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RESEARCH ARTICLE

Algebras and logics of partial quasiary predicates*

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Abstract. In the paper we investigate algebras and logics defined for classes of partial quasiary predicates. Informally speaking, such predicates are partial predicates defined over partial states (partial assignments) of variables. Conventional n-ary predicates can be considered as a special case of quasiary predicates. The notion of quasiary predicate, as well as the notion of quasiary function, is used in computer science to represent semantics of computer programs and their components. We define extended first-order algebras of partial quasiary predicates and investigate their properties. Based on such algebras we define a logic with irrefutability consequence relation. A sequent calculus is constructed for this logic, its soundness and completeness are proved.

Introduction

Logics of quasiary predicates can be considered as a natural generalization of classical predicate logic. The latter is based upon total n-ary predicates which represent fixed and static properties of subject domains. Though classical logics and its various extensions are widely used in computer science [1] some restrictions of such logics should be mentioned. For

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example, in computer science partial and non-deterministic predicates over complex data structures often appear. Therefore there is a need to construct such logical systems that better reflect the above-mentioned features. One of specific features for computer science is quasiarity of predicates. Such predicates are partial predicates defined over partial states (partial assignments) of variables and, consequently, they do not have fixed arity. Conventional n-ary predicates can be considered as a special case of quasiary predicates. Algebras of partial quasiary predicates form a semantic base for logics of such predicates. More detailed account on this topic can be found in [2, 3]. In these works we investigated basic algebras of partial quasiary predicates and constructed corresponding logics. But such algebras are not expressive enough to formulate some important properties of quasiary predicates. Therefore here we consider extended algebras and investigate their properties. This leads to a special logic of quasiary predicates. For this logic based on extended algebras we construct a sequent calculus and prove its soundness and completeness.

The logic construction, accepted here, consists of several phases: first, we construct predicate algebras, terms of which specify the language of a logic; then we define interpretation mappings and a consequence relation; at last, we construct a calculus for the defined logic. This scheme of logic construction determines a structure of the paper.

In Section 1 we define extended first-order algebras of quasiary predicates and study their properties. In Sections 2 we define and investigate an extended logic of quasiary predicates and irrefutability consequence relation. In Section 3 a sequent calculus is constructed; its soundness and completeness are proved. In the last section conclusions are formulated.

We use arrow $\xrightarrow{t}$ ($\xrightarrow{p}$) to denote the class of total (partial) mappings, arrow $\downarrow$ ($\uparrow$) to denote that a mapping is defined (undefined) on its argument, and symbol $\equiv$ to denote a strong equality.

1. Extended first-order algebras of partial quasiary predicates

Let V be a nonempty *set of variables* (*names*). Let A be a set of basic values ($A \neq \varnothing$). Given V and A, the class ${}^{V}\!A$ of *nominative sets* is defined as the class of all partial mappings from V to A, thus, ${}^{V}\!A = V \xrightarrow{p} A$. Informally speaking, nominative sets represent states of variables.

Though nominative sets are defined as mappings, we follow mathematical tradition and also use set-theoretic notation for these objects.

In particular, the notation $d = [v_i \mapsto a_i \mid i \in I]$ describes a nominative set d; the notation $v_i \mapsto a_i \in d$ means that $d(v_i)$ is defined and its value is a_i ($d(v_i) \downarrow= a_i$). The main operation for nominative sets is a total unary parametric *renomination* $r^{v_1,\ldots,v_n}_{x_1,\ldots,x_n} : {}^V\!A \xrightarrow{t} {}^V\!A$ where $v_1, \ldots, v_n, x_1, \ldots, x_n \in V$, $v_1, \ldots, v_n$ are distinct variables, $n \geqslant 0$, which is defined by the following formula:

$$r^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}(d) = [\, v \mapsto a \mid v \mapsto a \in d, v \notin \{v_1, \ldots, v_n\}] \\ \cup\ [\, v_i \mapsto a_i \mid x_i \mapsto a_i \in d,\ v_i \in \{v_1, \ldots, v_n\}].$$

Intuitively, given d this operation yields a new nominative set changing the values of $v_1, \ldots, v_n$ to the values of $x_1, \ldots, x_n$ respectively. We also use simpler notation for this formula: $r^{\bar{v}}_{\bar{x}}(d)$. Note that we treat parameter ${}^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}$ as a total mapping from $\{v_1, \ldots, v_n\}$ into $\{x_1, \ldots, x_n\}$ thus parameters obtained by pairs permutations are identical. We write $x \in \bar{v}$ to denote that x is a variable from $\bar{v}$; we write $\bar{v} \cup \bar{x}$ to denote the set of variables that occur in the sequences $\bar{v}$ and $\bar{x}$.

Operation of deleting a component with a variable v from a nominative set d is denoted $d|_{-v}$. Notation $d =_{-v} d'$ means that $d_{-v} = d'_{-v}$.

The set of *assigned variables* (*names*) in d is defined by the formula

$$asn(d) = \{v \in V \mid v \mapsto a \in d \text{ for some } a \in A\}.$$

Let $Bool = \{F, T\}$ be a set of Boolean values and let $Pr^V_A = {}^V\!A \xrightarrow{p} Bool$ be the set of all partial predicates over ${}^V\!A$. Such predicates are called *partial quasiary predicates.*

For $p \in Pr^V_A$ the truth and falsity domains of p are respectively

$$T(p) = \{d \in {}^V\!A \mid p(d) \downarrow= T\} \quad \text{and} \quad F(p) = \{d \in {}^V\!A \mid p(d) \downarrow= F\}.$$

A variable $u \in V$ is *unessential* for p if $p(d) \equiv p(d')$ for all $d, d' \in {}^V\!A$ such that $d =_{-u} d'$.

Operations over Pr^V_A are called *compositions.* Basic compositions over quasiary predicates with arity greater than 0 (non-trivial compositions) are *disjunction* $\vee$, *negation* $\neg$, *renomination* $R^{\bar{v}}_{\bar{x}}$, and *existential quantification* $\exists x$. We extend them with null-ary (trivial) composition εx called *variable unassignment predicate.* Thus, the extended set $CE(V)$ of first-order compositions consists of compositions $\vee$, $\neg$, $R^{\bar{v}}_{\bar{x}}$, $\exists x$, εz for all parameters $\bar{x}, \bar{v}, x, z$.

