Algebra and Discrete Mathematics
Volume **23** (2017). Number 2, pp. 249–262

RESEARCH ARTICLE

The R_∞ property for Houghton's groups

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Communicated by R. I. Grigorchuk

Abstract. We study twisted conjugacy classes of a family of groups which are called Houghton's groups $\mathcal{H}_n$ ($n \in \mathbb{N}$), the group of translations of n rays of discrete points at infinity. We prove that the Houghton's groups $\mathcal{H}_n$ have the R_∞ property for all $n \in \mathbb{N}$.

Introduction

Let G be a group and $\varphi : G \to G$ be a group endomorphism. We define an equivalence relation $\sim$ on G, called the Reidemeister action by φ, by

$$a \sim b \Leftrightarrow b = ha\varphi(h)^{-1} \text{ for some } h \in G.$$

The equivalence classes are called *twisted conjugacy classes* or *Reidemeister classes* and $R[\varphi]$ denotes the set of twisted conjugacy classes. The *Reidemeister number* $R(\varphi)$ of φ is defined to be the cardinality of $R[\varphi]$. We say that G has the R_∞ *property* if $R(\varphi) = \infty$ for every automorphism $\varphi : G \to G$.

In 1994, Fel'shtyn and Hill [10] conjectured that any injective endomorphism φ of a finitely generated group G with exponential growth would satisfy that $R(\varphi) = \infty$. Levitt and Lustig ([23]), and Fel'shtyn ([8]) showed that the conjecture holds for automorphisms when G is Gromov

*The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. 2013R1A1A2058693).

2010 MSC: 20E45, 20E36, 55M20.

Key words and phrases: Houghton's group, R_∞ property, Reidemeister number.

hyperbolic. However, in 2003, the conjecture was answered negatively by Gonçalves and Wong [15] who gave examples of finitely generated groups with exponential growth which do not have the R_∞ property. Since then, groups with the R_∞ property have been known including Baumslag-Solitar groups, lamplighter groups, Thompson's groups F and T, Grigorchuk group, mapping class groups, relatively hyperbolic groups, and some linear groups (see [2, 3, 6, 9, 11–14, 16, 17, 21, 24] and references therein). For a topological consequence of the R_∞ property, see [16, 21, 24]. In this article we show the following.

Theorem 1. *The Houghton's groups $\mathcal{H}_n$ have the R_∞ property for all $n \in \mathbb{N}$.*

It is shown that the conjugacy problem([1]) and the twisted conjugacy problem([5]) of $\mathcal{H}_n$ are solvable for $n \geqslant 2$. In 2010, Gonçalves and Kochloukova [11] proved that there is a finite index subgroup H of $\mathrm{Aut}(\mathcal{H}_n)$ such that $R(\varphi) = \infty$ for $\varphi \in H$ provided $n \geqslant 2$. Recently the structure of $\mathrm{Aut}(\mathcal{H}_n)$ is known from [4] (see Theorem 3 below). In [14], Gonçalves and Sankaran have studied also the R_∞ property of Houghton's groups.

In this paper we use simple but useful observations of the Reidmeister numbers and the structure of $\mathrm{Aut}(\mathcal{H}_n)$ to find equivalent conditions for two elements of $\mathcal{H}_n$ to determine the same twisted conjugacy class under mild assumptions. In Section 1, we will review definition and some facts about Houghton's groups $\mathcal{H}_n$ which are necessary mainly to the study of Reidemeister numbers for $\mathcal{H}_n$. In Section 2, we prove our main result for $n \geqslant 2$. The case of $n = 1$ is discussed in Section 3.

1. Houghton's groups $\mathcal{H}_n$

In this paper we use the following notational conventions. All bijections (or permutations) act on the right unless otherwise specified. Consequently gh means g followed by h. The conjugation by g is denoted by $\mu(g)$, $h^g = g^{-1}hg =: \mu(g)(h)$, and the commutator is defined by $[g, h] = ghg^{-1}h^{-1}$.

Our basic references are [19, 22] for Houghton's groups and [4] for their automorphism groups. Fix an integer $n \geqslant 1$. For each k with $1 \leqslant k \leqslant n$, let

$$R_k = \left\{ me^{i\theta} \in \mathbb{C} \mid m \in \mathbb{N},\ \theta = \tfrac{\pi}{2} + (k-1)\tfrac{2\pi}{n} \right\}$$

and let $X_n = \bigcup_{k=1}^{n} R_k$ be the disjoint union of n copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane. We shall use the notation $\{1, \cdots, n\} \times \mathbb{N}$ for X_n, letting (k, p) denote the point of R_k with distance p from the origin.

