Algebra and Discrete Mathematics
Volume **19** (2015). Number 2, pp. 193–199

RESEARCH ARTICLE

On one-sided interval edge colorings of biregular bipartite graphs

Rafayel Ruben Kamalian

Communicated by V. Mazorchuk

Abstract. A proper edge t-coloring of a graph G is a coloring of edges of G with colors $1, 2, \dots, t$ such that all colors are used, and no two adjacent edges receive the same color. The set of colors of edges incident with a vertex x is called a spectrum of x. Any nonempty subset of consecutive integers is called an interval. A proper edge t-coloring of a graph G is interval in the vertex x if the spectrum of x is an interval. A proper edge t-coloring φ of a graph G is interval on a subset R_0 of vertices of G, if for any $x \in R_0$, φ is interval in x. A subset R of vertices of G has an i-property if there is a proper edge t-coloring of G which is interval on R. If G is a graph, and a subset R of its vertices has an i-property, then the minimum value of t for which there is a proper edge t-coloring of G interval on R is denoted by $w_R(G)$. We estimate the value of this parameter for biregular bipartite graphs in the case when R is one of the sides of a bipartition of the graph.

We consider undirected, finite graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For any vertex $x \in V(G)$, we denote by $N_G(x)$ the set of vertices of a graph G adjacent to x. The degree of a vertex x of a graph G is denoted by $d_G(x)$, the maximum degree of a vertex of G by $\Delta(G)$. For a graph G and an arbitrary subset $V_0 \subseteq V(G)$, we denote by $G[V_0]$ the subgraph of G induced by the subset V_0 of its vertices.

2010 MSC: 05C15, 05C50, 05C85.

Key words and phrases: proper edge coloring, interval edge coloring, interval spectrum, biregular bipartite graph.

Using a notation $G(X, Y, E)$ for a bipartite graph G, we mean that G has a bipartition (X, Y) with the sides X, Y, and $E = E(G)$.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$.

A function $\varphi : E(G) \to [1, t]$ is called a proper edge t-coloring of a graph G, if all colors are used, and no two adjacent edges receive the same color.

The minimum $t \in \mathbb{N}$ for which there exists a proper edge t-coloring of a graph G is denoted by $\chi'(G)$ [26].

For a graph G and any $t \in [\chi'(G), |E(G)|]$, we denote by $\alpha(G, t)$ the set of all proper edge t-colorings of G. Let

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G, t).$$

If G is a graph, $x \in V(G)$, $\varphi \in \alpha(G)$, then let us set $S_G(x, \varphi) \equiv \{\varphi(e)/e \in E(G), e \text{ is incident with } x\}$.

We say that $\varphi \in \alpha(G)$ is persistent-interval in the vertex $x_0 \in V(G)$ of the graph G iff $S_G(x_0, \varphi) = [1, d_G(x_0)]$. We say that $\varphi \in \alpha(G)$ is persistent-interval on the set $R_0 \subseteq V(G)$ iff φ is persistent-interval in $\forall x \in R_0$.

We say that $\varphi \in \alpha(G)$ is interval in the vertex $x_0 \in V(G)$ of the graph G iff $S_G(x_0, \varphi)$ is an interval. We say that $\varphi \in \alpha(G)$ is interval on the set $R_0 \subseteq V(G)$ iff φ is interval in $\forall x \in R_0$.

We say that a subset R of vertices of a graph G has an i-property iff there exists $\varphi \in \alpha(G)$ interval on R; for a subset $R \subseteq V(G)$ with an i-property, the minimum value of t warranting existence of $\varphi \in \alpha(G, t)$ interval on R is denoted by $w_R(G)$.

Notice that the problem of deciding whether the set of all vertices of an arbitrary graph has an i-property is NP-complete [7, 8, 17]. Unfortunately, even for an arbitrary bipartite graph (in this case the interest is strengthened owing to the application of an i-property in timetablings [6, 17]) the problem keeps the complexity of a general case [3, 12, 25]. Some positive results were obtained for graphs of certain classes with numerical or structural restrictions [9, 11, 13–15, 17, 19–22, 28, 29]. The examples of bipartite graphs whose sets of vertices have not an i-property are given in [6, 13, 16, 23, 25].