Compositions have the following types:

$$\vee : Pr^V_A \times Pr^V_A \xrightarrow{t} Pr^V_A; \quad \neg, R^{\bar{v}}_{\bar{x}}, \exists x : Pr^V_A \xrightarrow{t} Pr^V_A, \quad \varepsilon x : Pr^V_A$$

and are defined by the following formulas ($p, q \in Pr_A^V$):

- $T(p \vee q) = T(p) \cup T(q)$; $F(p \vee q) = F(p) \cap F(q)$;
- $T(\neg p) = F(p)$; $F(\neg p) = T(p)$;
- $T(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^V\!A \mid r_{\bar{x}}^{\bar{v}}(d) \in T(p)\}$;
  $F(R_{\bar{x}}^{\bar{v}}(p)) = \{d \in {}^V\!A \mid r_{\bar{x}}^{\bar{v}}(d) \in F(p)\}$;
- $T(\exists xp) = \{d \in {}^V\!A \mid d\nabla x \mapsto a \in T(p)$ for some $a \in A\}$;
  $F(\exists xp) = \{d \in {}^V\!A \mid d\nabla x \mapsto a \in F(p)$ for all $a \in A\}$;
- $T(\varepsilon z) = \{d \in {}^V\!A \mid z \notin asn(d)\}$; $F(\varepsilon z) = \{d \in {}^V\!A \mid z \in asn(d)\}$.

Here $d\nabla x \mapsto a = [v \mapsto c \in d \mid v \neq x] \cup [x \mapsto a]$.

Please note that definitions of compositions are similar to strong Kleene's connectives and quantifiers.

A pair $AQE(V, A) =< Pr_A^V; CE(V) >$ is called *an extended first-order algebra of partial quasiary predicates.*

Let us consider semantic properties of such algebras. Compositions $\vee, \neg, R_{\bar{x}}^{\bar{v}}, \exists x, \varepsilon z$ specify four types of properties related to propositional compositions $\vee$ and $\neg$, to renomination composition $R_{\bar{x}}^{\bar{v}}$, to existential quantifier $\exists x$, and to variable unassignment composition (predicate) εz.

Properties of propositional compositions are traditional. In particular, disjunction composition is associative, commutative, and idempotent; negation composition is involutive $\neg\neg p = p$.

Renomination composition is a new composition specific for logics of quasiary predicates. Its properties are not well-known therefore we describe them in more detail. We formulate six equalities ($R\vee$, $R\neg$, RR, $R\exists$, $R\varepsilon s$, $R\varepsilon$) for distributive properties and three equalities (R, RI, RU) for normalization properties. Note, that here only those properties are presented which will induce corresponding sequent rules.

Lemma 1. *The following properties of renomination compositions hold:*

$R\vee$: $R_{\bar{x}}^{\bar{v}}(p \vee q) = R_{\bar{x}}^{\bar{v}}(p) \vee R_{\bar{x}}^{\bar{v}}(q)$;

$R\neg$: $R_{\bar{x}}^{\bar{v}}(\neg p) = \neg R_{\bar{x}}^{\bar{v}}(p)$;

RR: $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(p)) = R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (p)$;

$R\exists$: $R_{\bar{x}}^{\bar{v}}(\exists yp) = \exists z R_{\bar{x}}^{\bar{v}}(R_z^y(p))$, $z \notin \bar{v} \cup \{y\}$, *$z$ is unessential for p;*

$R\varepsilon s$: $R_{\bar{x}}^{\bar{v}}(\varepsilon z) = \varepsilon z$, $z \notin \bar{v}$;

$R\varepsilon$: $R_{\bar{x},y}^{\bar{v},z}(\varepsilon z) = \varepsilon y$;

R: $R(p) = p$;

RI: $R_{z,\bar{x}}^{z,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$;

RU: $R_{z,\bar{x}}^{y,\bar{v}}(p) = R_{\bar{x}}^{\bar{v}}(p)$, *$y$ is unessential for* $R_{\bar{x}}^{\bar{v}}(p)$.

Here $R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}}$ represents two successive renomination $R_{\bar{y}}^{\bar{w}}$ and $R_{\bar{x}}^{\bar{v}}$.

Proof. We prove the lemma by showing that truth and falsity domains for predicates in the left- and right-hand sides of equalities coincide. Let us consider property $R\exists$ only.

For the truth domain we have ($d \in {}^{V}\!A$):

$d \in T(R_{\bar{x}}^{\bar{v}}(\exists yp)) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d) \in T(\exists yp) \Leftrightarrow r_{\bar{x}}^{\bar{v}}(d)\nabla y \mapsto a \in T(p)$ for some $a \in A \Leftrightarrow (r_{\bar{x}}^{\bar{v}}(d)\nabla y \mapsto a)\nabla z \mapsto a \in T(p)$ for some $a \in A$ (since z is unessential for p) $\Leftrightarrow$ (since $z \notin \{y\}$) $(r_{\bar{x}}^{\bar{v}}(d)\nabla z \mapsto a)\nabla y \mapsto a \in T(p)$ for some $a \in A \Leftrightarrow (r_{\bar{x}}^{\bar{v}}(d)\nabla z \mapsto a) \in T(R_z^y(p))$ for some $a \in A \Leftrightarrow$ (since $z \notin \bar{x}$) $d\nabla z \mapsto a \in T(R_{\bar{x}}^{\bar{v}}(R_z^y(p)))$ for some $a \in A \Leftrightarrow d \in T(\exists z R_{\bar{x}}^{\bar{v}}(R_z^y(p)))$.

In the same way the coincidence of the falsity domains is proved. □

As to variable unassignment composition (predicate), we admit that εz is a total predicate ($T(\varepsilon y) \cup F(\varepsilon y) = {}^{V}\!A$) for which any y ($y \in V, y \neq z$) is unessential.

Lemma 2. *The following properties of quantification compositions hold ($x \neq y$):*

$T\exists v : T(R_y^x(p)) \cap F(\varepsilon y) \subseteq T(\exists xp)$;

$F\exists v : F(\exists xp) \cap F(\varepsilon y) \subseteq F(R_y^x(p))$;

$T\exists u : T(R_y^x(p)) \subseteq T(\varepsilon y) \cup T(\exists xp)$;

$F\exists u : F(\exists xp) \subseteq T(\varepsilon y) \cap F(R_y^x(p))$.

Proof. To prove $T\exists v$ consider arbitrary $d \in T(R_y^x(p)) \cap F(\varepsilon y)$. This means that y is assigned in d with some value a and $d\nabla x \mapsto a \in T(p)$, therefore $d \in T(\exists xp)$.

Property $F\exists v$ is proved in the same manner.

Properties $T\exists u$ and $F\exists u$ are derived respectively from $T\exists v$ and $F\exists v$ using equalities $T(\varepsilon y) \cup F(\varepsilon y) = {}^{V}\!A$ and $T(\varepsilon y) \cap F(\varepsilon y) = \varnothing$. □

Algebras $AQE(V, A)$ (for various A) form a semantic base for a pure extended first-order logic of partial quasiary predicates L^{QE} (called also extended quasiary logic) being constructed here. Let us proceed with formal definitions.

2. Extended quasiary logic

To define a logic we should define its semantic component, syntactic component, and interpretational component [2, 3]. Semantics of the logic under consideration is specified by algebras of the type $AQE(V, A)$ (for various A), so, we start with syntactic component of the logic.