A bijection $g : X_n \to X_n$ is called an *eventual translation* if the following holds:

> *There exist an n-tuple* $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ *and a finite set* $K_g \subset X_n$ *such that*
>
> $$(k, p) \cdot g := (k, p + m_k) \quad \forall (k, p) \in X_n - K_g.$$

An eventual translation acts as a translation on each ray outside a finite set. For each $n \in \mathbb{N}$ the *Houghton's group* $\mathcal{H}_n$ is defined to be the group of all eventual translations of X_n.

Let g_i be the translation on the ray of $R_1 \cup R_{i+1}$ by 1 for $1 \leqslant i \leqslant n-1$. Namely,

$$(j, p) \cdot g_i = \begin{cases} (1, p-1) & \text{if } j = 1 \text{ and } p \geqslant 2, \\ (i+1, 1) & \text{if } (j, p) = (1, 1), \\ (i+1, p+1) & \text{if } j = i+1, \\ (j, p) & \text{otherwise.} \end{cases}$$

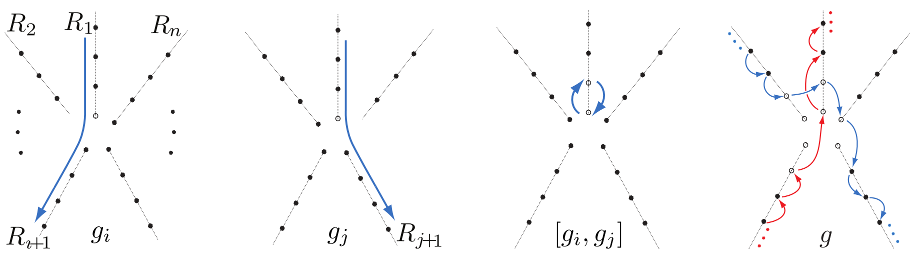

Figure 1. Some examples of $\mathcal{H}_n$.

Figure 1 illustrates some examples of elements of $\mathcal{H}_n$, where points which do not involve arrows are meant to be fixed. Finite sets K_{g_i} and K_{g_j} are singleton sets. The commutator $[g_i, g_j]$ of two distinct elements g_i and g_j is the transposition exchanging $(1, 1)$ and $(1, 2)$. We will denote this transposition by α. The last element g is rather generic and K_g consists of eight points. Johnson provided a finite presentation for $\mathcal{H}_3$ in [20] and the third author gave a finite presentation for $\mathcal{H}_n$ with $n \geqslant 3$ in [22] as follows:

Theorem 2 ([22, Theorem C]). *For $n \geqslant 3$, $\mathcal{H}_n$ is generated by*

$$g_1, \cdots, g_{n-1}, \alpha$$

with relations

$$\alpha^2 = 1,\ (\alpha\alpha^{g_1})^3 = 1,\ [\alpha, \alpha^{g_1^2}] = 1,\ \alpha = [g_i, g_j],\ \alpha^{g_i^{-1}} = \alpha^{g_j^{-1}}$$

for $1 \leqslant i \neq j \leqslant n-1$.

From the definition of Houghton's groups, the assignment $g \in \mathcal{H}_n \mapsto (m_1, \cdots, m_n) \in \mathbb{Z}^n$ defines a homomorphism $\pi = (\pi_1, \cdots, \pi_n) : \mathcal{H}_n \to \mathbb{Z}^n$. Then we have:

Lemma 1 ([22, Lemma 2.3]). *For $n \geqslant 3$, we have* $\ker \pi = [\mathcal{H}_n, \mathcal{H}_n]$.

Note that $\pi(g_i) \in \mathbb{Z}^n$ has only two nonzero values -1 and 1,

$$\pi(g_i) = (-1, 0, \cdots, 0, 1, 0, \cdots, 0)$$

where 1 occurs in the $(i+1)$st component. Since the image of $\mathcal{H}_n$ under π is generated by those elements, we have that

$$\pi(\mathcal{H}_n) = \left\{ (m_1, \cdots, m_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} m_i = 0 \right\},$$

which is isomorphic to the free Abelian group of rank $n-1$. Consequently, $\mathcal{H}_n$ ($n \geqslant 3$) fits in the following short exact sequence

$$1 \longrightarrow \mathcal{H}_n' = [\mathcal{H}_n, \mathcal{H}_n] \longrightarrow \mathcal{H}_n \stackrel{\pi}{\longrightarrow} \mathbb{Z}^{n-1} \longrightarrow 1.$$

