Algebra and Discrete Mathematics
Volume **23** (2017). Number 1, pp. 35–46
© Journal "Algebra and Discrete Mathematics"
SURVEY ARTICLE

Galois orders of symmetric differential operators

Vyacheslav Futorny* and João Schwarz**

Dedicated to the memory of Sergey Ovsienko

Abstract. In this survey we discuss the theory of Galois rings and orders developed in ([20], [22]) by Sergey Ovsienko and the first author. This concept allows to unify the representation theories of Generalized Weyl Algebras ([4]) and of the universal enveloping algebras of Lie algebras. It also had an impact on the structure theory of algebras. In particular, this abstract framework has provided a new proof of the Gelfand-Kirillov Conjecture ([24]) in the classical and the quantum case for gl_n and sl_n in [18] and [21], respectively. We will give a detailed proof of the Gelfand-Kirillov Conjecture in the classical case and show that the algebra of symmetric differential operators has a structure of a Galois order.

1. Motivation

Throughout the paper $\boldsymbol{k}$ will denote an algebraically closed field of zero characteristic. All considered rings are algebras over $\boldsymbol{k}$. In representation theory one often considers the following question: given an embedding of algebras $\Gamma \subseteq U$, relate representations of U and Γ. The functors of restriction and induction are very powerful tools in this study. In particular, in the representation theory of Lie algebras a concept of a Harish-Chandra module relates the universal enveloping algebra of

*Supported in part by CNPq grant (301320/2013-6) and by Fapesp grant (2014/09310-5)
**Supported in part by Fapesp grant (2014/25612-1)
2010 MSC: 13N10, 16D30, 16S32, 16S85.
Key words and phrases: Weyl algebra, invariant differential operators, Galois order, filed of fractions.

a reductive Lie algebra U and the universal enveloping algebra of its reductive subalgebra Γ [9]. On the other hand, when Γ is the universal enveloping algebra of a Cartan subalgebra, one obtains the so called generalized weight representations. The classification of irreducible weight modules whose weight spaces are finite dimensional was done in [13] and [31]. The problem remains open in general. To approach this problem, Drozd, Futorny and Ovsienko introduced the category of Gelfand-Tsetlin modules over $U(gl_n)$ with respect to the Gelfand-Tsetlin subalgebra (a certain maximal commutative subalgebra) ([10]). This approach was inspired by a remarkable paper of Gelfand and Tsetlin ([24]) which gave a construction of irreducible representations of gl_n using as a basis a combinatorial object — Gelfand-Tsetlin tableaux, very much in the spirit of the representation theory of the symmetric groups [26]. A similar idea was used by Okunkov and Vershik in [37]. Using the natural embedding of S_{m-1} in S_m one introduces a subalgebra analogous to the Gelfand-Tsetling subalgebra in the gl_n case. Namely, in this case U is $\boldsymbol{k}S_n$ and Γ is the maximal commutative subalgebra generated by the Jucys-Murphy elements:

$$(1i) + \ldots + (i-1\,i)\; i = 1, \ldots, n.$$

Then $Specm\,\Gamma$ parametrizes the irreducible representations of S_n, and the Young tableaux can be recovered.

For an account of the recent research in this area, including generalizations for Lie algebras of types B, C and D, see [34]. An excellent exposition of the classical material can be found in [41].

To understand better the phenomena of the Gelfand-Tsetlin formulas, the notion of an astract Harish-Chandra subalgebra and Harish-Chandra module were introduced for an arbitrary associative algebras in [11]. In [20] it was noticed that using the Gelfand-Tsetlin formulas one can embed $U(gl_n)$ into the skew group ring over a field L (a similar construction was also done by Khomenko [28]), where L is a finite Galois extension of the field of fractions of the Gelfand-Tsetlin subalgebra.

The appearence of skew group rings is also a phenomenon in the representation theory of another class of algebras - the Generalized Weyl Algebras [4]. In particular cases of the first Weyl Algebra and $U(sl_2)$ (see [6]), and their quantum analogues, it is known that their irreducible modules are completely described modulo a classification of irreducible elements in certain skew polynomial rings in one variable over a skew field.

