Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **19** (2015). Number 2, pp. 162–171

On inverse subsemigroups of the semigroup of orientation-preserving or orientation-reversing transformations

Paula Catarino*, **Peter M. Higgins**, **Inessa Levi**

Communicated by V. Mazorchuk

Abstract. It is well-known [16] that the semigroup $\mathcal{T}_n$ of all total transformations of a given n-element set X_n is covered by its inverse subsemigroups. This note provides a short and direct proof, based on properties of digraphs of transformations, that every inverse subsemigroup of order-preserving transformations on a finite chain X_n is a semilattice of idempotents, and so the semigroup of all order-preserving transformations of X_n is not covered by its inverse subsemigroups. This result is used to show that the semigroup of all orientation-preserving transformations and the semigroup of all orientation-preserving or orientation-reversing transformations of the chain X_n are covered by their inverse subsemigroups precisely when $n \leqslant 3$.

1. Introduction

In a regular semigroup S every element α has an inverse β in S meaning that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. In an inverse semigroup S every element of S has a unique inverse in S. An inverse β of an element α in a

*The first author is a Member of the Research Center of Mathematics, CM-UTAD, Portugal.

2010 MSC: 20M20, 05C25.

Key words and phrases: semigroup, semilattice, inverse subsemigroup, strong inverse, transformation, order-preserving transformation, orientation-preserving transformation, orientation-reversing transformation.

semigroup S is said to be a *strong inverse* of α if the subsemigroup $\langle \alpha, \beta \rangle$ of S generated by α and β is an inverse subsemigroup of S. A semigroup S is covered by its inverse subsemigroups precisely when every element in S has a strong inverse in S.

This note addresses the following question: what regular semigroups are covered by their inverse subsemigroups?

For example, the semigroup $\mathcal{T}_n$ of all total transformations of a given n-element set X_n and the semigroup $\mathcal{PT}_n$ of all total and partial transformations of X_n are both regular but not inverse. B. M. Schein [16] noted that the above question was formulated in 1964 during the VI Vsesouznyi Algebra Colloquium in Minsk, USSR, in terms of the semigroups $\mathcal{T}_n$ and $\mathcal{PT}_n$. In his 1971 paper [16], B. M. Schein showed, generalizing the results by L. M. Gluskin [9], that $\mathcal{T}_n$ and $\mathcal{PT}_n$ are covered by their inverse subsemigroups. A detailed proof of this result may be found in P. M. Higgins' book [11]. Note that this result does not hold for the semigroup of all total transformations of an infinite set, see, for example, [11, Exercise 6.2.8].

Let $X_n = \{1, 2, \cdots, n\}$ be a chain with respect to the standard order, and let $\mathcal{O}_n$ be the semigroup of all order-preserving transformations α on X_n, that is transformations satisfying the condition $x\alpha \leqslant y\alpha$ whenever $x < y$, for all $x, y \in X_n$. Let $\{i_n\}$ denote the identity permutation of X_n. The semigroup $\mathcal{O}_n$ was introduced by A. Ya. Aizenstat [1], where she gave a presentation for $\mathcal{O}_n \setminus \{i_n\}$ in terms of $2n - 2$ idempotent generators. She described in [2] the congruences on $\mathcal{O}_n$. There is a large body of literature on properties of the semigroup $\mathcal{O}_n$. For example, it is shown in [10] that the minimal number of generators of $\mathcal{O}_n \setminus \{i_n\}$ is n; combinatorial properties of $\mathcal{O}_n$ were studied in [13], [12] and [14]. It is well known that $\mathcal{O}_n$ is a regular semigroup.

It was shown recently by A. Vernitski [18] that all the inverse subsemigroups of $\mathcal{O}_n$ are semilattices. Indeed he proved that a finite inverse semigroup can be represented by order-preserving mappings if and only if it is a semilattice of idempotents. Vernitski's paper is concerned with the study of the pseudovariety of all finite semigroups whose inverse subsemigroups consist of a single element, and the quasivariety of all finite semigroups whose inverse subsemigroups are semilattices. The proof uses the Krohn-Rhodes Theorem on wreath products of monoids. In the present paper we provide a simple self-contained proof of the result based on digraphs associated with transformations (Theorem 2.7).