2.1. Syntactic component

A syntactic component specifies the language of L^{QE}. For simplicity, we use the same notation for symbols of compositions and compositions themselves. Let $CEs(V)$ be *a set of composition symbols* that represent compositions in algebras defined above. Thus, $CEs(V)$ consists of symbols $\vee$, $\neg$, $R_{\bar{x}}^{\bar{v}}$, $\exists x$, εz for all parameters $\bar{x}, \bar{v}, x, z$.

Let Ps be a set of *predicate symbols*, V and U be infinite sets ($U \subseteq V$). U is called a set of *unessential variables*. This set does not affect the set of formulas, but restricts their interpretations. Having U we obtain more possibilities for formula transformations. A tuple $\Sigma^{QE} = (V, U, CEs(V), Ps)$ is a language *signature*. Given Σ^{QE}, we define inductively the language of L^{QE} – *the set of formulas* $Fr(\Sigma^{QE})$. Formulas P and εz are *atomic* ($P \in Ps$, $z \in V$); *composite formulas* are of the form $\Phi \vee \Psi$, $\neg\Phi$, $R_{\bar{x}}^{\bar{v}}\Phi$, and $\exists x\Phi$ where Φ and Ψ are formulas. Formulas of the form $R_{\bar{x}}^{\bar{v}}P$ ($P \in Ps$) are called *primitive*. Such formula is *normal* [2] if none of the normalization rules can be applied to it.

2.2. Interpretational component

Given Σ^{QE} and nonempty set A we can define an extended algebra of partial quasiary predicates $AQE(V, A) =< Pr_A^V; CE(V) >$. Composition symbols have fixed interpretation. We also need interpretation I^{Ps} : $Ps \xrightarrow{t} Pr_A^V$ of predicate symbols; for obtained predicates *all variables* $u \in U$ *should be unessential.*

Formulas and interpretations in L^{QE} are called L^{QE}-*formulas* and L^{QE}-*interpretations* respectively. Usually the prefix L^{QE} is omitted. Given a formula Φ and an interpretation J we can speak of an *interpretation of* Φ in J. It is denoted by Φ_J.

Predicates εz specify cases when z is assigned or unassigned. This property can be used for construction of sequent rules for quantifiers. Note that notation $E!z$ in free logic [4] corresponds to negation of εz.

In the sequel we adopt the following convention: a, b denote elements from A; x, y, s, z, v, w (maybe with indexes) denote variables (names) from V; d, d', d_1, d_2 denote nominative sets from ${}^VA$; p, q denote predicates from $AQE(V, A)$; P denotes a predicate symbol from Ps; Φ, Ψ, Ξ, Ω denote L^{QE}-formulas; Γ, Δ, Υ denotes sets of L^{QE}-formulas; J denotes L^{QE}-interpretation. The set of all variables (names) that occur in Φ is denoted $nm(\Phi)$. Variables from $U \backslash nm(\Phi)$ are called *fresh unessential variables* for Φ and their set is denoted $fu(\Phi)$. We use natural extensions of this notation for a case of several formulas and sets of formulas like $nm(R_{\bar{v}}^{\bar{u}}, \exists x\Phi, \Gamma, \Delta)$

and $fu(R_{\bar{v}}^{\bar{u}}, \exists x\Phi, \Gamma, \Delta)$. A set $\varepsilon(\Gamma) = \{x \mid \varepsilon x \in \Gamma\}$ is the set of *unassigned variables* in $\Gamma \to \Delta$ and $\varepsilon(\Delta) = \{x \mid \varepsilon x \in \Delta\}$ is the set of *assigned variables* in $\Gamma \to \Delta$. A set $uns(\Gamma \to \Delta) = nm(\Gamma \cup \Delta) \setminus (\varepsilon(\Gamma) \cup \varepsilon(\Delta))$ is the set of *unspecified variables* in $\Gamma \to \Delta$.

2.3. Consequence relation for sets of formulas

Let $\Gamma \subseteq Fr(\Sigma^{QE})$ and $\Delta \subseteq Fr(\Sigma^{QE})$ be sets of formulas. Δ is a *consequence* of Γ in interpretation J (denoted by $\Gamma \ {}_J\!\models \Delta$), if

$$\bigcap_{\Phi\in\Gamma} T(\Phi_J) \cap \bigcap_{\Psi\in\Delta} F(\Psi_J) = \varnothing.$$

This formula is also written in a simpler form as $T(\Gamma) \cap F(\Delta) = \varnothing$. Δ is a *logical consequence* of Γ (denoted by $\Gamma \models \Delta$), if $\Gamma \ {}_J\!\models \Delta$ in every interpretation J.

This relation of logical consequence is *irrefutability* relation.

Here we consider only those properties of the consequence relation which induce sequent rules for the logic under consideration. Such properties are constructed upon semantic properties of compositions (Lemma 1, Lemma 2). To do this the following lemma is often used.

Lemma 3. *Let J be an arbitrary interpretation. Then*

- *if $T(\Phi_J) = T(\Psi_J)$ then $\Phi, \Gamma \ {}_J\!\models \Delta \Leftrightarrow \Psi, \Gamma \ {}_J\!\models \Delta$;*
- *if $F(\Phi_J) = F(\Psi_J)$ then $\Gamma \ {}_J\!\models \Phi, \Delta \Leftrightarrow \Gamma \ {}_J\!\models \Psi, \Delta$;*
- *if $T(\Phi_J) = T(\Psi_J) \cap T(\Omega)$ then $\Phi, \Gamma \ {}_J\!\models \Delta \Leftrightarrow \Psi, \Omega, \Gamma \ {}_J\!\models \Delta$;*
- *if $F(\Phi_J) = F(\Psi_J) \cup F(\Omega)$ then*

$$\Gamma {}_J\!\models \Phi, \Delta \Leftrightarrow \Gamma {}_J\!\models \Psi, \Delta \quad \textit{and} \quad \Gamma {}_J\!\models \Omega, \Delta.$$

Proof. $\Phi, \Gamma \ {}_J\!\models \Delta$ means that $T(\Phi_J) \cap T(\Gamma_J) \cap F(\Delta_J) = \varnothing$. For the first property this is equivalent to $T(\Psi_J) \cap T(\Gamma_J) \cap F(\Delta_J) = \varnothing$ since $T(\Phi_J)$=$T(\Psi_J)$. The last condition is equivalent to $\Psi, \Gamma \ {}_J\!\models \Delta$.

Other properties are proved in the same way. □

Using this lemma we can prove the following properties:

$\vee_L$: $\Phi\vee\Psi, \Gamma \models \Delta \Leftrightarrow \Phi, \Gamma \models \Delta$ and $\Psi, \Gamma \models \Delta$;

$\vee_R$: $\Gamma \models \Delta, \Phi\vee\Psi \Leftrightarrow \Gamma \models \Delta, \Phi, \Psi$.