The main motivation of the development of this theory was an evolution of the ideas in [11] in the “semi-commutative” case for a pair of an associative algebra and its commutative subalgebra, and understanding of the role of skew group rings in the representation theory of infinite dimensional algebras. A key concept introduced in [20] is a notion of a noncommutative Galois order for skew monoid rings (cf. [33], Chapter 5). Known examples of Galois algebras include:

- Generalized Weyl algebras over integral domains with infinite order automorphisms, which include algebras,such as the n-th Weyl algebra A_n, the quantum plane, the q-deformed Heisenberg algebra, quantized Weyl algebras, the Witten—Woronowicz algebra among the others;
- The universal enveloping algebra $U(gl_n)$ with respect to its Gelfand—Tsetlin subalgebra.
- It was shown in [16], [19] that shifted Yangians and finite W - algebras associated with gl_n are Galois orders with respect to the corresponding Gelfand—Tsetlin subalgebras;
- Certain invariant rings on the differential operators on the torus [20].

Representation theory of Galois orders was developed in [22]. In the case of gl_n the Galois order structure of the universal enveloping algebra led to a significant breakthrough in its representation theory in the remarkable paper [38].

2. Basic definitions

Let R be a ring, $\mathfrak{M}$ a monoid acting on R by ring automorphisms. Consider the skew monoid ring $R * \mathfrak{M}$. Let G be a finite group acting on $\mathfrak{M}$ by conjugation:. We can define an action of G on $R * \mathfrak{M}$ as $g(rm) = g(r)g(m)$, $g \in G, r \in R$, $m \in \mathfrak{M}$. We denote the ring of invariants by the action of G by $\mathfrak{K}$.

Any element of $R * \mathfrak{M}$ can be written in the form $x = \sum_{m \in \mathfrak{M}} x_m m$. Define $supp\, x$ as the set of those $m \in \mathfrak{M}$ for which x_m is not zero.

From now on we will restrict ourselves to the following case: R will be a field L, a finite Galois extension of a field K such that $G = Gal(L, K)$. The monoid $\mathfrak{M}$ will be assumed to have the following property: if $m, m' \in \mathfrak{M}$ and their restrictions to K coincide, then $m = m'$.

Definition. A finitely generated Γ-ring U embedded in $\mathfrak{K}$ is called a Galois ring over Γ if $KU = KU = \mathfrak{K}$.

Note that Γ is not required to be central in U.

3. Structure of Galois rings

In this section we recall the structure theory of Galois rings following [20]. A very useful characterization of Galois rings in the following

Proposition 1. *[[20], Proposition 4.1] Assume that a* Γ*-ring* $U \subset \mathfrak{K} = (L * \mathfrak{M})^G$ *is generated by* $u_1, \dots, u_k$*. If* $\bigcup_{i=1}^k \operatorname{supp} u_i$ *generates* $\mathfrak{M}$ *as a monoid then* U *is a Galois ring. In particular, if* $LU = L * \mathfrak{M}$ *then* U *is a Galois ring.*

Theorem 1 ([20], Theorem 4.1)**.** *Let* U *be a Galois ring over* Γ *in* $\mathfrak{K}$*,* $S = \Gamma \setminus \{0\}$*. Then*

- $U \cap K$ *is a maximal commutative subalgebra in* U *and the center of* $U \cap K^{\mathfrak{K}}$*.*
- S *is a left and right Ore denominator set, and the localization of* U *by* S *both at the left and the right are isomorphic to* $\mathfrak{K}$*.*

Definition. A Galois ring is called a right (left) Galois order over Γ if for every right (left) finite dimensional K vector space $W \subset \mathfrak{K}$, $W \cap \Gamma$ is a finitely generated right (left) Γ-module. If it is both left and right, we will simply say Galois order.

We have the following caracterization of Galois orders.