A transformation $\alpha \in \mathcal{T}_n$ is said to be *orientation-preserving* (*orientation-reversing*) if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ is a cyclic permutation of a non-decreasing (non-increasing) sequence. The semigroup

$\mathcal{OP}_n$ of all orientation-preserving transformations and the semigroup $\mathcal{P}_n$ of all orientation-preserving or orientation-reversing transformations were introduced independently by D. B. McAlister [15] and P. M. Catarino and P. M. Higgins [5]. Clearly, $\mathcal{O}_n$ is a subsemigroup of $\mathcal{OP}_n$, which in turn is a subsemigroup of $\mathcal{P}_n$.

For a transformation $\alpha \in \mathcal{T}_n$ the rank of α, denoted by rank(α), is the number of elements in the image set $X_n\alpha$ of α. It was shown in [4] and [15] that $\mathcal{OP}_n$ is generated by an idempotent in $\mathcal{O}_n$ of rank $n-1$ and the cyclic group generated by the n-cycle $(1, 2, 3, \ldots, n)$. It was also shown [15] that $\mathcal{P}_n$ is generated by an idempotent in $\mathcal{O}_n$ of rank $n-1$ and the dihedral group D_n. It follows that minimal generating sets of $\mathcal{OP}_n$ and $\mathcal{P}_n$ have sizes 2 and 3 respectively. The semigroups $\mathcal{OP}_n$ and $\mathcal{P}_n$ are regular [5].

The introduction of the semigroups $\mathcal{OP}_n$ and $\mathcal{P}_n$ generated a large body of fruitful research by a number of authors. For example, P. M. Catarino [4] exhibited a presentation of $\mathcal{OP}_n$ in terms of $2n-1$ generators, by extending A. Ja. Aizenstat's [1] presentation for $\mathcal{O}_n$ by a single generator and $2n$ relations. R. E. Arthur and N. Ruškuc [3] gave a presentation for $\mathcal{OP}_n$ in terms of the minimal number of generators (two) and $n+2$ relations. In the same article they also gave a presentation of $\mathcal{P}_n$ on three generators and $n+6$ relations. The congruences of $\mathcal{OP}_n$ and $\mathcal{P}_n$ were described by V. H. Fernandes, G. M. S. Gomes and M. M. Jesus [8]. The pseudovariety generated by all semigroups of orientation-preserving transformations on a finite cycle was introduced and studied by P. M. Catarino and P. M. Higgins in [6]. More recently, combinatorial properties of semigroups of total and partial orientation-preserving transformations were studied by A. Umar [17], and all maximal subsemigroups of $\mathcal{OP}_n$ and $\mathcal{P}_n$ were described by I. Dimitrova, V. H. Fernandez and J. Koppitz [7].

In the present paper we use the result that every inverse subsemigroup of $\mathcal{O}_n$ is a semilattice of idempotents (Theorem 2.7 below) to show that $\mathcal{OP}_n$ and $\mathcal{P}_n$ are covered by their respective inverse subsemigroups if and only if $n \leqslant 3$.

2. Results

Every transformation α of X_n may be viewed as a digraph on n vertices, in which for $x, y \in X_n$ we have that xy is an arc of the digraph of α precisely when $x\alpha = y$. A comprehensive discussion on digraphs associated with transformations may be found in [11, Section 1.6]; we summarize here the results used in the proofs below.

The *orbits* of a mapping α in $\mathcal{T}_n$ are the classes of the equivalence relation $\sim$ on X_n defined by $x \sim y$ if and only if there exist non-negative integers k, m such that $x\alpha^k = y\alpha^m$. The sets of vertices of connected components of a digraph of α correspond to orbits of α. Each component of a digraph of a transformation is *functional*, that is, it consists of a unique cycle together with a number of trees rooted around this cycle. A cycle on m distinct vertices of X_n is to be referred to as an m-cycle. If the cycle of a component consists of a single vertex x, then x is a fixed point of α, that is $x\alpha = x$.