The subject of this research is a parameter $w_R(G)$ of a bipartite graph $G = G(X, Y, E)$ in the case when R is one of the sides of the bipartition of G (the exact value of this parameter for an arbitrary bipartite graph is not known as yet). We obtain an upper bound of the parameter being discussed for biregular [2–5, 24] bipartite graphs, and the exact values of it in the case of the complete bipartite graph $K_{m,n}$ ($m \in \mathbb{N}, n \in \mathbb{N}$) as well.

The terms and concepts that we do not define can be found in [27].

First we recall some known results.

Theorem 1 ([7, 8, 17]). *If R is one of the sides of a bipartition of an arbitrary bipartite graph $G = G(X, Y, E)$, then: 1) there exists $\varphi \in \alpha(G, |E|)$ interval on R, 2) for $\forall t \in [w_R(G), |E|]$, there exists $\psi_t \in \alpha(G, t)$ interval on R.*

Theorem 2 ([1, 7, 8]). *Let $G = G(X, Y, E)$ be a bipartite graph. If for $\forall e = (x, y) \in E$, where $x \in X, y \in Y$, the inequality $d_G(y) \leqslant d_G(x)$ is true, then $\exists \varphi \in \alpha(G, \Delta(G))$ persistent-interval on X.*

Corollary 1 ([1, 7, 8]). *Let $G = G(X, Y, E)$ be a bipartite graph. If $\max_{y \in Y} d_G(y) \leqslant \min_{x \in X} d_G(x)$, then $\exists \varphi \in \alpha(G, \Delta(G))$ persistent-interval on X.*

Remark 1. Note that Corollary 1 follows from the result of [10].

Let $H = H(\mu, \nu)$ be a $(0, 1)$-matrix with μ rows, ν columns, and with elements h_{ij}, $1 \leqslant i \leqslant \mu$, $1 \leqslant j \leqslant \nu$. The i-th row of H, $i \in [1, \mu]$, is called collected, iff $h_{ip} = h_{iq} = 1$, $t \in [p, q]$ imply $h_{it} = 1$, and the inequality $\sum_{j=1}^{\nu} h_{ij} \geqslant 1$ is true. Similarly, the j-th column of H, $j \in [1, \nu]$, is called collected, iff $h_{pj} = h_{qj} = 1$, $t \in [p, q]$ imply $h_{tj} = 1$, and the inequality $\sum_{i=1}^{\mu} h_{ij} \geqslant 1$ is true. If all rows and all columns of H are collected, then for i-th row of H, $i \in [1, \mu]$, we define the number $\varepsilon(i, H) \equiv \min\{j / h_{ij} = 1\}$.

H is called a collected matrix (see Figure 1), iff all its rows and all its columns are collected, $h_{11} = h_{\mu\nu} = 1$, and $\varepsilon(1, H) \leqslant \varepsilon(2, H) \leqslant \cdots \leqslant \varepsilon(\mu, H)$.

H is called a b-regular matrix ($b \in \mathbb{N}$), iff for $\forall i \in [1, \mu]$, $\sum_{j=1}^{\nu} h_{ij} = b$. H is called a c-compressed matrix ($c \in \mathbb{N}$), iff for $\forall j \in [1, \nu]$, $\sum_{i=1}^{\mu} h_{ij} \leqslant c$.

Lemma 1 ([18]). *If a collected n-regular ($n \in \mathbb{N}$) matrix $P = P(m, w)$ with elements p_{ij} ($1 \leqslant i \leqslant m, 1 \leqslant j \leqslant w$) is n-compressed, then $w \geqslant \lceil \frac{m}{n} \rceil \cdot n$.*

Proof. We use induction on $\lceil \frac{m}{n} \rceil$.

If $\lceil \frac{m}{n} \rceil = 1$, the statement is trivial.

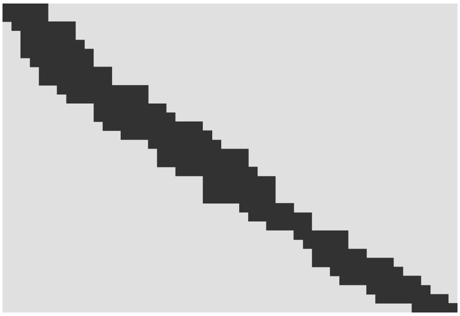

FIGURE 1. An example of the visual image of a collected matrix. The dark area is filled by 1s, the light area — by 0s.