Proposition 2. *Let* U *is a Galois ring over* Γ*.*

- *If* Γ *is noetherian and* U *a left (right) projective* Γ*-module then* U *is a left (right) Galois order.*
- *If* Γ *is a finitely generated domain over* $\boldsymbol{k}$ *and* U *a Galois order over* Γ *then* Γ *is a Harish-Chandra algebra in* U*.*

4. Noncommutative Noether's Problem and the Gelfand-Kirillov Conjecture

In this section we show how the theory of Galois rings can be used to prove the Gelfand-Kirillov Conjecture for gl_n. The noncommutative version of the the classical Noether's problem will also be required.

Definition. Let V be a finite dimensional vector space, of dimension n over $\boldsymbol{k}$, G a finite subgroup of $GL(V)$. It acts on $S(V^*)$ by $\boldsymbol{k}$-algebra automorphisms: $g.f(v) = f(g^{-1}v)$, $g \in G, f \in S(V^*), v \in V$. After fixing a basis of V, $S(V^*)$ can be identified with $\boldsymbol{k}[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ are the duals of the basis elements in V^*. Automorphisms of the polynomial algebra arising this way will be called *linear*.

Hence the group G acts also on the field of rational functions $K = \boldsymbol{k}(x_1, \ldots, x_n)$ by extension. Then one can ask:

Noether's Problem ([36])**.** If G is a finite group of linear automorphisms, when K^G is a purely transcendental extension of $\boldsymbol{k}$?

The following are some important cases when the Noether's Problem has a positive solution:

- $n = 1, n = 2$ or $n = 3$ (these are classical results due to Luroth, Castelnuovo and Burnside).
- When V is a direct sum of one dimensional G-submodules. In particular, for abelian G (Theorem of Fischer).
- The action of G by pseudo-reflections (by the Chevalley-Shephard-Todd Theorem)
- for alternating groups A_3, A_4 and A_5 (by Maeda), permuting variables as usual. The question remains open for $n > 5$.

There are also counter-examples to the Noether's Problem, cf. [40], [15].

We will introduce now the Noncommutative Noether's Problem for the Weyl algebra, $A_n(\boldsymbol{k})$ with generators $x_i, \partial_i, i = 1, \ldots, n$, subject to the relations $x_i x_j = x_j x_i$, $\partial_i \partial_j = \partial_j \partial_i$ and $\partial_i x_j - x_j \partial_i = \delta_{ij}$ for all i, j. Recall that $A_n(\boldsymbol{k})$ is a left and right noetherian simple domain which admits a total ring of fractions (skew field), $F_n(\boldsymbol{k})$, called the *Weyl field*. For our purposes it will be useful to identify the Weyl algebra with the ring of differential operators on the polynomial algebra in n variables.

Let A be a finitely generated commutative, regular $\boldsymbol{k}$-algebra. Then the ring of differential operators $D(A)$ on A is the subalgebra of $End_{\boldsymbol{k}}(A)$ generated by the $\boldsymbol{k}$-linear derivations of A and the scalar multiplications l_a that sends $x \to ax$, $\forall a \in A$. The set of multiplications gives an isomorphism of A with a subring of $D(A)$, allowing A to be viewed as a subring of it.

For our purposes it will be useful to identify the Weyl algebra with the ring of differential operators on the polynomial algebra in n variables. Let V a finite dimensional vector space of dimension n, G be a finite subgroup of $GL(V)$. As previously, this induces an action on $S(V^*) = \boldsymbol{k}[x_1, \ldots, x_n]$. This action can be extended to the ring of differential operators on $S(V^*)$: if d is such an operator, $g.d(x) = g(d(g^{-1}x))$, where $x \in S(V^*)$. This induces a $\boldsymbol{k}$-automorphism of the Weyl algebra $A_n(\boldsymbol{k})$. Such $\boldsymbol{k}$-automorphisms will be called linear.

The following Noncommutative Noether's Problem was formulated by Alev and Dumas, [3].

Noncommutative Noether's Problem. For a finite group of linear automorphisms G, when $F_n(\boldsymbol{k})^G$ is isomorphic to $F_n(\boldsymbol{k})$?