The above abelianization, first observed by C. H. Houghton in [19], is the characteristic property of $\{\mathcal{H}_n\}$ for which he introduced those groups in the same paper. We may regard π as a homomorphism $\mathcal{H}_n \to \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ given by

$$\pi : g_i \mapsto (-1, 0, \cdots, 0, 1, 0, \cdots, 0) \mapsto (0, \cdots, 0, 1, 0, \cdots, 0).$$

In particular, $\pi(g_1), \cdots, \pi(g_{n-1})$ form a set of free generators for $\mathbb{Z}^{n-1}$.

By definition, $\mathcal{H}_1$ is the symmetric group itself on X_1 with finite support, which is not finitely generated. Furthermore, $\mathcal{H}_2$ is

$$\mathcal{H}_2 = \langle g_1, \alpha \mid \alpha^2 = 1, (\alpha\alpha^{g_1})^3 = 1, [\alpha, \alpha^{g_1^k}] = 1 \text{ for all } |k| > 1 \rangle,$$

which is finitely generated, but not finitely presented. It is not difficult to see that $\mathcal{H}_2' = \mathsf{FAlt}_2$.

From now on we use the following notations:

- Sym_n is the full symmetric group of X_n;
- FSym_n is the symmetric group of X_n with finite support;
- FAlt_n is the alternating group of X_n with finite support.

For each n the group FAlt_n can be seen as the kernel of the sign homomorphism $\mathsf{FSym}_n \to \{\pm 1\}$. The following fact is necessary for our discussion, see [7].

Remark 1. For any $\sigma \in \mathsf{Sym}_n$, the conjugation by σ induces automorphisms $\mu(\sigma) : \mathsf{FSym}_n \to \mathsf{FSym}_n$ and $\mu(\sigma) : \mathsf{FAlt}_n \to \mathsf{FAlt}_n$. Then $\mu : \mathsf{Sym}_n \to \mathrm{Aut}(\mathsf{FAlt}_n)$ and $\mu : \mathsf{Sym}_n \to \mathrm{Aut}(\mathsf{FSym}_n)$ are isomorphisms.

Every automorphism of $\mathcal{H}_n$ restricts to an automorphism of the characteristic subgroup $\mathcal{H}_n'' = [\mathsf{FSym}_n, \mathsf{FSym}_n] = \mathsf{FAlt}_n$, which induces a homomorphism res : $\mathrm{Aut}(\mathcal{H}_n) \to \mathrm{Aut}(\mathsf{FAlt}_n)$. One can show this map is injective by using the fact that FAlt_n is generated by 3-cycles. The embedding

$$\mathrm{Res} : \mathrm{Aut}(\mathcal{H}_n) \xrightarrow{\mathrm{res}} \mathrm{Aut}(\mathsf{FAlt}_n) \xrightarrow{\mu^{-1}} \mathsf{Sym}_n$$

implies that each automorphism of $\mathcal{H}_n$ is given by a conjugation of an element in Sym_n. Moreover the composition preserves the normality $\mathcal{H}_n = \mathrm{Inn}(\mathcal{H}_n) \lhd \mathrm{Aut}(\mathcal{H}_n)$.

Proposition 1 ([4, Proposition 2.1])**.** *For $n \geqslant 1$, the automorphism group $\mathrm{Aut}(\mathcal{H}_n)$ is isomorphic to the normalizer of $\mathcal{H}_n$ in the group Sym_n.*

We need an explicit description for the normalizer $N_{\mathsf{Sym}_n}(\mathcal{H}_n)$ to study $\mathrm{Aut}(\mathcal{H}_n)$. Consider an element $\sigma_{ij} \in \mathsf{Sym}_n$ for $1 \leqslant i \neq j \leqslant n$ defined by

$$(\ell, p) \cdot \sigma_{ij} = \begin{cases} (j, p) & \text{if } \ell = i \\ (i, p) & \text{if } \ell = j \\ (\ell, p) & \text{otherwise} \end{cases}$$