They induce sequent rules $\vee_L$ and $\vee_R$ (see the next section).

Lemma 4. *Let $u \in fu(\Phi)$. Then for any interpretation J variable u is unessential for Φ_J.*

Proof goes by induction on the structure of Φ.

This result can be generalized on sets of formulas.

Lemma 5. *The following properties related to quantification compositions hold:*

$\exists \mathrm{E}_L : \exists x\Phi,\ \Gamma \models \Delta \Leftrightarrow R^x_z(\Phi),\ \Gamma \models \varepsilon z, \Delta,$ *if* $z \in fu(\exists x\Phi, \Gamma, \Delta)$*;*

$\exists \mathrm{E}1_R : \Gamma \models \exists x\Phi, \varepsilon y, \Delta \Leftrightarrow \Gamma \models R^x_y(\Phi), \exists x\Phi, \varepsilon y, \Delta$*;*

$\exists \mathrm{E}2_R : \Gamma \models \exists x\Phi, \Delta \Leftrightarrow \Gamma \models R^x_z(\Phi), \varepsilon z, \exists x\Phi, \Delta,$

if $\varepsilon(\Delta) = \varnothing$ *and* $z \in fu(\exists x\Phi, \Gamma, \Delta)$*;*

$\exists \mathrm{E}3_R : \Gamma \models \exists x\Phi, \Delta \Leftrightarrow \varepsilon y, \Gamma \models \exists x\Phi, \Delta$ *and* $\Gamma \models R^x_y(\Phi), \varepsilon y, \exists x\Phi, \Delta,$

if $y \in uns(\Gamma \to \Delta)$.

Proof. For $\exists \mathrm{E}_L$ we should prove that

$$T(\exists x\Phi_J) \cap T(\Gamma_J) \cap F(\Delta_J) = \varnothing$$
$$\Leftrightarrow T(R^x_z(\Phi)_J) \cap T(\Gamma_J) \cap F(\varepsilon z_J) \cap F(\Delta_J) = \varnothing.$$

$\Rightarrow$) Let $T(\exists x\Phi_J) \cap T(\Gamma_J) \cap F(\Delta_J) = \varnothing$. By $\mathrm{T}\exists v$, $T(R^x_z(\Phi)_J) \cap F(\varepsilon z_J) \subseteq T(\exists x\Phi_J)$, therefore $T(R^x_z(\Phi)_J) \cap T(\Gamma_J) \cap F(\varepsilon z_J) \cap F(\Delta_J) = \varnothing$.

$\Leftarrow$) Let $T(R^x_z(\Phi)_J) \cap T(\Gamma_J) \cap F(\varepsilon z_J) \cap F(\Delta_J) = \varnothing$. Assume that $T(\exists x\Phi_J) \cap T(\Gamma_J) \cap F(\Delta_J) \neq \varnothing$. Then there exists d such that $d \in T(\exists x\Phi_J) \cap T(\Gamma_J) \cap F(\Delta_J)$. We have $d \in T(\exists x\Phi_J)$, $d \in T(\Gamma_J)$, and $d \in F(\Delta_J)$. Since $d \in T(\exists x\Phi_J)$ we have $d\nabla x \mapsto a \in T(\Phi_J)$ for some $a \in A$. But $z \in fu(\exists x\Phi, \Gamma, \Delta)$ therefore $(d\nabla x \mapsto a)\nabla z \mapsto a \in T(\Phi_J)$, $d\nabla z \mapsto a \in T(\Gamma_J)$, and $d\nabla z \mapsto a \in F(\Delta_J)$. Since $(d\nabla x \mapsto a)\nabla z \mapsto a = (d\nabla z \mapsto a)\nabla x \mapsto a$, $d\nabla z \mapsto a \in T(R^x_z(\Phi)_J)$; by definition of εz we have $d\nabla z \mapsto a \in F(\varepsilon z_J)$, thus, $d\nabla z \mapsto a \in T(R^x_z(\Phi)_J) \cap T(\Gamma_J) \cap F(\varepsilon z_J) \cap F(\Delta_J)$, that contradicts to the assumption.

For $\exists \mathrm{E}1_R$ we should prove that

$$T(\Gamma_J) \cap F(\exists x\Phi_J) \cap F(\varepsilon y) \cap F(\Delta_J) = \varnothing$$
$$\Leftrightarrow T(\Gamma_J) \cap F(R^x_y(\Phi)_J) \cap F(\exists x\Phi_J) \cap F(\varepsilon y) \cap F(\Delta_J) = \varnothing$$

$\Rightarrow$) This part is obvious.

$\Leftarrow$) Let $T(\Gamma_J) \cap F(R^x_y(\Phi)_J) \cap F(\exists x\Phi_J) \cap F(\varepsilon y) \cap F(\Delta_J) = \varnothing$. Assume that $T(\Gamma_J) \cap F(\exists x\Phi_J) \cap F(\varepsilon y) \cap F(\Delta_J) \neq \varnothing$. Then there exists d such that $d \in T(\Gamma_J) \cap F(\exists x\Phi_J) \cap F(\varepsilon y) \cap F(\Delta_J)$. We have $d \in T(\Gamma_J)$, $d \in F(\exists x\Phi_J)$, $d \in F(\varepsilon y)$, and $d \in F(\Delta_J)$. Since $d \in F(\varepsilon y)$ then $y \mapsto a \in d$ for some a. Since $d \in F(\exists x\Phi_J)$ then $d\nabla x \mapsto a \in F(\Phi_J)$. Therefore $d \in F(R^x_y(\Phi)_J)$, that contradicts to the assumption.

For $\exists\mathrm{E}2_R$ we should prove that

$$T(\Gamma_J) \cap F(\exists x\Phi_J) \cap F(\Delta_J) = \varnothing$$
$$\Leftrightarrow T(\Gamma_J) \cap F(R_z^x(\Phi)_J) \cap F(\varepsilon z_J) \cap F(\exists x\Phi_J) \cap F(\Delta_J) = \varnothing.$$

$\Rightarrow$) This part is obvious.

$\Leftarrow$) The proof of this part goes similar to proof of $\exists\mathrm{E}_L$. Assuming that $d \in T(\Gamma_J) \cap F(\exists x\Phi_J) \cap F(\Delta_J)$ we prove that for any a a new $d' = d\nabla z \mapsto a$ belongs to $T(\Gamma_J)\cap F(R_z^x(\Phi)_J)\cap F(\varepsilon z_J)\cap F(\exists x\Phi_J)\cap F(\Delta_J)$, that contradicts to the assumption.

Property $\exists\mathrm{E}3_R$ is proved in the same manner. □

3. Sequent calculus for L^{QE}

For the logic L^{QE} we build a *calculus of sequent type*. Sequents are pairs of the form $\Gamma \to \Delta$, where Γ and Δ are countable sets of formulas. Formulas of Γ are called T-formulas of the sequent, formulas of Δ are called F-formulas.