Some cases with known positive solution for the Noncommutative Noether's Problem are:

- For $n = 1$ or $n = 2$ and arbitrary G (Alev, Dumas, [3]).
- When V is a direct sum of one dimensional G-submodules (Alev, Dumas, [3]).
- When G acts by pseudo-reflections (Eshmatov, Futorny, Ovsienko, Schwarz, [12]).

Positive solution of the Noncommutative Noether's Problem in the context of the structure theory of Galois rings provides a new proof of the celebrated Gelfand-Kirillov conjecture for the gl_n and sl_n cases.

The Gelfand-Kirillov conjecture [24] states that if g be a finite dimensional algebraric Lie algebra then the skew field of fractions of the universal enveloping algebra $U(g)$ is isomorphic to a Weyl field over a purely transcendental extension (of finite transcendence degree) of $\boldsymbol{k}$. The important cases with a positive solution are:

- $g = gl_n, sl_n$ or nilpotent [24];
- g is solvable [5], [27], [32];
- g has dimension at most 8 [2].

The first counter-example to this conjecture was found by Alev, Ooms, Van den Bergh in [1]. For simple finite dimensional Lie algebras the question was almost solved by Premet [39]: the conjecture is true for algebras of type A and G_2, unkown for type C and false for all other types.

We are going to present two proofs of the Noncommutative Noether's Problem in the case of the symmetric group. One of them is a simplified version of the proof found in [16] and [12], while the other is elementary — it involves only the Cramer's rule.

Let $\Delta = (\prod_{i<j}(x_i - x_j))^2$. It is clearly an S_n-invariant element and $F_n(\boldsymbol{k}) = Frac\, A_n(\boldsymbol{k})_\Delta$, the skew field of fractions of the localized algebra by Δ. In the following we denote the polynomial algebra in n variables just by Λ for the sake of simplicity. The following holds:

Proposition 3.

- *Let S be any multiplicatively closed set in Λ. Then $D(\Lambda_S) = (D(\Lambda))_S$.*
- $(D(\Lambda)_\Delta)^{S_n} \cong ((D(\Lambda))^{S_n})_\Delta$.
- $(\Lambda_\Delta)^{S_n} = (\Lambda^{S_n})_\Delta$.
- $Frac\, A_n(\boldsymbol{k})^{S_n} \cong (Frac\, A_n(\boldsymbol{k}))^{S_n}$.

Proof. The first item follows from Theorem 15.1.25 of [33]. For the second statement note that if $d \in (D(\Lambda)_\Delta)^{S_n}$, then $d_1 = \Delta^k d \in D(\Lambda)^{S_n}$ for some $k \geqslant 0$. The third item is proved similarly. The fourth statement follows from [35], Theorem 5.3(4). □

Now we need the following crucial lemma:

Lemma 1. $(D(\Lambda_\Delta))^{S_n} = D(\Lambda_\Delta^{S_n})$.

Proof. First we follow [12]. Recall that if X is a normal irreducible affine variety and G a finite group of automorphisms that acts freely on X then $D(X)^G \cong D(O(X/G))$ (cf. [8]). This applies to S_n acting on Λ_Δ, and hence the lemma follows.

Now we show how to obtain this result algebraically. Note that S_n has no non-trivial inner automorphisms. Therefore, $A_n(\boldsymbol{k})_\Delta^{S_n}$ is simple by [35], Corollary 2.6.

Let σ_i be the i-th symmetrical polynomial in $x_1, \ldots, x_n$, $i = 1, \ldots, n$, $\Lambda^{S_n} = \boldsymbol{k}[\sigma_1, \ldots, \sigma_n] \subset \Lambda$. Let M be the $n \times n$ matrix whose ij's entry is $\partial_j(\sigma_i)$, and let J be it's determinant.

Claim. $J = \prod_{i<j}(x_i - x_j)$.