Now assume that $\lceil \frac{m}{n} \rceil = \lambda_0 \geqslant 2$, and the statement is true for all collected n'-regular n'-compressed matrixes $P'(m', w')$ with $\lceil \frac{m'}{n'} \rceil \leqslant \lambda_0 - 1$.

First of all let us prove that $\varepsilon(n+1, P) \geqslant n+1$. Assume the contrary: $\varepsilon(n+1, P) \leqslant n$. Since P is a collected n-regular matrix, we obtain $\sum_{i=1}^{m} p_{in} \geqslant \sum_{i=1}^{n+1} p_{in} \geqslant n+1$, which is impossible because $P(m, w)$ is an n-compressed matrix. This contradiction shows that $\varepsilon(n+1, P) \geqslant n+1$.

Now let us form a new matrix $P'(m-n, w-(\varepsilon(n+1, P)-1))$ by deleting from the matrix P the elements p_{ij}, which satisfy at least one of the inequalities $i \leqslant n$, $j \leqslant \varepsilon(n+1, P) - 1$.

It is not difficult to see that $P'(m-n, w-(\varepsilon(n+1, P)-1))$ is a collected n-regular n-compressed matrix with $\lceil \frac{m-n}{n} \rceil = \lambda_0 - 1$. By the induction hypothesis, we have

$$w - (\varepsilon(n+1, P) - 1) \geqslant \left\lceil \frac{m-n}{n} \right\rceil \cdot n,$$

which means that

$$w \geqslant (\lambda_0 - 1)n + \varepsilon(n+1, P) - 1 \geqslant (\lambda_0 - 1)n + n = \lambda_0 n = \left\lceil \frac{m}{n} \right\rceil \cdot n. \quad \square$$

Now, for arbitrary positive integers m, l, n, k, where $m \geqslant n$ and $ml = nk$, let us define the class $Bip(m, l, n, k)$ of biregular bipartite graphs:

$$Bip(m, l, n, k) \equiv \left\{ G = G(X, Y, E) \;\middle|\; \begin{array}{l} |X| = m, |Y| = n, \\ \text{for } \forall x \in X, d_G(x) = l, \\ \text{for } \forall y \in Y, d_G(y) = k. \end{array} \right\}$$

Remark 2. Clearly, if $G \in Bip(m, l, n, k)$, then $\chi'(G) = k$.

Theorem 3. *If* $G = G(X,Y,E) \in Bip(m,l,n,k)$*, then* $w_Y(G) = k$*,* $w_X(G) \leqslant l \cdot \lceil \frac{m}{l} \rceil$.

Proof. The equality follows from Remark 2. Let us prove the inequality.

Let $X = \{x_1, \dots, x_m\}$. For $\forall r \in [1, \lfloor \frac{m}{l} \rfloor]$, define $X_r \equiv \{x_{(r-1)l+1}, \dots, x_{rl}\}$. Define $X_{1+\lfloor \frac{m}{l} \rfloor} \equiv X \backslash \left(\bigcup_{i=1}^{\lfloor \frac{m}{l} \rfloor} X_i \right)$. For $\forall r \in [1, \lfloor \frac{m}{l} \rfloor]$, define $Y_r \equiv \bigcup_{x \in X_r} N_G(x)$. Define $Y_{1+\lfloor \frac{m}{l} \rfloor} \equiv \bigcup_{x \in X_{1+\lfloor \frac{m}{l} \rfloor}} N_G(x)$. For $\forall r \in [1, \lceil \frac{m}{l} \rceil]$, define $G_r \equiv G[X_r \cup Y_r]$.

Consider the sequence $G_1, G_2, \dots, G_{\lceil \frac{m}{l} \rceil}$ of subgraphs of the graph G. From Corollary 1, we obtain that for $\forall i \in [1, \lceil \frac{m}{l} \rceil]$, there is $\varphi_i \in \alpha(G_i, l)$ persistent-interval on X_i.