for all $p \in \mathbb{N}$. Each element σ_{ij} defines a transposition on n rays isometrically. The subgroup of Sym_n generated by all σ_{ij} is isomorphic to the symmetric group Σ_n on the n rays. Note that Σ_n acts on $\mathcal{H}_n$ by conjugation. One can show that $N_{\mathsf{Sym}_n}(\mathcal{H}_n)$ coincides with $\mathcal{H}_n \rtimes \Sigma_n$ by using the ray structure (end structure) of the underlying set X_n. An eventual translation g preserves each ray up to a finite set. Let R_i^* denote the set of all points of R_i but finitely many. It is not difficult to see that if $\phi \in \mathsf{Sym}_n$ normalizes $\mathcal{H}_n$ then

$$(R_i^*)\phi = R_j^*$$

for $1 \leqslant i, j \leqslant n$. Thus ϕ defines an element σ of Σ_n, and we see that $\phi\sigma^{-1} \in \mathcal{H}_n$ since $(R_i^*)\phi\sigma^{-1} = (R_j^*)\sigma^{-1} = R_i^*$ for each i. Consequently, $N_{\mathsf{Sym}_n}(\mathcal{H}_n)$ has the internal semidirect product of $\mathcal{H}_n$ by Σ_n. Therefore we have:

Theorem 3 ([4, Theorem 2.2]). *For $n \geqslant 2$, we have*

$$\mathrm{Aut}(\mathcal{H}_n) \cong \mathcal{H}_n \rtimes \Sigma_n$$

where Σ_n is the symmetric group that permutes n rays isometrically.

2. The R_∞ property for $\mathcal{H}_n$, $n \geqslant 2$

We consider the Houghton's groups $\mathcal{H}_n$ with $n \geqslant 2$. Let ϕ be an automorphism on $\mathcal{H}_n$. Remark that, when $n \geqslant 3$, ϕ induces an automorphism ϕ' on the commutator subgroup $\mathcal{H}_n' = \mathsf{FSym}_n$ and an automorphism $\bar{\phi}$ on $\mathbb{Z}^{n-1}$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathsf{FSym}_n & \xrightarrow{i} & \mathcal{H}_n & \xrightarrow{\pi} & \mathbb{Z}^{n-1} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \\
1 & \longrightarrow & \mathsf{FSym}_n & \xrightarrow{i} & \mathcal{H}_n & \xrightarrow{\pi} & \mathbb{Z}^{n-1} & \longrightarrow & 1
\end{array}$$

But when $n = 2$, $\mathcal{H}_2' = \mathsf{FAlt}_2$ and $\mathcal{H}_2/\mathcal{H}_2' = \mathbb{Z} \oplus \mathbb{Z}_2$. Since FSym_2 is a normal subgroup of $\mathcal{H}_2$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathsf{FAlt}_2 & \longrightarrow & \mathcal{H}_2 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle =} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathsf{FAlt}_2 & \longrightarrow & \mathsf{FSym}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Let $\phi \in \mathrm{Aut}(\mathcal{H}_2)$. Then ϕ restricts to an element ϕ' of $\mathrm{Aut}(\mathcal{H}_2') = \mathrm{Aut}(\mathsf{FAlt}_2) = \mathrm{Aut}(\mathsf{FSym}_2)$, and hence induces an automorphism $\bar{\phi}$ on $\mathbb{Z}$

so that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{FSym}_2 & \longrightarrow & \mathcal{H}_2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} & & \\ 1 & \longrightarrow & \mathsf{FSym}_2 & \longrightarrow & \mathcal{H}_2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

These diagrams induce an exact sequence of Reidemeister sets

$$\mathcal{R}[\phi'] \xrightarrow{\hat{i}} \mathcal{R}[\phi] \xrightarrow{\hat{\pi}} \mathcal{R}[\bar{\phi}] \longrightarrow 1.$$

Because $\hat{\pi}$ is surjective, we have that if $R(\bar{\phi}) = \infty$, then $R(\phi) = \infty$. Consequently, we have

Lemma 2. *Let ϕ be an automorphism on $\mathcal{H}_n$, $(n \geqslant 2)$. If $R(\bar{\phi}) = \infty$, then $R(\phi) = \infty$.*

By Theorem 3, $\phi = \mu(\gamma\sigma)$ for some $\gamma \in \mathcal{H}_n$ and $\sigma \in \Sigma_n$. First, we will show that when $\phi = \mu(\sigma)$ for $\sigma \in \Sigma_n$ the Reidemeister number of ϕ is infinity. When $\sigma = \mathrm{id}$, ϕ and hence $\bar{\phi}$ are identities. It is easy to see from definition that $R(\bar{\phi}) = R(\mathrm{id}) = \infty$, and so $R(\phi) = \infty$.