Indeed, J has degree $n(n-1)/2$. Clearly, $\prod_{i<j}(x_i - x_j)$ divides J. Since both have the same degree we have $J = a\prod_{i<j}(x_i - x_j)$ for some scalar a. Note that in both polynomials the monomial $x_1^n x_2^{n-1} \ldots x_n$ appears with coefficient 1. So $a = 1$.

Let $d \in D(\Lambda_\Delta)^{S_n}$, and $f \in \Lambda_\Delta^{S_n}$. For all $\pi \in S_n$, $\pi(d(f)) = (\pi.d)(\pi f) = d(f)$, that is, $d(f)$ also belongs to $\Lambda_\Delta^{S_n}$. In this way, by restricting the domain, we have a ring homomorphism $\phi : D(\Lambda_\Delta)^{S_n} \to D(\Lambda_\Delta^{S_n})$. We need is to show it is an isomorphism. The injectivity follows from the simplicity of $D(\Lambda_\Delta)^{S_n}$, as shown above. We discuss the surjectivity of ϕ. The ring $D(\Lambda_\Delta^{S_n})$ is generated over $\Lambda_\Delta^{S_n}$ by $\partial'_1, \ldots, \partial'_n$ such that $\partial'_i(\sigma_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. Hence, it is enough to construct S_n-invariant differential operators $d_1, \ldots, d_n : \Lambda_\Delta \to \Lambda_\Delta$, whose restriction onto $\Lambda_\Delta^{S_n}$ coincide with $\partial'_1, \ldots, \partial'_n$ above.

Let

$$E_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

be a vector of size n, with 1 in the position i and 0 elsewhere, and let

$$F_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in} \end{pmatrix}$$

be a solution of the system $MF_i = E_i$. By the Kramer rule, $f_{ij} \in \Lambda_\Delta$, $1 \leqslant i, j \leqslant n$.

Let $d_i = \sum_{k=1}^n f_{ik}\partial_k$. We have $d_i(\sigma_j) = \delta_{ij}$, and that $d_i \in D(\Lambda_\Delta) = D(\Lambda)_\Delta$. What is left is to show that d_i is S_n-invariant.

It is sufficient to show that for any $\pi \in S_n$ we have $\pi f_{ij} = f_{i\pi(j)}$ for $1 \leqslant i, j \leqslant n$, since $\pi(\partial_i) = \partial_{\pi(i)}$. We shall use the Kramer's rule. Let v_i be the vector

$$\begin{pmatrix} \partial_i(\sigma_1) \\ \vdots \\ \partial_i(\sigma_n) \end{pmatrix}.$$

It is clear that $\pi(v_i) = v_{\pi(i)}$ and

$$f_{ij} = \frac{\det(v_1, \ldots, E_i, \ldots, v_n)}{\det(v_1, \ldots, v_n)},$$

with E_i in the j's position.

Then

$$\pi f_{ij} = \frac{\det(v_{\pi(1)}, \ldots, E_i, \ldots, v_{\pi(n)})}{\det(v_{\pi(1)}, \ldots, v_{\pi(n)})}$$

$$= \operatorname{sign}(\pi)\det(v_1, \ldots, E_i, \ldots, v_n)/\operatorname{sign}(\pi)\det(v_1, \ldots, v_n),$$

now with E_i in the position $\pi(j)$. This clearly equals $f_{i\pi(j)}$. □

Now we are in the position to prove the Gelfand-Kirillov conjecture.

Proof of the Gelfand-Kirillov conjecture. The Galois ring structure of the universal enveloping algebra $U(gl_n)$ over the Gelfand-Tsetlin subalgebra Γ implies the embedding of $U(gl_n)$ into the tensor product

$$\mathcal{A}_1^{S_1} \otimes \mathcal{A}_2^{S_2} \otimes \ldots \otimes \mathcal{A}_{n-1}^{S_{n-1}} \otimes \Bbbk[t_1, \ldots, t_n]^{S_n},$$

where $\mathcal{A}_k$ is a certain localization of the k-th Weyl algebra A_k. Since $(F_k)^{S_k} \simeq Frac(A_k^{S_k}) \simeq F_k$ by the Noether's Problem, we have that

$$Frac(U(gl_n)) \simeq F_1 \otimes \ldots \otimes F_{n-1} \otimes \boldsymbol{k}(y_1, \ldots, y_k) \simeq F_{\frac{(n(n-1)}{2}} \otimes \boldsymbol{k}(y_1, \ldots, y_k),$$

that $U(gl_n)$ is birationally equivalent to A_m over $\Bbbk(y_1, \ldots, y_n)$, $m = n(n-1)/2$ (see [18], Proposition 5.2 for details). □

