}\} & \text{if } \forall K_j^{-1} \in \mathrm{Antikey}(f) : T_i \setminus \{a_{i+1}\} \nsubseteq K_j^{-1} \\ T_i & \text{otherwise.} \end{cases}$$

Step $q+1$: Let $K = T_q$.

It is easy to see that the time comlexity of Algorithm 2 is $O(|U|^2 \cdot p)$. Therefore, our algorithm is very effective.

Lemma 3. *The sets T_i $(i = 0, 1, \ldots, q)$ are the keys and T_q is a minimal key of closure operation f.*

Proof. (Prood by induction) It is easy to see that T_0 is a key. If T_i is a key and $T_{i+1} = T_i$, then it is clear that T_{i+1} is a key. If $T_{i+1} = T_i \setminus \{a_{i+1}\}$ and $f(T_{i+1}) \neq U$, then, by Theorem 1, there exists a $K_j^{-1} \in \mathrm{Antikey}(f)$ such that $f(T_{i+1}) \subseteq K_j^{-1}$. Thus, $T_{i+1} \subseteq K_j^{-1}$. Which contradicts with the fact $\forall K_j^{-1} \in \mathrm{Antikey}(f) : T_{i+1} \nsubseteq K_j^{-1}$. Therefore, T_{i+1} is a key.

Now assume that $Y \subset T_q$. It is clear that if $a \notin Y$, then $f(Y) \neq U$. If $a \in Y$, then there exists an $a_i \in X$ such that $a_i \in T_q \setminus Y$. According to Algorithm 2, there exists a $K_t^{-1} \in \mathrm{Antikey}(f)$ such that $T_{i-1} \setminus \{a_i\} \subseteq K_t^{-1}$. Then we obtain

$$Y \subseteq T_q \setminus \{a_i\} \subseteq T_{i-1} \setminus \{a_i\} \subseteq K_t^{-1}.$$

Note that $T_q \subseteq T_i$ $(0 \leqslant i \leqslant q-1)$. This implies that $f(Y) \neq U$. Consequently, we have $T_q \in \mathrm{Key}(f)$. □

Example 5. Let $U = \{a, b, c, d, e, f\}$ and $f \in \mathrm{Cl}(U)$ with $\mathrm{Antikey}(f) = \{\{b, c, e\}, \{a, b, f\}, \{b, c, f\}, \{a, d, e\}, \{a, d, f\}, \{b, d, e\}, \{a, c, e\}, \{b, d, f\}\}$.

Consider $X = \{a, d, f, c\}$. Then we have

$$T_0 = \{a, d, f, c\}, \qquad T_1 = \{d, f, c\}, \qquad T_2 = \{d, f, c\},$$
$$T_3 = \{d, c\}, \qquad T_4 = \{d, c\}.$$

Hence, $K = \{d, c\}$ is a minimal key of f.

Note that Algorithm 2 also give $K \in \mathrm{Key}(f)$ if X is an arbitrary key of f. It is best to choose X such that $|X|$ is minimal. The condition $\forall K_j^{-1} \in \mathrm{Antikey}(f) : T_i \backslash \{a_{i+1}\} \nsubseteq K_j^{-1}$ in Algorithm 2 may be replaced by the condition $\forall K_j^{-1} \in \{K_t^{-1} \in \mathrm{Antikey}(f) : a \in K_t^{-1}\} : T_i \backslash \{a_{i+1}\} \nsubseteq K_j^{-1}$. Then Algorithm 2 will be more effective.

The following result is the basis for the algorithm to find all the minimal keys of closure operations.

Lemma 4. *Let* $f, f' \in \mathrm{Cl}(U)$ *such that* $\mathrm{Key}(f') \subseteq \mathrm{Key}(f)$. *Suppose that* $\mathrm{Antikey}(f) = \{K_1^{-1}, K_2^{-1}, \ldots, K_p^{-1}\}$. *Then* $\mathrm{Key}(f') \subset \mathrm{Key}(f)$ *and* $\mathrm{Key}(f') \neq \varnothing$ *if and only if there exists a* $X \in \mathrm{Antikey}(f')$ *such that* $X \nsubseteq K_j^{-1}$, $\forall j = 1, 2, \ldots, p$.