Semantic properties of relation $\models$ have their syntactic analogues – sequent rules. If a rule has additional condition (sometimes denoted C) it is written on the right of the rule. We present three groups of sequent rules associated with three groups of non-trivial compositions.

Sequent rules for propositional compositions:

$$\vee_L\ \frac{\Phi, \Gamma \to \Delta \qquad \Psi, \Gamma \to \Delta}{\Phi \vee \Psi, \Gamma \to \Delta}\ ; \qquad \vee_R\ \frac{\Gamma \to \Phi, \Psi, \Delta}{\Gamma \to \Phi \vee \Psi, \Delta}\ ;$$

$$\neg_L\ \frac{\Gamma \to \Phi, \Delta}{\neg\Phi, \Gamma \to \Delta}\ ; \qquad \neg_R\ \frac{\Phi, \Gamma \to \Delta}{\Gamma \to \neg\Phi, \Delta}\ .$$

Sequent rules for renomination compositions:

$$\mathrm{R}\vee_L\ \frac{R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi), \Gamma \to \Delta}{R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi), \Gamma \to \Delta}\ ; \qquad \mathrm{R}\vee_R\ \frac{\Gamma \to R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi), \Delta}{\Gamma \to R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi), \Delta}\ ;$$

$$\mathrm{R}\neg_L\ \frac{\neg R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \to \Delta}{R_{\bar{x}}^{\bar{v}}(\neg\Phi), \Gamma \to \Delta}\ ; \qquad \mathrm{R}\neg_R\ \frac{\Gamma \to \neg R_{\bar{x}}^{\bar{v}}(\Phi), \Delta}{\Gamma \to R_{\bar{x}}^{\bar{v}}(\neg\Phi), \Delta}\ ;$$

$$\mathrm{RR}_L\ \frac{R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi), \Gamma \to \Delta}{R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)), \Gamma \to \Delta}\ ; \qquad \mathrm{RR}_R\ \frac{\Gamma \to R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi), \Delta}{\Gamma \to R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)), \Delta}\ ;$$

$$\mathrm{R}\exists_L\ \frac{\exists u R_{\bar{x}}^{\bar{v}} R_u^y(\Phi), \Gamma \to \Delta}{R_{\bar{x}}^{\bar{v}}(\exists y\Phi), \Gamma \to \Delta}\ , C_{\mathrm{R}\exists}; \quad \mathrm{R}\exists_R\ \frac{\Gamma \to \exists u R_{\bar{x}}^{\bar{v}} R_u^y(\Phi), \Delta}{\Gamma \to R_{\bar{x}}^{\bar{v}}(\exists y\Phi), \Delta}\ , C_{\mathrm{R}\exists};$$

$$\mathrm{R}\varepsilon s_L \ \frac{\varepsilon z, \Gamma \to \Delta}{R_{\bar{x}}^{\bar{v}}(\varepsilon z), \Gamma \to \Delta}, z \notin \bar{v}; \qquad \mathrm{R}\varepsilon s_R \ \frac{\Gamma \to \varepsilon z, \Delta}{\Gamma \to R_{\bar{x}}^{\bar{v}}(\varepsilon z), \Delta}, z \notin \bar{v};$$

$$\mathrm{R}\varepsilon_L \ \frac{\varepsilon y, \Gamma \to \Delta}{R_{\bar{x},y}^{\bar{v},z}(\varepsilon z), \Gamma \to \Delta}; \qquad \mathrm{R}\varepsilon_R \ \frac{\Gamma \to \varepsilon y, \Delta}{\Gamma \to R_{\bar{x},y}^{\bar{v},z}(\varepsilon z), \Delta};$$

$$\mathrm{R}_L \ \frac{\Phi, \Gamma \to \Delta}{R(\Phi), \Gamma \to \Delta}; \qquad \mathrm{R}_R \ \frac{\Gamma \to \Phi, \Delta}{\Gamma \to R(\Phi), \Delta};$$

$$\mathrm{RI}_L \ \frac{R_{\bar{x}}^{\bar{v}}(\Phi), \Gamma \to \Delta}{R_{z,\bar{x}}^{z,\bar{v}}(\Phi), \Gamma \to \Delta}; \qquad \mathrm{RI}_R \ \frac{\Gamma \to R_{\bar{x}}^{\bar{v}}(\Phi), \Delta}{\Gamma \to R_{z,\bar{x}}^{z,\bar{v}}(\Phi), \Delta};$$

$$\mathrm{RU}_L \ \frac{R_{\bar{u}}^{\bar{v}}(\Phi), \Gamma \to \Delta}{R_{z,\bar{u}}^{y,\bar{v}}(\Phi), \Gamma \to \Delta}, C_{\mathrm{RU}}; \qquad \mathrm{RU}_R \ \frac{\Gamma \to R_{\bar{u}}^{\bar{v}}(\Phi), \Delta}{\Gamma \to R_{z,\bar{u}}^{y,\bar{v}}(\Phi), \Delta}, C_{\mathrm{RU}}.$$

Here $C_{\mathrm{R}\exists}$ is $u \in fu(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))$, C_{RU} is $y \in fu(\Phi)$.

Sequent rules for quantification compositions:

$$\exists\mathrm{E}_L \ \frac{R_z^x(\Phi), \Gamma \to \varepsilon z, \Delta}{\exists x\Phi, \Gamma \to \Delta}, z \in fu(\exists x\Phi, \Gamma, \Delta);$$

$$\exists\mathrm{E}1_R \ \frac{\Gamma \to R_y^x(\Phi), \exists x\Phi, \varepsilon y, \Delta}{\Gamma \to \exists x\Phi, \varepsilon y, \Delta};$$

$$\exists\mathrm{E}2_R \ \frac{\Gamma \to R_z^x(\Phi), \varepsilon z, \exists x\Phi, \Delta}{\Gamma \to \exists x\Phi, \Delta}, \varepsilon(\Delta) = \varnothing, z \in fu(\exists x\Phi, \Gamma, \Delta);$$

$$\exists\mathrm{E}3_R \ \frac{\varepsilon y, \Gamma \to \exists x\Phi, \Delta \qquad \Gamma \to R_y^x(\Phi), \varepsilon y, \exists x\Phi, \Delta}{\Gamma \to \exists x\Phi, \Delta}, y \in uns(\Gamma \to \Delta).$$

Rule $\exists\mathrm{E}1_R$ is applied when at least one variable is assigned. Rule $\exists\mathrm{E}2_R$ is applied when there are no assigned variables (in this case a fresh unassigned variables is assigned). This means the first application of quantification elimination (therefore $\varepsilon(\Delta) = \varnothing$). Rule $\exists\mathrm{E}3_R$ is applied when an unspecified variable is involved into quantifier elimination. In this case two branches appear: with this variable being unassigned and assigned.