5. Symmetric differential operators

Let $t_i = \partial_i x_i \in A_n(\boldsymbol{k})$, $i = 1, \ldots, n$. It is well known that $\boldsymbol{k}[t_1, \ldots, t_n]$ is a maximal commutative algebra of $A_n(\boldsymbol{k})$, and $A_n(\boldsymbol{k})$ is a free left and right module over $\boldsymbol{k}[t_1, \ldots, t_n]$ (which can be seen, for example, using the theorem from [17]). In this section we construct a new example of a Galois order given by the algebra of symmetric differential operators. Set $\Gamma = k[x_1, \ldots, x_n]^{S_n}$.

Theorem 2. Γ *is a Harish-Chandra subalgebra of* $A_n(\boldsymbol{k})^{S_n}$ *and* $A_n(\boldsymbol{k})^{S_n}$ *is a Galois order over* Γ.

Proof. By the result of Levasseur and Stafford ([29], Theorem 5) we have that $A_n(\boldsymbol{k})^{S_n}$ is generated as an algebra by $\boldsymbol{k}[x_1, \ldots, x_n]^{S_n}$ and $\boldsymbol{k}[\partial_1, \ldots, \partial_n]^{S_n}$. Denote $K = Frac\,\Gamma$ and $L = Frac\,\boldsymbol{k}[x_1, \ldots, x_n]$. Let $\mathbb{Z}^n$ be generated by $\delta_1, \ldots, \delta_n$, acting on L in the following way: $\delta_i(t_j) = t_j - \delta_{ij}$. Consider an action of S_n on $\mathbb{Z}^n$ by conjugation, and set $\mathfrak{K} = (L * \mathbb{Z}^n)^{S_n}$. Recall that $A_n(\boldsymbol{k})^{S_n}$ is simple. Hence we have an embedding

$$A_n(\boldsymbol{k})^{S_n} \to \mathfrak{K}$$

induced by the homomorphism $A_n(\boldsymbol{k}) \to L * \mathbb{Z}^n$, which sends x_i to δ_i and ∂_i to $t_i \delta_i^{-1}$.

Consider the elements $x_1 + \ldots + x_n$ and $\partial_1 + \ldots + \partial_n$. Their images in $\mathfrak{K}$ have supports that generate $\mathbb{Z}^n$ as a monoid. So, by Proposition 1, $A_n(\boldsymbol{k})^{S_n}$ is a Galois ring over Γ. Moreover, the canonical embedding of Γ modules

$$A_n(\boldsymbol{k})^{S_n} \to A_n(\boldsymbol{k})$$

splits, with inverse being the symmetrizer map $\frac{1}{n!} \sum_{\pi \in S_n} \pi$. Since $A_n(\boldsymbol{k})$ is free over $\boldsymbol{k}[t_1, \ldots, t_n]$, and the latter algebra is free over Γ we have that $A_n(\boldsymbol{k})^{S_n}$ is a left and right projective Γ module. Applying Proposition 2 we conclude that $A_n(\boldsymbol{k})^{S_n}$ is a Galois order over Γ and Γ is a Harish-Chandra subalgebra. □

We finish with the following conjecture.

Conjecture. $A_n(\boldsymbol{k})^{S_n}$ is a free left (right)- module over Γ.

Remark. One way to prove the conjecture above would be to use the analog of the Kostant theorem from [17]. For that one would need to show in particular that the associated graded algebra of $A_n(\boldsymbol{k})^{S_n}$ is a complete intersection ring. However, we were communicated by Gregor Kemper, that this fails already for $n = 3$.