Proof. Assume that $\mathrm{Key}(f') \neq \varnothing$ and $\mathrm{Key}(f') \subset \mathrm{Key}(f)$. This implies that there exists a minimal key $K \in \mathrm{Key}(f) \setminus \mathrm{Key}(f')$. It is easy to see that $\mathrm{Key}(f') \cup \{K\}$ is a Sperner system. Hence, there exists the biggest set X such that $K \subseteq X$ and $\mathrm{Key}(f') \cup \{X\}$ is still a Sperner system. This means $X \in \mathrm{Antikey}(f')$. Since $K \in \mathrm{Key}(f)$, we have $K \nsubseteq K_j^{-1}$, $\forall j = 1, 2, \ldots, p$. Consequently, $X \nsubseteq K_j^{-1}$, $\forall j = 1, 2, \ldots, p$.

Conversely, assume that there exists a $X \in \mathrm{Antikey}(f')$ such that $X \nsubseteq K_j^{-1}$, $\forall j = 1, 2, \ldots, p$. Because $\mathrm{Antikey}(f') \neq \varnothing$, we have $\mathrm{Key}(f') \neq \varnothing$, and for all $Y \in \mathrm{Key}(f')$, $Y \nsubseteq X$. Clearly, if there exists a $K_j^{-1} \in \mathrm{Antikey}(f)$ such that $K_j^{-1} \subset X$, then X is a key of f. If $f(X) \neq U$, then by Theorem 1 there is a $K_j^{-1} \in \mathrm{Antikey}(f)$ such that $f(X) \subseteq K_j^{-1}$. Hence, $X \subseteq K_j^{-1}$, which contradicts the fact $X \nsubseteq K_j^{-1}$, $\forall j = 1, 2, \ldots, p$. Thus, X is a key of f. This means there exists a $K \subseteq X$ and $K \in \mathrm{Key}(f) \setminus \mathrm{Key}(f')$. Consequently, $\mathrm{Key}(f') \subset \mathrm{Key}(f)$. □

Based on Lemma 4 we present the algorithm for finding all minimal keys of closure operations.

Algorithm 3. (Finding all minimal keys)

Input: $f \in \mathrm{Cl}(U)$ with $\mathrm{Antikey}(f) = \{K_1^{-1}, K_2^{-1}, \ldots, K_p^{-1}\}$.

Output: $\mathrm{Key}(f)$.

Step 1: Using Algorithm 2 we construct a minimal key $K_1 \in \mathrm{Key}(f)$. We set $\mathrm{Key}(f_1) = \{K_1\}$ with $f_1 \in \mathrm{Cl}(U)$.

Step $i+1\,(i = 1, 2, \ldots)$: We compute $\mathrm{Antikey}(f_i)$ with $f_i \in \mathrm{Cl}(U)$. If there is a $X \in \mathrm{Antikey}(f_i)$ such that $X \not\subseteq K_j^{-1}, \forall j = 1, 2, \ldots, p$, then by Algorithm 2 we determine a $K_{i+1} \in \mathrm{Key}(f)$ and $K_{i+1} \subseteq X$. Set $\mathrm{Key}(f_{i+1}) = \mathrm{Key}(f_i) \cup \{K_{i+1}\}$.

In the converse case, we set $\mathrm{Key}(f) = \mathrm{Key}(f_i)$. The algorithm stops.

It is easy to see that the time comlexity of Algorithm 3 is exponential in the number of elements of set U.