One useful observation in calculating $R(\mu(\sigma))$ is that a product $\sigma = \sigma_1\sigma_2$ induces a bijection

$$R[\mu(\sigma_1)] \longleftrightarrow R[\mu(\sigma)], \tag{1}$$

which follows from

$$b = ha\bar{h}^{\sigma_1} \Leftrightarrow b\sigma_2 = h(a\sigma_2)\bar{h}^{\sigma_1\sigma_2}$$

for all $a, b, h \in \mathcal{H}_n$. Note that any product for σ induces a bijection between the twist conjugacy classes of σ and of the first term in the product. Recall that a cycle decomposition of a permutation σ allows one to write σ as a product of disjoint cycles. Since disjoint cycles commute there exists a bijection between $R[\mu(\sigma)]$ and $R[\mu(\sigma_1)]$ for any cycle σ_1 in a cycle decomposition of σ. The following observation plays a crucial role in the sequel.

Remark 2. For a cycle σ_1 in a cycle decomposition of $\sigma \in \Sigma_n$, we have that $R(\mu(\sigma_1)) = \infty$ if and only if $R(\mu(\sigma)) = \infty$.

Recall that the *cycle type* of a permutation $\tau \in \mathsf{FSym}_n$ encodes the data of how many cycles of each length are present in a cycle decomposition of τ. Note that two permutations τ and τ' have the same cycle type if and only if they are conjugate in FSym_n. In particular two cycles determine the same conjugacy class if and only if they have the same length. We extend this to establish a criterion for twisted conjugacy classes of cycles with respect to an automorphism $\phi = \mu(\sigma)$ when $\sigma \in \Sigma_n$ is a cycle.

Lemma 3. *Suppose $\sigma \neq \mathrm{id} \in \Sigma_n$ is a cycle and $n \geqslant 2$. A pair of cycles τ and τ' on the same ray determine the same twisted conjugacy class of $\phi = \mu(\sigma)$ if and only if they have the equal length. In particular $R(\phi) = \infty$.*

Proof. Suppose that τ and τ' are cycles on the same ray of the equal length. We first consider the case when σ permutes rays as an ℓ-cycle $(1\,2 \cdots \ell)$ for some $2 \leqslant \ell \leqslant n$, and τ and τ' are disjoint cycles on R_1. Two cycles τ and τ' can be written as

$$\tau = (p_1 \cdots p_m) \text{ and } \tau' = (q_1 \cdots q_m)$$

(by suppressing the ray notation) where $m \geqslant 2$. We need to find an element $h \in \mathcal{H}_n$ such that $\tau' = h\tau\mu(\sigma)(h)^{-1}$, or equivalently

$$h^\sigma = \tau'^{-1} h \tau. \tag{2}$$

Let h_1 be the $2m$-cycle on R_1 given by

$$h_1 = (p_1 q_1\, p_2\, q_2 \cdots p_m\, q_m).$$

It is direct to check that

$$\tau'^{-1} h_1 \tau = (q_m \cdots q_1)(p_1 q_1\, p_2\, q_2 \cdots p_m\, q_m)(p_1 \cdots p_m) = h_1. \tag{3}$$

Consider $h \in \mathcal{H}'_n$ defined by

$$h = h_1^{\sigma^{\ell-1}} \cdots h_1^\sigma h_1.$$

Note that h is a product of ℓ disjoint $2m$-cycles each of which is an 'isometric translation' of h_1 to the ray $R_\ell, \cdots, R_2, R_1$. More precisely $(k+1, p)h_1^{\sigma^k} = (1,p)h_1\sigma^k$ for all $(1,p) \in \mathrm{supp}(h_1)$ and $k = 1, \cdots, \ell - 1$. One crucial observation is that

$$h^\sigma = h.$$

The above follows from that σ is a ℓ-cycle and that components of h have pairwise disjoint supports. Moreover, τ' commutes with $h_1^{\sigma^{\ell-1}} \cdots h_1^\sigma$, so we have

$$\begin{aligned} h^\sigma = h = h_1^{\sigma^{\ell-1}} \cdots h_1^\sigma h_1 &= h_1^{\sigma^{\ell-1}} \cdots h_1^\sigma (\tau'^{-1} h_1 \tau) \\ &= \tau'^{-1} (h_1^{\sigma^{\ell-1}} \cdots h_1^\sigma h_1) \tau = \tau'^{-1} h \tau. \end{aligned}$$

Therefore h satisfies the condition (2), and hence $[\tau] = [\tau']$ in $R[\mu(\sigma)]$.