References

[1] Alev, J.; Ooms, A.; Van den Bergh, M.; A class of counterexamples to the Gelfand–Kirillov conjecture, Trans. Amer. Math. Soc. 348 (1996) 1709–1716.

[2] Alev, J.; Ooms, A.; Van den Bergh, M.; The Gelfand–Kirillov conjecture for Lie algebras of dimension at most eight, J. Algebra 227 (2000) 549–581. Corrigendum in J. Algebra 230 (2000) p. 749.

[3] Alev, J.; Dumas, F.; Opérateurs différentiels invariants et problème de Noether, Studies in Lie Theory (eds. J. Bernstein, V. Hinich and A. Melnikov), Birkhäuser, Boston, 2006.

[4] Bavula, V.; Generalized Weyl algebras and their representations, Algebra i Analiz 4 (1992) 75–97. English translation: St. Petersburg Math. J. 4 (1993) 71–92.

[5] Borho, W.; Gabriel, P.; Rentschler, R.; Primideale in Einhullenden Auflösbarer Lie-Algebren, Lecture Notes in Mathematics v. 357, Springer-Verlag, Berlin, New York, 1973.

[6] Block, R.; The irreducible representations of the Lie algebra sl(2) and of the Weyl algebra, Adv. Math. 39 (1981) 69–110

[7] Burnside, W.; Theory of Groups of Finite Order, Cambridge University Press, Cambridge, 1914 (reprinted by Dover, New York, 1955).

[8] Cannings, R.; Holland, M. P.; Differential operators on varieties with a quotient subvariety. J. Algebra 170 (1994), no. 3, 735-753

[9] Jacques Dixmier, Alg‘ebres enveloppantes (French), Gauthier-Villars Editeur, Paris-Brussels-Montreal, Que., 1974. Cahiers Scientifiques, Fasc. XXXVII.

[10] Drozd, Y; Ovsienko, A; Futorny, F; On Gelfand-Zetlin modules, Proceedings of the Winter School on Geometry and Physics (Srnı, 1990), 1991, pp. 143–147.

[11] Drozd, Y; Ovsienko, A; Futorny, F; Harish-Chandra subalgebras and Gelfand–Zetlin modules, in: Finite Dimensional Algebras and Related Topics, in: NATO ASI Ser. C., Math. Phys. Sci., vol. 424, 1994, pp. 79–93.

[12] Eshmatov, F.; Futorny, V.; Ovsienko, S.; Schwarz, J.; Noncommutative Noether's Problem for Unitary Reflection Groups, Proceedings of the American Mathematical Society, to appear.

[13] Fernando, S. L.; Lie algebra modules with finite-dimensional weight spaces. I, Trans. Amer. Math. Soc. 322 (1990), no. 2, 757–781.

[14] Fischer, E.; Zur theorie der endlichen Abelschen Gruppe, Math. Ann. 77 (1916), 81-88.

[15] Formanek, E.; Rational Function Fields. Noether's Problem and Related Questions, Journal of Pure and Applied Algebra 31 (1984), 28-36.

[16] Futorny, V.; Molev, A.; Ovsienko, A.; Harish-Chandra modules for Yangians, Repr. Theory AMS 9 (2005) 426–454

[17] Futorny, V.; Ovsienko, S.; Konstant's theorem for special filtered algebras, Bulletin of the London Mathematical Society v.37 (2005), 187-199.

[18] Futorny, V.; Molev, A.; Ovsienko, S.; The Gelfand-Kirillov conjecture and Gelfand-Tsetlin modules for finite W-algebras, Advances in Mathematics 223 (2010), 773-796.

[19] Futorny, V.; Molev, A.; Ovsienko, S.; The Gelfand-Kirillov conjecture and Gelfand-Tsetlin modules for finite W-algebras, Advances in Mathematics 223 (2010), 773-796.