We now consider again Example 5. We already know that $K_1 = \{d, c\} \in \mathrm{Key}(f)$. Set $\mathrm{Key}(f_1) = \{\{d, c\}\}$. Then we have $\mathrm{Antikey}(f_1) = \{\{a, b, d, e, f\},\ \{a, b, c, e, f\}\}$. Because $\{a, b, d, e, f\} \in \mathrm{Antikey}(f_1)$ and $\{a, b, d, e, f\} \not\subseteq K_j^{-1}$ for all $K_j^{-1} \in \mathrm{Antikey}(f)$ we consider $X = \{a, b, d, e, f\}$. Then we obtain

$$T_0 = \{a, b, d, e, f\}, \quad T_1 = \{b, d, e, f\}, \quad T_2 = \{d, e, f\},$$
$$T_3 = \{e, f\}, \quad T_4 = \{e, f\}, \quad T_5 = \{e, f\}.$$

Thus, $K_2 = \{e, f\} \in \mathrm{Key}(f)$. We now set $\mathrm{Key}(f_2) = \mathrm{Key}(f_1) \cup \{K_2\} = \{\{c, d\}, \{e, f\}\}$. Then we have

$$\mathrm{Antikey}(f_2) = \{\{a, b, c, e\}, \{a, b, c, f\}, \{a, b, d, e\}, \{a, b, d, f\}\}.$$

The same as above, we obtain $K_3 = \{a, b\} \in \mathrm{Key}(f)$. Set $\mathrm{Key}(f_3) = \mathrm{Key}(f_2) \cup \{K_3\} = \{\{c, d\}, \{e, f\}, \{a, b\}\}$. It implies $\mathrm{Antikey}(f_3) = \{\{c, e, a\}, \{c, e, b\}, \{c, f, a\}, \{c, f, b\}, \{d, e, a\}, \{d, e, b\}, \{d, f, a\}, \{d, f, b\}\}$.

We set $\mathrm{Key}(f) = \mathrm{Key}(f_3)$. Therefore, the set of all minimal keys of f is

$$\{\{c, e, a\}, \{c, e, b\}, \{c, f, a\}, \{c, f, b\}, \{d, e, a\}, \{d, e, b\}, \{d, f, a\}, \{d, f, b\}\}.$$

5. Some observations on closeness of the closure operations

Let U be a finite set and $Ma(U)$ denotes the set of all mappings $\mathcal{P}(U) \to \mathcal{P}(U)$. We consider $f_1, f_2 \in Ma(U)$. A mapping $g : \mathcal{P}(U) \to \mathcal{P}(U)$ such that $g(X) = f_1(X) \cap f_2(X)$ for every $X \subseteq U$ is called *intersection* of f_1 and f_2, denoted by $g = f_1 \wedge f_2$.

A mapping $h : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by $h(X) = f_1(X) \cup f_2(X)$ for every $X \subseteq U$ is called *union* of f_1 and f_2, denoted by $h = f_1 \vee f_2$.

A mapping $k : \mathcal{P}(U) \to \mathcal{P}(U)$ defined by $k(X) = f_1(f_2(X))$ for each $X \subseteq U$ is called *composition* of f_1 and f_2, denoted by $k = f_1 f_2$.

Let U_1 and U_2 be two disjoint finite sets, and two mappings $f_1 \in Ma(U_1)$, $f_2 \in Ma(U_2)$. A mapping $l : \mathcal{P}(U_1 \cup U_2) \to \mathcal{P}(U_1 \cup U_2)$ defined by $l(X) = f_1(X \cap U_1) \cup f_2(X \cap U_2)$ for all $X \subseteq U_1 \cup U_2$ is called a *direct product* of f_1 and f_2, denoted by $l = f_1 \times f_2$.

It is known [3, 9] that the class of closure operations is closed under intersection and direct product operations. However, the class of the closure operations is not closed under union and composition operations. Two sufficient and necessary conditions for the closure operations class to be closed under the composition operation are proposed in [8]. In this section we first show that the class of closure operations is not closed under the union operation.

Proposition 2. *The union of two closure operations is not a closure operation.*

Proof. We consider the following counterexample: let $U = \{a, b, c\}$ and two mappings $f_a, g_a : \mathcal{P}(U) \to \mathcal{P}(U)$, as follows:

$$f_a(X) = X \cup \{a\},$$

and

$$g_a(X) = \begin{cases} X & \text{if } a \notin X \\ U & \text{otherwise} \end{cases}$$

for every $X \subseteq U$.