Clearly, for $\forall e \in E(G)$, there exists the unique $\xi(e)$, satisfying the conditions $\xi(e) \in [1, \lceil \frac{m}{l} \rceil]$ and $e \in E(G_{\xi(e)})$.

Define a function $\psi : E(G) \to [1, l \cdot \lceil \frac{m}{l} \rceil]$. For an arbitrary $e \in E(G)$, set $\psi(e) \equiv (\xi(e) - 1) \cdot l + \varphi_{\xi(e)}(e)$.

It is not difficult to see that $\psi \in \alpha(G, l \cdot \lceil \frac{m}{l} \rceil)$ and ψ is interval on X. Hence, $w_X(G) \leqslant l \cdot \lceil \frac{m}{l} \rceil$. □

Theorem 4. *Let* R *be an arbitrary side of a bipartition of the complete bipartite graph* $G = K_{m,n}$*, where* $m \in \mathbb{N}$*,* $n \in \mathbb{N}$*. Then*

$$w_R(G) = (m + n - |R|) \cdot \left\lceil \frac{|R|}{m + n - |R|} \right\rceil.$$

Proof. Without loss of generality we can assume that G has a bipartition (X, Y), where $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_n\}$, and $m \geqslant n$.

Case 1. $R = Y$. In this case the statement follows from Theorem 3; thus $w_Y(G) = m$.

Case 2. $R = X$.

The inequality $w_X(G) \leqslant n \cdot \lceil \frac{m}{n} \rceil$ follows from Theorem 3. Let us prove that $w_X(G) \geqslant n \cdot \lceil \frac{m}{n} \rceil$.

Consider an arbitrary proper edge $w_X(G)$-coloring φ of the graph G, which is interval on X.

Clearly, without loss of generality, we can assume that

$$\min(S_G(x_1, \varphi)) \leqslant \min(S_G(x_2, \varphi)) \leqslant \ldots \leqslant \min(S_G(x_m, \varphi)).$$

Let us define a $(0,1)$-matrix $P(m, w_X(G))$ with m rows, $w_X(G)$ columns, and with elements p_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant w_X(G)$. For $\forall i \in [1, m]$, and for $\forall j \in [1, w_X(G)]$, set

$$p_{ij} = \begin{cases} 1, & \text{if } j \in S_G(x_i, \varphi) \\ 0, & \text{if } j \notin S_G(x_i, \varphi). \end{cases}$$

It is not difficult to see that $P(m, w_X(G))$ is a collected n-regular n-compressed matrix. From Lemma 1, we obtain $w_X(G) \geqslant n \cdot \lceil \frac{m}{n} \rceil$. □

From Theorems 1 and 3, taking into account the proof of Case 2 of Theorem 4, we also obtain

Corollary 2. *If $G \in Bip(m, l, n, k)$, then*

1) *for* $\forall t \in \left[l \cdot \lceil \frac{m}{l} \rceil, ml\right]$, *there exists* $\varphi_t \in \alpha(G, t)$ *interval on* X,

2) *for* $\forall t \in [k, nk]$, *there exists* $\psi_t \in \alpha(G, t)$ *interval on* Y.

References

[1] A.S. Asratian, *Investigation of some mathematical model of Scheduling Theory*, Doctoral Dissertation, Moscow University, 1980 (in Russian).

[2] A.S. Asratian, C.J. Casselgren, *A sufficient condition for interval edge colorings of* $(4, 3)$*-biregular bipartite graphs*, Research report LiTH-MAT-R-2006-07, Linköping University, 2006.

[3] A.S. Asratian, C.J. Casselgren, *Some results on interval edge colorings of* (α, β)*-biregular bipartite graphs*, Research report LiTH-MAT-R-2006-09, Linköping University, 2006.

[4] A.S. Asratian, C.J. Casselgren, *On interval edge colorings of* (α, β)*-biregular bipartite graphs*, Discrete Math 307 (2007), pp.1951-1956.

[5] A.S. Asratian, C.J. Casselgren, J. Vandenbussche, D.B. West, *Proper path-factors and interval edge-coloring of* $(3, 4)$*-biregular bigraphs*, J. of Graph Theory 61 (2009), pp.88-97.

[6] A.S. Asratian, T.M.J. Denley, R. Haggkvist, *Bipartite graphs and their applications*, Cambridge Tracts in Mathematics, 131, Cambridge University Press, 1998.