The above written rules specify *QE-calculus*.

Based on definition of $\models$ and properties of compositions we obtain the following properties for sequent rules of QE-calculus.

Lemma 6. *Let* $\dfrac{\Gamma' \to \Delta'}{\Gamma \to \Delta}$ *and* $\dfrac{\Gamma' \to \Delta' \qquad \Gamma'' \to \Delta''}{\Gamma \to \Delta}$ *be sequent rules of QE-calculus. Then*

$$\Gamma' \models \Delta' \Leftrightarrow \Gamma \models \Delta; \qquad \Gamma' \models \Delta' \text{ and } \Gamma'' \models \Delta'' \Leftrightarrow \Gamma \models \Delta.$$

To define derivability in QE-calculus we should first introduce the notion of closed sequent. Such sequents are *axioms* of QE-calculus. For QE-calculus we have two conditions for a sequent to be closed:

- *classical closedness* (c-closedness);
- *unassigned closedness* (u-closedness).

Classical closedness is defined in a usual way: sequent $\Gamma \to \Delta$ is *closed* if there exists Φ such that $\Phi \in \Gamma$ and $\Phi \in \Delta$.

Unassigned closedness is defined in more difficult way. Given sequent $\Gamma \to \Delta$ and $R_{s_1,..,s_k,y_1,...,y_n,v_1,...,v_m}^{r_1,..,r_k,x_1,..,x_n,z_1,..z_m}\Phi$ such that $\{x_1,\ldots,x_n\} \cap \varepsilon(\Gamma) = \varnothing$ and $\{r_1,\ldots,r_k,s_1,\ldots,s_k,y_1,\ldots,y_n\} \subseteq \varepsilon(\Gamma)$, an expression $R_{\perp,\,...,\,\perp,v_1,...,v_m}^{x_1,...,x_n,z_1,...,z_m}\Phi$ is called a $\perp$-form of $R_{s_1,..,s_k,y_1,...,y_n,v_1,...,v_m}^{r_1,..,r_k,x_1,..,x_n,z_1,..,z_m}\Phi$. Then we define two formulas $R_{\bar{x}}^{\bar{v}}(\Phi)$ and $R_{\bar{y}}^{\bar{s}}(\Phi)$ be u-equivalent with respect to $\Gamma \to \Delta$ if their $\perp$-forms coincide. At last, we say that $\Gamma \to \Delta$ is *u-closed* if there exist two u-equivalent formulas $R_{\bar{x}}^{\bar{v}}(\Phi)$ and $R_{\bar{y}}^{\bar{s}}(\Phi)$ such that $R_{\bar{x}}^{\bar{v}}(\Phi) \in \Gamma$ and $R_{\bar{y}}^{\bar{s}}(\Phi) \in \Delta$.

Lemma 7. *If* $\Gamma \to \Delta$ *is closed then* $\Gamma \models \Delta$.

Proof. To prove this lemma we should consider two cases: $\Gamma \models \Delta$ is c-closed or u-closed. For the first case the lemma is obvious. Assume that $\Gamma \to \Delta$ is u-closed. Then $\Gamma \to \Delta$ can be presented in the form $\varepsilon(\Gamma)$, $R_{\bar{x}}^{\bar{v}}(\Phi)$, $\Sigma \to \Upsilon$, $R_{\bar{y}}^{\bar{s}}(\Phi)$, where $R_{\bar{x}}^{\bar{v}}(\Phi)$ and $R_{\bar{y}}^{\bar{s}}(\Phi)$ are u-equivalent.

Let J be an interpretation and $d \in {}^{V}\!A$. Two cases are possible:

- $\varepsilon u_J(d) = T$ for all $u \in \varepsilon(\Gamma)$;
- $\varepsilon u_J(d) = F$ for some $u \in \varepsilon(\Gamma)$.

For the first case $R_{\bar{x}}^{\bar{v}}(\Phi)_J(d) \equiv R_{\bar{y}}^{\bar{s}}(\Phi)_J(d)$ by u-equivalence, therefore $d \notin T(R_{\bar{x}}^{\bar{v}}(\Phi)_J) \cap F(R_{\bar{y}}^{\bar{s}}(\Phi)_J)$; for the second case $d \notin T(\varepsilon(\Gamma_J))$.

Since d was chosen arbitrarily we have that

$$T(\varepsilon(\Gamma_J)) \cap T(R_{\bar{x}}^{\bar{v}}(\Phi)_J) \cap T(\Sigma_J) \cap F(\Upsilon_J) \cap F(R_{\bar{y}}^{\bar{s}}(\Phi)_J) = \varnothing.$$

J was chosen arbitrarily therefore $\varepsilon(\Gamma), R_{\bar{x}}^{\bar{v}}(\Phi), \Sigma \models \Upsilon, R_{\bar{y}}^{\bar{s}}(\Phi)$. This means that $\Gamma \models \Delta$. □

Derivation in QE-calculus has the form of tree, the vertices of which are sequents. Such trees are called sequent trees. A sequent tree is *closed*, if every its leaf is a closed sequent. A sequent $\Gamma \rightarrow \Delta$ is *derivable*, if there is a closed sequent tree with the root $\Gamma \rightarrow \Delta$. Sequent calculus is constructed in such a way that a sequent $\Gamma \rightarrow \Delta$ has a derivation if and only if $\Gamma \models \Delta$.

Let us consider a procedure of construction of the sequent tree for a given sequent $\Gamma \rightarrow \Delta$. Such procedure is defined in the same way as for other sequent calculi for countable sequents [5] therefore we present only its general description without details. In the case of logic of quasiary predicates, the procedure of construction of a sequent tree is more complicated. The reason is that the value of a predicate p on d can be different depending on whether the component with some variable is assigned in d or not. Therefore sets of assigned, unassigned, and unspecified variables should be examined. This feature manifests itself in the sequent rules $\exists E1_R$, $\exists E2_R$, and $\exists E3_R$.

During construction of a sequent tree the following cases are possible:

- all sequents on the leaves are closed; we have a finite closed tree;
- procedure is not completed; we have a finite or infinite unclosed tree. Such tree has at least one path called *unclosed* all vertices of which are unclosed sequents.

The first case leads to soundness of QE-calculus.

Theorem 1 (soundness)**.** *Let* $\Gamma \rightarrow \Delta$ *be derivable. Then* $\Gamma \models \Delta$*.*

Proof. If $\Gamma \rightarrow \Delta$ is derivable then a finite closed tree was constructed. By the procedure of sequent tree construction we have that for any leaf of this tree its sequent $\Gamma' \rightarrow \Delta'$ is closed. Thus, by Lemma 7, $\Gamma' \models \Delta'$ holds. By Lemma 6, sequent rules preserve relation of logical consequence. Therefore for the root of the tree $\Gamma \rightarrow \Delta$ we also have that $\Gamma \models \Delta$. $\square$

For the second case, formulas of the unclosed path form Hintikka's model for which a counter example can be constructed. To do this we first formulate the properties of formulas of an unclosed path.