Applying appropriate conjugations one can extend the above observations to show that $[\tau]=[\tau']$ in $R[\mu(\sigma)]$ for any cycle $\sigma\in\Sigma_n$ and for any two disjoint cycles τ and τ' on the same ray with the equal length. Therefore, by the transitivity of the class, we can see that two cycles (not necessarily disjoint) on a ray belong to the same class for $\phi=\mu(\sigma)$ as long as they have the same length. Indeed, if two m-cycles τ and τ' are not disjoint, one takes another m-cycle τ_0 which is disjoint with τ and τ' to have $[\tau]=[\tau_0]=[\tau']$. Thus we are done with one direction.

For the converse, suppose there exists $h\in\mathcal{H}_n$ satisfying the condition (2) for a cycle $\sigma\in\Sigma_n$ even when cycles τ and τ' on the same ray have different lengths m and m' respectively. Assume $m'>m$. Let ℓ be the order of σ. Applying the identity (2) ℓ times, we have

$$h=h^{\sigma^\ell}=(\tau'^{-1})^{\sigma^{\ell-1}}\cdots(\tau'^{-1})^{\sigma}\tau'^{-1}h\tau\tau^{\sigma}\cdots\tau^{\sigma^{\ell-1}}. \tag{4}$$

Let $c'=(\tau'^{-1})^{\sigma^{\ell-1}}\cdots(\tau'^{-1})^{\sigma}\tau'^{-1}$ and $c=\tau\tau^{\sigma}\cdots\tau^{\sigma^{\ell-1}}$ be the products of first and last ℓ terms on the RHS of (4). Note that each component of c' is an 'isometric translation' of τ'^{-1} to different ℓ rays (and similarly for each component of c). To draw a contradiction, we use the fact that the size of $\mathrm{supp}(c')$ is strictly greater than that of $\mathrm{supp}(c)$. For details we need to examine how $h=c'hc$ acts on $\mathrm{supp}(c')$. Being a disjoint union, $\mathrm{supp}(c')=\bigcup_{0\leqslant k\leqslant \ell-1}(\mathrm{supp}(\tau'))\sigma^k$, $\mathrm{supp}(c')$ has size $\ell\times m'$, while $\mathrm{supp}(c)$ has size $\ell\times m$. For each $P\in\mathrm{supp}(c')$, we have

$$(P)h=(P)c'hc=(P')hc \quad\text{or}\quad (P)hc^{-1}=(P')h$$

where P' is a point in the same ray of P but distinct from P. We claim that $(P)h$ belongs to $\mathrm{supp}(c)$. Otherwise c^{-1} fixes $(P)h$, forcing $(P)h=(P')h$. Since $P\in\mathrm{supp}(c')$ was arbitrary, a bijection h maps $\mathrm{supp}(c')$ to $\mathrm{supp}(c)$. We conclude that there does not exists $h\in\mathcal{H}_n$ satisfying the condition (2) for cycles τ and τ' on the same ray with different lengths. □

Theorem 4. *The Houghton's groups $\mathcal{H}_n$ have the R_∞ property for all $n\geqslant 2$.*

Proof. Theorem 3 says that an automorphism ϕ of $\mathcal{H}_n$ is determined by $\phi=\mu(g\sigma)$ for some $g\in\mathcal{H}_n$ and $\sigma\in\Sigma_n$. As we noted earlier, we may assume that $\sigma\neq 1$. Note that

$$g\sigma=\sigma(\sigma^{-1}g\sigma)=\sigma g'$$

with $g'\in\mathcal{H}_n$. The product in RHS yields a bijection between $R[\mu(g\sigma)]$ and $R[\mu(\sigma)]$ as in (1). Consider a cycle σ_1 in a cycle decomposition of σ.