[7] A.S. Asratian, R.R. Kamalian, *Interval colorings of edges of a multigraph*, Appl. Math. 5 (1987), Yerevan State University, pp.25-34 (in Russian).

[8] A.S. Asratian, R.R. Kamalian, *Investigation of interval edge-colorings of graphs*, Journal of Combinatorial Theory. Series B 62 (1994), N.**1**, pp.34-43.

[9] M.A. Axenovich, *On interval colorings of planar graphs*, Congr. Numer. 159 (2002), pp.77-94.

[10] D.P. Geller and A.J.W. Hilton, *How to color the lines of a bigraph*, Networks, 4(1974), pp.281-282.

[11] K. Giaro, *Compact task scheduling on dedicated processors with no waiting periods*, PhD thesis, Technical University of Gdansk, EIT faculty, Gdansk, 1999 (in Polish).

[12] K. Giaro, *The complexity of consecutive* Δ*-coloring of bipartite graphs:* 4 *is easy,* 5 *is hard*, Ars Combin. 47(1997), pp.287-298.

[13] K. Giaro, M. Kubale and M. Malafiejski, *On the deficiency of bipartite graphs*, Discrete Appl. Math. 94 (1999), pp.193-203.

[14] H.M. Hansen, *Scheduling with minimum waiting periods*, Master's Thesis, Odense University, Odense, Denmark, 1992 (in Danish).

[15] D. Hanson, C.O.M. Loten, B. Toft, *On interval colorings of bi-regular bipartite graphs*, Ars Combin. 50(1998), pp.23-32.

[16] T.R. Jensen, B. Toft, *Graph Coloring Problems*, Wiley Interscience Series in Discrete Mathematics and Optimization, 1995.

[17] R.R. Kamalian, *Interval Edge Colorings of Graphs*, Doctoral dissertation, the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, 1990 (in Russian).

[18] R.R. Kamalian, *On one-sided interval colorings of bipartite graphs*, the Herald of the RAU, N.**2**, Yerevan, 2010, pp.3-11 (in Russian).

[19] R.R. Kamalian, *Interval colorings of complete bipartite graphs and trees*, Preprint of the Computing Centre of the Academy of Sciences of Armenia, Yerevan, 1989 (in Russian).

[20] M. Kubale, *Graph Colorings*, American Mathematical Society, 2004.

[21] P.A. Petrosyan, *Interval edge-colorings of complete graphs and n-dimensional cubes*, Discrete Math. 310 (2010), pp.1580-1587.

[22] P.A. Petrosyan, *On interval edge-colorings of multigraphs*, The Herald of the RAU, N.**1**, Yerevan, 2011, pp.12-21 (in Russian).

[23] P.A. Petrosyan, H.H. Khachatrian, *Interval non-edge-colorable bipartite graphs and multigraphs*, J. of Graph Theory 76 (2014), pp.200-216.

[24] A.V. Pyatkin, *Interval coloring of $(3,4)$-biregular bipartite graphs having large cubic subgraphs*, J. of Graph Theory 47 (2004), pp.122-128.

[25] S.V. Sevast'janov, *Interval colorability of the edges of a bipartite graph*, Metody Diskret. Analiza 50(1990), pp.61-72 (in Russian).

[26] V.G. Vizing, *The chromatic index of a multigraph*, Kibernetika 3 (1965), pp.29-39.

[27] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, New Jersey, 1996.

[28] F. Yang, X. Li, *Interval coloring of $(3,4)$-biregular bigraphs having two $(2,3)$-biregular bipartite subgraphs*, Appl. Math. Letters 24(2011), pp.1574-1577.

[29] Y. Zhao and J.G. Chang, *Consecutive Edge-Colorings of Generalized θ-Graphs*, J. Akiyama et al. (Eds.): CGGA 2010, LNCS 7033, 2011, pp.214-225.

Contact information

R. R. Kamalian Institute for Informatics and Automation Problems of the National Academy of Sciences of RA, 0014 Yerevan, Republic of Armenia
E-Mail(s): rrkamalian@yahoo.com

Received by the editors: 17.12.2012
and in final form 10.02.2015.