Let $\wp$ be an unclosed path in a sequent tree, L and R be respectively sets of all T-formulas and F-formulas of sequents of a path $\wp$.

All sequents of the path $\wp$ are unclosed, therefore the c- and u-closedness conditions are not satisfied. From this follows unclosedness conditions of a pair $H = (L, R)$:

HC) for every Φ it is not possible that $\Phi \in L$ and $\Phi \in R$;

HCU) there does not exist a pair of u-equivalent formulas of the form $R_{\bar{x}}^{\bar{v}}\Phi$ and $R_{\bar{y}}^{\bar{u}}\Phi$ such that $R_{\bar{x}}^{\bar{v}}\Phi \in L$ and $R_{\bar{y}}^{\bar{u}}\Phi \in R$.

Let $W = nm(L \cup R) \setminus \varepsilon(L)$. We assume that $W \neq \varnothing$; the case with $W = \varnothing$ can be considered as in propositional logic.

For our derivation procedure the following conditions, derived from the sequent rules of QE-calculus, should hold.

H∨) If $\Phi \vee \Psi \in L$ then $\Phi \in L$ or $\Psi \in L$;
if $\Phi \vee \Psi \in R$ then $\Phi \in R$ and $\Psi \in R$.

H¬) If $\neg\Phi \in L$ then $\Phi \in R$; if $\neg\Phi \in R$ then $\Phi \in L$.

HR∨) If $R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi) \in L$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi) \in L$;
if $R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi) \in R$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi) \in R$.

HR¬) If $R_{\bar{x}}^{\bar{v}}(\neg\Phi) \in L$ then $\neg R_{\bar{x}}^{\bar{v}}(\Phi) \in L$;
if $R_{\bar{x}}^{\bar{v}}(\neg\Phi) \in R$ then $\neg R_{\bar{x}}^{\bar{v}}(\Phi) \in R$.

HRR) If $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)) \in L$ then $R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi) \in L$;
if $R_{\bar{x}}^{\bar{v}}(R_{\bar{y}}^{\bar{w}}(\Phi)) \in R$ then $R_{\bar{x}}^{\bar{v}} \circ_{\bar{y}}^{\bar{w}} (\Phi) \in R$.

HR∃) If $R_{\bar{x}}^{\bar{v}}(\exists y\Phi) \in L$ then $\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi) \in L$ for some $z \in fu(R_{\bar{x}}^{\bar{v}}(\exists x\Phi))$;
if $R_{\bar{x}}^{\bar{v}}(\exists y\Phi) \in R$ then $\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi) \in R$ for some $z \in fu(R_{\bar{x}}^{\bar{v}}(\exists x\Phi))$.

HRεs) If $R_{\bar{x}}^{\bar{v}}(\varepsilon z) \in L$ and $z \notin \bar{v}$ then $\varepsilon z \in L$;
if $R_{\bar{x}}^{\bar{v}}(\varepsilon z) \in R$ and $z \notin \bar{v}$ then $\varepsilon z \in R$.

HRε) If $R_{\bar{x},y}^{\bar{v},z}(\varepsilon z) \in L$ then $\varepsilon y \in L$; if $R_{\bar{x},y}^{\bar{v},z}(\varepsilon z) \in R$ then $\varepsilon z \in R$.

HR) If $R(\Phi) \in L$ then $\Phi \in L$; if $R(\Phi) \in R$ then $\Phi \in R$.

HRI) If $R_{z,\bar{x}}^{z,\bar{v}}(\Phi) \in L$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \in L$; if $R_{z,\bar{x}}^{z,\bar{v}}(\Phi) \in R$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \in R$.

HRU) If $R_{z,\bar{x}}^{y,\bar{v}}(\Phi) \in L$ and $y \in fu(R_{\bar{x}}^{\bar{v}}(\Phi))$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \in L$;
if $R_{z,\bar{x}}^{y,\bar{v}}(\Phi) \in R$ and $y \in fu(R_{\bar{x}}^{\bar{v}}(\Phi))$ then $R_{\bar{x}}^{\bar{v}}(\Phi) \in R$.

H∃) If $\exists x\Phi \in L$ then there exists $y \in W$ such that $R_y^x(\Phi) \in L$;
if $\exists x\Phi \in R$ then $R_y^x(\Phi) \in R$ for every $y \in W$.

Let us demonstrate the correctness of the last property. Indeed, let $\exists x\Phi \in L$, then on some derivation step of path $\wp$ the $\exists\mathrm{E}_L$ rule was applied to T-formula $\exists x\Phi$ giving T-formula $R_y^x(\Phi)$. Therefore $R_y^x(\Phi) \in L$ and $\varepsilon y \in R$, thus $y \in W$. So, for some $y \in W$ we have $R_y^x(\Phi) \in L$.

Dually, let $\exists x\Phi \in R$; take any $y \in W$. Then necessarily one of the rules $\exists\mathrm{E}1_R$, $\exists\mathrm{E}2_R$, or $\exists\mathrm{E}3_R$ was applied to $\exists x\Phi$ with such y generating a formula $R_y^x(\Phi) \in R$. Note that in the rule $\exists\mathrm{E}3_R$ the first branch generates a formula $\varepsilon y \in L$, thus, such $y \notin W$.

A pair of formula sets $H = (L, R)$, for which the above formulated conditions (with letter 'H' in their labels) hold, is called a *quasiary model pair*, or *quasiary Hintikka's pair*.

A pair (L, R) of arbitrary sets of formulas is called *satisfiable* if there exist a set A, an interpretation J, and $\delta \in {}^V\!A$ such that:

- for all $\Phi \in L$ $\Phi_J(\delta) \downarrow= T$;
- for all $\Phi \in R$ $\Phi_J(\delta) \downarrow= F$.

Lemma 8. *Let $H = (L, R)$ be a quasiary Hintikka's pair. Then H is satisfiable.*

Proof. Given such pair $H = (L, R)$, we construct an extended quasiary algebra, an interpretation in this algebra, and a nominative set that confirm satisfiability of H.

Choose any set A such that $|A| = |W|$. The set A "mimics" W. This specifies an algebra $AQE(V, A)$.

Let $\delta \in {}^{V}A$ be such data that $asn(\delta) = W$ and δ itself (considered as a mapping) realizes a bijection from W to A.

First, we prescribe interpretation of *variable unassignment predicates* according to their definition:

- if $\varepsilon y \in L$ then define $\varepsilon y_J(\delta) = T$ (this means that $y \notin asn(\delta)$),
- if $\varepsilon y \in R$ then define $\varepsilon y_J(\delta) = F$ (this means that $y \in asn(\delta)$).