Remark 2 together with Lemma 3 implies $R[\mu(\sigma)] = R[\mu(\sigma_1)] = \infty$. Therefore we have $R[\phi] = R[\mu(g\sigma)] = R[\mu(\sigma)] = \infty$ for all $\phi \in \mathrm{Aut}(\mathcal{H}_n)$ when $n \geqslant 2$. □

We remark that Lemma 2 can be used extensively to establish Theorem 4. As observed in commuting diagrams above an automorphism $\phi = \mu(g\sigma)$ of $\mathcal{H}_n$ induces an automorphism $\overline{\phi}$ on the abelianization $\mathbb{Z}^{n-1}$, which is freely generated by $\pi(g_1), \ldots, \pi(g_{n-1})$. Since $\mu(g)$ fixes the generates $g_1, \ldots, g_{n-1}$, se see that $\overline{\phi} = \mu(\sigma)$. The Reidemeister number of an automorphism on $\mathbb{Z}^{n-1}$ ($n \geqslant 2$) is well understood. By [18, Theorem 6.11], $R(\overline{\phi}) = \infty$ if and only if $\overline{\phi}$ has eigenvalue 1. By using induction on n one can show that $\overline{\phi} = \mu(\sigma)$ has eigenvalue 1 unless σ is an n-cycle on the rays $R_1, \ldots, R_n$. Now Lemma 3 implies that $R(\mu(\sigma)) = \infty$, and so $R(\phi) = R(\overline{\phi}) = \infty$.

3. The group $\mathcal{H}_1$ and its R_∞ property

In this section, we will study the R_∞ property for the group $\mathcal{H}_1$. We remark that $\mathcal{H}_1 = \mathsf{FSym}_1$ is generated by the transpositions exchanging two consecutive points of R_1. Let ϕ be an automorphism of $\mathcal{H}_1$. Since $\mathrm{Aut}(\mathcal{H}_1) = \mathrm{Aut}(\mathsf{FSym}_1) \cong \mathsf{Sym}_1$, we have that $\phi = \mu(\gamma)$ for some $\gamma \in \mathsf{Sym}_1$.

Lemma 4. *Let $\varphi : G \to G$ be an endomorphism. Then for any $g \in G$ we have $[g] = [\varphi(g)]$ in $R[\varphi]$.*

Proof. The Lemma follows from

$$\varphi(g) = (g^{-1}) g \varphi(g^{-1})^{-1}.$$

□

An infinite cycle $\gamma \in \mathsf{Sym}_1$ is given by a bijection $\gamma : \mathbb{Z} \to R_1$. For convenience we use the 1-to-1 correspondence to denote points of $\mathrm{supp}(\gamma) \subset R_1$ by integers, that is, each point of $\mathrm{supp}(\gamma)$ is denoted by its *preimage*. With this notation, each infinite cycle can be realized as the translation on $\mathbb{Z}$ by $+1$. Remark that if $h \in \mathsf{FSym}_1$ with $\mathrm{supp}(h) \subset \mathrm{supp}(\gamma)$ then the conjugation $\mu(\gamma)$ *shifts* $\mathrm{supp}(h)$ to $\mathrm{supp}(h^\gamma)$ by $+1$;

$$(k)h = k' \;\Leftrightarrow (k+1)h^\gamma = k' + 1 \tag{5}$$

for all $k \in \mathrm{supp}(h)$. We say that an infinite cycle γ *conjugates* a permutation $\tau \in \mathsf{FSym}_1$ to τ' if τ' can be written as a conjugation of τ by a power of γ.

Lemma 5. *For an infinite cycle $\gamma \in \mathsf{Sym}_1$, two transpositions τ and τ' with* $\operatorname{supp}(\tau) \subset \operatorname{supp}(\gamma)$ *and* $\operatorname{supp}(\tau') \subset \operatorname{supp}(\gamma)$ *determine the same conjugacy class for $\phi = \mu(\gamma)$ if and only if γ conjugates τ to τ'. In particular $R(\mu(\gamma)) = \infty$.*

Proof. Assume that $\tau' = \tau^{\gamma^m}$ or $\tau' = \phi^m(\tau)$, for some m. By Lemma 4, we have $[\tau] = [\phi(\tau)] = \cdots = [\phi^m(\tau)] = [\tau']$.

For the converse, suppose that there exists $h \in \mathsf{FSym}_1$ satisfying

$$h^\gamma = \tau'^{-1} h \tau = \tau' h \tau \tag{6}$$

for two transpositions τ and τ' with the condition on their supports, one of which γ does not conjugate to the other. By the shift (5), they can be written as $\tau = (0\,\ell)$ and $\tau' = (m\, m{+}\ell')$ for some $m \geqslant 0$ and $\ell \neq \ell' > 0$. By Lemma 4, which implies $[(0\,\ell')] = [(m\ m{+}\ell')]$ for all $m \in \mathbb{Z}$, we may further assume that $\tau' = (0\,\ell')$ and $\ell < \ell'$.