Clearly, $f_a = t_{\{a\}}$. Therefore, it is easy to see that $f_a, g_a \in \mathrm{Cl}(U)$.

We now consider $X = \{b\}$ and set $h = f_a \vee g_a$. Then we get

$$h(X) = f_a(X) \cup g_a(X) = \{a, b\} \cup \{b\} = \{a, b\},$$
$$h(h(X)) = h(\{a, b\}) = \{a, b\} \cup U = U.$$

This implies that $h \notin \mathrm{Cl}(U)$.

Note that it is easy to see h satisfies (C1) and (C2). □

It can be seen that if $f_1, f_2 \in \mathrm{Cl}(U)$ and $f_1(X) \subseteq f_2(X)$ or $f_2(X) \subseteq f_1(X)$ for all $X \subseteq U$, then $f_1 \vee f_2 \in \mathrm{Cl}(U)$.

It is known [3] that the closeness of closure operations class under direct product operation is proved by the concept of represent matrix of closure operations. However, in this section we shall prove this result only by the definition of closure operation. The proof shows the essence of closure operations.

Proposition 3. *The direct product of two closure operations is a closure operation.*

Proof. Suppose that $f_1 \in \mathrm{Cl}(U_1), f_2 \in \mathrm{Cl}(U_2), U_1 \cap U_2 = \varnothing$ and $U = U_1 \cup U_2$. We shall prove $f_1 \times f_2 \in \mathrm{Cl}(U)$.

Clearly, we first have $X = (X \cap U_1) \cup (X \cap U_2)$ for all $X \subseteq U_1 \cup U_2$. Furthermore, $X \cap U_1 \subseteq f_1(X \cap U_1)$ and $X \cap U_2 \subseteq f_2(X \cap U_2)$. This implies that $(X \cap U_1) \cup (X \cap U_2) \subseteq f_1(X \cap U_1) \cup f_2(X \cap U_2)$. Hence, $X \subseteq f_1(X \cap U_1) \cup f_2(X \cap U_2)$.

Next, we have $X \cap U_1 \subseteq Y \cap U_1$ and $X \cap U_2 \subseteq Y \cap U_2$ for all $X \subseteq Y \subseteq U$. Then by using (C2) of f_1 and f_2, we obtain $f_1(X \cap U_1) \cup f_2(X \cap U_2) \subseteq f_1(Y \cap U_1) \cup f_2(Y \cap U_2)$.

Lastly, we set $l = f_1 \times f_2$. Then we obtain

$$\begin{aligned} f_1(l(X) \cap U_1) &= f_1((f_1(X \cap U_1) \cup f_2(X \cap U_2)) \cap U_1) \\ &= f_1((f_1(X \cap U_1) \cap U_1) \cup (f_2(X \cap U_2) \cap U_1)) \\ &= f_1(f_1(X \cap U_1) \cap U_1) \\ &= f_1(X \cap U_1). \end{aligned}$$

By using the symmetry, we also have $f_2(l(X) \cap U_2) = f_2(X \cap U_2)$. Thus, we get $l(l(X)) = f_1(l(X) \cap U_1) \cup f_2(l(X) \cap U_2) = f_1(X \cap U_1) \cup f_2(X \cap U_2) = l(X)$. □

Now let $f_1, f_2, \ldots, f_n$ be closure operations on the disjoint ground sets $U_1, U_2, \ldots, U_n$ respectively. Then the direct product of $f_1, f_2, \ldots, f_n$, denoted as $f_1 \times f_2 \times \cdots \times f_n$, is defined as following

$$f_1 \times f_2 \times \cdots \times f_n(X) = \bigcup_{i=1}^{n} f_i(X) \cap U$$

with $X \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

By the induction we also obtain the following result for n closure operations.

Corollary 1. *The direct product of n closure operations is a closure operation.*

6. Conclusion

The paper first proposes some combinatorial characteristics of minimal key and antikey of closure operations. After that it give effective algorithms finding minimal keys and antikeys of closure operations. Lastly, the paper investigates the closeness of closure operations class under the union and direct product operations.

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