Then we prescribe interpretation to predicates symbols. Values of corresponding predicates are determined by atomic and normal primitive formulas. Also, unessential variables should be taken into account.

Atomic formulas of the form P where $P \in Ps$ define

- $P_J(\delta) = T$ if $P \in L$,
- $P_J(\delta) = F$ if $P \in R$.

Normal primitive formulas of the form $R_{\bar{x}}^{\bar{v}}(P)$ define

- $P_J(r_{\bar{x}}^{\bar{v}}(\delta)) = T$ if $R_{\bar{x}}^{\bar{v}}(P) \in L$,
- $P_J(r_{\bar{x}}^{\bar{v}}(\delta)) = F$ if $R_{\bar{x}}^{\bar{v}}(P) \in R$.

Also, we extend predicate interpretations specifying variables from U as unessential.

The predicates are defined unambiguously due to unclosedness conditions HC and HCU.

Indeed, for the case of atomic formula P this follows from HC, for the case of two different normal primitive formulas $R_{\bar{x}}^{\bar{v}}P$ and $R_{\bar{y}}^{\bar{u}}P$ we obtain different nominative sets $r_{\bar{x}}^{\bar{v}}(\delta)$ and $r_{\bar{y}}^{\bar{u}}(\delta)$, thus, no ambiguity can arise.

The proof goes on by induction on the formula structure with respect to definition of $H = (L, R)$.

For atomic and normal primitive formulas the satisfiability statements follow from their definitions.

Let us prove induction steps for these statements. We consider the main cases only and omit simpler cases.

Let $\Phi \vee \Psi \in L$. By H$\vee$ we have $\Phi \in L$ or $\Psi \in L$. By induction hypothesis $\Phi_J(\delta) = T$ or $\Psi_J(\delta) = T$, therefore $(\Phi \vee \Psi)_J(\delta) = T$. Let $\Phi \vee \Psi \in R$. By H$\vee$ we have $\Phi \in R$ and $\Psi \in R$. By induction hypothesis $\Phi_J(\delta) = F$ and $\Psi_J(\delta) = F$, therefore $(\Phi \vee \Psi)_J(\delta) = F$.

Let $R_{\bar{x}}^{\bar{v}}(\Phi\vee\Psi) \in L$. By HR$\vee$ we have $R_{\bar{x}}^{\bar{v}}(\Phi)\vee R_{\bar{x}}^{\bar{v}}(\Psi) \in L$. By induction hypothesis $(R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi))_J(\delta) = T$, therefore $(R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi))_J(\delta) = T$. Let $R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi) \in R$. By HR$\vee$ we have $R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi) \in R$. By induction hypothesis$(R_{\bar{x}}^{\bar{v}}(\Phi) \vee R_{\bar{x}}^{\bar{v}}(\Psi))_J(\delta) = F$, therefore $(R_{\bar{x}}^{\bar{v}}(\Phi \vee \Psi))_J(\delta) = F$.

Let $R_{\bar{x}}^{\bar{v}}(\exists y\Phi) \in L$. By HR$\exists$ we have $\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi) \in L$, where $z \in fu(R_{\bar{x}}^{\bar{v}}(\exists x\Phi))$. By induction hypothesis $(\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi))_J(\delta) = T$, therefore $(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))_J(\delta) = T$. Let $R_{\bar{x}}^{\bar{v}}(\exists y\Phi) \in R$. By HR$\exists$ we have $\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi) \in R$, where $z \in fu(R_{\bar{x}}^{\bar{v}}(\exists\Phi))$. By induction hypothesis $(\exists z R_{\bar{x}}^{\bar{v}} R_z^y(\Phi))_J(\delta) = F$, therefore $(R_{\bar{x}}^{\bar{v}}(\exists y\Phi))_J(\delta) = F$.

Let $\exists x\Phi \in L$. By H$\exists$ there exists $y \in W$ such that $R_y^x(\Phi) \in L$. By induction hypothesis $(R_y^x(\Phi))_J(\delta) = T$. From this $\Phi_J(\delta\nabla x \mapsto \delta(y)) = T$. But $\delta(y)\downarrow$ according to $\delta \in {}^W A$ and $y \in W$, therefore for $a = \delta(y)$ we have $\Phi_J(\delta\nabla x \mapsto a) = T$, thus, $(\exists x\Phi)_J(\delta) = T$. Let $\exists x\Phi \in R$. By H$\exists$ for all $y \in W$ we have $R_y^x(\Phi) \in R$. By induction hypothesis $(R_y^x(\Phi))_J(\delta) = F$ for all $y \in W$. From this $\Phi_J(\delta\nabla x \mapsto \delta(y)) = F$ for all $y \in W$. By $\delta \in {}^W A$, we have $\delta(y)\downarrow$ for all $y \in W$. Since δ is a bijection $W \to A$, then every $b \in A$ can be represented in the form $b = \delta(y)$ for some $y \in W$. So, $\Phi_J(\delta\nabla x \mapsto b) = F$ for every $b \in A$, therefore we have $(\exists x\Phi)_J(\delta) = F$. □

Theorem 2 (completeness)**.** *Let* $\Gamma \models \Delta$. *Then* $\Gamma \to \Delta$ *is derivable.*

Proof. Assume that $\Gamma \models \Delta$ and $\Gamma \to \Delta$ is not derivable. In this case a derivation tree for $\Gamma \to \Delta$ is not closed. Thus, an unclosed path $\wp$ exists in this derivation tree. Let L and R be respectively the sets of all T-formulas and F-formulas of this path. By Lemma 8, $H = (L, R)$ is satisfiable for some set A, some interpretation J, and some $\delta \in {}^V A$. This means that $\Phi_J(\delta) = T$ for all $\Phi \in L$ and $\Phi_J(\delta) = F$ for all $\Phi \in R$. Since $\Gamma \subseteq L$ and $\Delta \subseteq R$, then for all $\Phi \in \Gamma$ we have $\Phi_J(\delta) = T$ and for all $\Psi \in \Delta$ we have $\Psi_J(\delta) = F$. This contradicts to $\Gamma \models \Delta$. □

QE-calculus is a new generalized version of *QG*-calculus presented in [6]. *QG*-calculus was constructed for a basic quasiary logic with special rather complicated consequence relation, but here we adopted a traditional definition of this relation.

The obtained results can be used in logics for program reasoning. Some steps of construction of such quasiary program logics were presented in [7].

Conclusion

In the paper we have investigated algebras and logics defined for classes of partial quasiary predicates. Quasiary predicates and quasiary functions are used to represent semantics of computer programs and their components. Based on algebras of such predicates we have defined a corresponding extended quasiary logic with irrefutability consequence relation. A sequent calculus has been constructed for this logic, its soundness and completeness have been proved.

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