We first claim that $(-1)h = -1$. If $-1 \in \operatorname{supp}(h)$, the identity (6) says

$$(-1)h^\gamma = (-1)\tau' h \tau = (-1)h\tau \neq -1$$

since τ and τ' fix all negative integers. So $-1 \in \operatorname{supp}(h^\gamma)$. Now the shift

$$k \in \operatorname{supp}(h) \ \Leftrightarrow\ k+1 \in \operatorname{supp}(h^\gamma)$$

implies $-2 \in \operatorname{supp}(h)$. Observe that the same argument establishes simultaneous induction on k for

$$-k \in \operatorname{supp}(h) \text{ and } -k \in \operatorname{supp}(h^\gamma)$$

for all positive k with the above base cases when $k = 1$. This means that $\operatorname{supp}(h)$ must contain all negative integers. It contradicts that $h \in \mathsf{FSym}_1$. Therefore h fixes -1, or equivalently h^γ fixes 0. One can also show h fixes $\ell' + 1$ by verifying

$$\ell' + k \in \operatorname{supp}(h) \text{ and } \ell + k \in \operatorname{supp}(h^\gamma)$$

for all positive k if we are given the base case $\ell' + 1 \in \operatorname{supp}(h)$ (and $\ell' + 1 \in \operatorname{supp}(h^\gamma)$, which follows immediately by (6)). So we also have $(\ell' + 1)h = (\ell' + 1)$, and hence $(\ell' + 1)h^\gamma = (\ell' + 1)$ by (6).

From the fixed point $0 = (0)h^\gamma$ we have

$$(0)\tau' h \tau = 0 \ \Leftrightarrow\ (\ell')h = \ell.$$

The shift (5) says $\ell' + 1 \in \operatorname{supp}(h^\gamma)$. However this contradicts that $(\ell' + 1)h^\gamma = (\ell' + 1)$. Therefore $\tau' = (0\,\ell')$ does not belong to the class of $\tau = (0\,\ell)$ unless $\ell = \ell'$. □

Lemma 6. *Suppose two permutations* $\tau, \tau' \in \mathsf{FSym}_1$ *are disjoint with a permutation* $\gamma \in \mathsf{FSym}_1$*. Then* τ *and* τ' *belong to the same class in* $R[\mu(\gamma)]$ *if and only if they have the same cycle type. In particular* $R(\mu(\gamma)) = \infty$*.*

Proof. The statement follows from *cycle type criterion* for usual conjugacy classes of the symmetric group on the fixed points of $\gamma \in \mathsf{FSym}_1$. Any permutations on $R_1' = R_1 \setminus \mathrm{supp}(\gamma)$ with finite supports are conjugate if and only if they have the same cycle type. For two permutations τ and τ' on R_1' there exists a permutation $h \in \mathsf{FSym}_1$ on R_1' such that

$$\tau' = h\tau h^{-1}$$

if and only if τ and τ' have the same cycle type. Since $h^\gamma = h$ one can replace h^{-1} by $(h^{-1})^\gamma$ in the identity to establish $\tau' = h\tau(h^{-1})^\gamma$. □

Theorem 5. *The group* $\mathcal{H}_1$ *has the* R_∞ *property.*

Proof. Recall $\mathrm{Aut}(\mathcal{H}_1) = \mathrm{Aut}(\mathsf{FSym}_1) \cong \mathsf{Sym}_1$. Each automorphism ϕ is given by $\phi = \mu(\gamma)$ for some $\gamma \in \mathsf{Sym}_1$. Consider the orbits of $\mathrm{supp}(\gamma)$ to form a partition of $\mathrm{supp}(\gamma)$. Observe that γ restricts to a cycle on each orbit. Thus we see that a cycle decomposition of γ is well defined and so γ can be expressed as a product of commuting cycles. If γ has an infinite orbit then it contains an infinite cycle γ_1 so that γ can be written as

$$\gamma = \gamma_1\gamma_2. \tag{7}$$

We have a bijection $R[\mu(\gamma_1)] \leftrightarrow R[\mu(\gamma)]$ from Remark 2. By Lemma 5, we know that $R[\mu(\gamma_1)] = \infty$, and hence $R[\mu(\gamma)] = \infty$. If all orbits of γ are finite then we can express γ as a product (7) with a finite cycle γ_1. From Lemma 6, we see that $R[\mu(\gamma_1)] = \infty$, and so $R[\mu(\gamma)] = \infty$ due to the same bijection as above. We have proved that $R[\phi] = \infty$ for all automorphisms of $\mathcal{H}_1$. □

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Received by the editors: 15.05.2017.