[20] Futorny, V.; Ovsienko, S.; Galois orders in skew monoid rings, J. of Algebra, 324 (2010), 598-630.

[21] Futorny, V.; Hartwig, J. T.; Solution of a q-difference Noether Problem and the quantum Gelfand-Kirillov Conjecture for gl_N, Mathematische Zeitschrift 276 (2014), 1-37.

[22] Futorny, V; Ovsienko, S; Fibers of characters in Gelfand-Tsetlin categories, Transactions of The American Mathematical Society, v. 366 (2014), 4173-4208.

[23] Gelfand, I.M.; Tsetlin, M. S.; Finite dimensional representations of the group of unimodular matrices, Doklady Akad. Nauk SSSR 71 (1950) 1017–1020.

[24] Gelfand, I. M.; Kirillov, K. K., Sur les corps liés aux algebres enveloppantes des algebres de Lie, Inst. Hautes Etudes Sci. Publ. Math. 31 (1966), 5-19.

[25] Humphreys, J. E.; Representations of Semisimple Lie Algebras in the BGG Category O, Graduate Studies in Mathematics vol. 94, American Mathematical Society, 2008.

[26] James, G; Kerber, A; The Representation Theory of the Symmetric Group, Addison Wesley, 1980.

[27] Joseph, A.; Proof of the Gelfand-Kirillov conjecture for solvable Lie algebras, Proc. Amer.Math. Soc. 45 (1974), 1-10.

[28] Khomenko, A; Some applications of Gelfand–Zetlin modules, in: Representations of Algebras and Related Topics, in: Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 205–213.

[29] Levasseur, T.; Stafford, J.; Invariant differential operators and an homomorphism of Harish-Chandra, J. Amer. Math. Soc. 8, 365-372, 1995.

[30] Maeda, T.; Noether's problem for A_5, Journal of Algebra 125 (1989), 418-430.

[31] Mathieu, O; Classification of irreducible weight modules (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) 50 (2000), no. 2, 537–592.

[32] McConnell, J.C.; Representations of solvable Lie algebras and the Gelfand-Kirillov conjecture, Proc. London Math. Soc. 29 (1974), 453-484.

[33] McConnell, J.C.; Robson J.C.; Noncommutative noetherian rings, revised edition, Graduate Studies in Mathematics 30, American Mathematical Society, Providence, 2001.

[34] Molev, A; Gelfand-Tsetlin bases for classical Lie algebras, Handbook of algebra. Vol. 4, Handb. Algebr., vol. 4, Elsevier-North-Holland, Amsterdam, 2006, pp. 109–170.

[35] Montgomery, S.; Fixed Rings of Finite Automorphism Groups of Associative Rings, Lecture Notes in Mathematics v.818, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[36] Noether, E.; Rational Funktionenkrper, Jahrbericht Deutsch. Math.-Verein. 22 (1913), 316-319. Reprinted in "Collected Papers", Springer, Berlin, 141-144, 1982.

[37] Okunkov, A.; Vershik, A.; A new approach to representation theory of symmetric groups, Selecta Math. (N.S.) 2 (1996) 581–605.

[38] Ovsienko, S.; Finiteness statements for Gelfand-Tsetlin modules, Proceedings of Third International Algebraic Conference in Ukraine (Ukrainian), Natsional. Akad. Nauk Ukrainy, Inst. Mat., Kiev, (2002), 323-338.

[39] Premet, A.; Modular Lie Algebras and the Gelfand-Kirillov conjecture, Invent. Math. 181 (2010), 395-420.

[40] Saltman, D.; Noether's problem over an algebraically closed field, Invent. Math. 77 (1984), 71-84.

[41] Zhelobenko, D.P.; Compact Lie Groups and Their Representations, Nauka, Moscow, 1970; Transl. Math. Monogr., vol. 40, Amer. Math. Soc., Providence, RI, 1973.

Contact information

V. Futorny, J. Schwarz

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo SP, Brasil

E-Mail(s): futorny@ime.usp.br, jfschwarz.0791@gmail.com

Received by the editors: 21.03.2017.