Lemma 2.1. *Let α be a transformation in $\mathcal{T}_n$ and suppose that all the cycles in the digraph of α are 1-cycles. Then for any positive integer k, the orbits and fixed points of α and α^k are identical.*

Proof. Assume that x and y are in the same orbit with respect to some power α^k of α, that is $x \sim y$ with respect to α^k. Then there exist positive integers s and t such that $x\,(\alpha^k)^s = y\,(\alpha^k)^t$, whence $x\,\alpha^{ks} = y\,\alpha^{kt}$ and so $x \sim y$ with respect to α. Conversely, assume that $x \sim y$ with respect to α. By our assumption, the component C of the digraph of α containing vertices x and y has a unique 1-cycle, say, with a vertex z. Therefore z is a fixed point of α, and so $x\,\alpha^t = y\,\alpha^t = z$ for any positive integer $t \geqslant l$, where l is the length of the longest directed path in C. Hence $x\,\alpha^{kl} = y\,\alpha^{kl} = z$ or $x\,(\alpha^k)^l = y\,(\alpha^k)^l$. Thus $x \sim y$ with respect to α^k also. We conclude that the vertex set of C is a common orbit for all positive powers of α. Moreover z is a fixed point of α if and only if the same is true of all such powers. $\square$

The following result follows directly from Lemma 2.1.

Corollary 2.2. *Let α be a transformation in $\mathcal{T}_n$ and suppose that all the cycles in the digraph of α are 1-cycles. Let ε be an idempotent in $\mathcal{T}_n$ such that $\varepsilon = \alpha^r$, for some positive integer r. Then the orbits and fixed points of α and ε are identical.*

Lemma 2.3. *Let α be a transformation in $\mathcal{T}_n$ and suppose that all the cycles in the digraph of α are 1-cycles. If $\beta \in \mathcal{T}_n$ is any strong inverse of α then the orbits and fixed points of α and β are identical.*

Proof. Observe that since β is a strong inverse of α, the subsemigroup $S = \langle \alpha, \beta \rangle$ of $\mathcal{T}_n$ generated by α and β is an inverse semigroup. Therefore for any positive integer t we have that β^t is the unique inverse of α^t in S. Taking $t = r$ so that $\varepsilon = \alpha^r$ is an idempotent as in Corollary 2.2 we have that β^r is the unique inverse of $\alpha^r = \varepsilon$. Since an idempotent is its own unique inverse in S, we have that $\beta^r = \varepsilon$ also, and so $\alpha^r = \beta^r$. It follows immediately from Lemma 2.1 that the orbits and fixed points of α, β and ε are identical. □

It follows from the definition of an order-preserving transformation on a finite chain that the iterative sequence of images $x, x\,\alpha, \ldots, x\,\alpha^k, \ldots$ of a point $x \in X_n$ under a transformation $\alpha \in \mathcal{O}_n$ must terminate in a fixed point, whence it follows that the cycles of the components of the digraph of α are merely fixed points. This observation leads to Proposition 2.4 below, see a proof in [12, Proposition 1.5]. From this we also note that the semigroup $\mathcal{O}_n$ is *aperiodic*, meaning that all of its subgroups are trivial as it follows from the previous observation that the cyclic subgroup of the monogenic subsemigroup $\langle\alpha\rangle$ of $\mathcal{O}_n$ has only one member.

Proposition 2.4 ([12, Proposition 1.5]). *The cycle of each component of $\alpha \in \mathcal{O}_n$ consists of a unique fixed point.*

Therefore, as it was noted in [12], the digraph of a mapping in $\mathcal{O}_n$ consists of components, each of which is a directed tree with all arcs directed towards the root, which represents a fixed point of the mapping. The next result follows from Proposition 2.4 and Lemma 2.3.

Corollary 2.5. *Let α, β be transformations in $\mathcal{O}_n$. If β is a strong inverse of α then α and β have the same orbits and their components have the same roots.*

Recall that any order-preserving transformation has a strong inverse in $\mathcal{T}_n$. However, as the next result shows, an order-preserving transformation does not have an order-preserving strong inverse unless the transformation is an idempotent.

Theorem 2.6. *Let $\alpha \in \mathcal{O}_n$. Then*

1) *α has a strong inverse in $\mathcal{O}_n$ if and only if α is an idempotent.*
2) *If α is a non-idempotent with at least two fixed points, then α has no strong inverse in $\mathcal{OP}_n$.*

Proof. Since the first statement of the theorem is clearly true in the forward direction, we assume that there exists a non-idempotent $\alpha \in \mathcal{O}_n$ that has a strong inverse β in $\mathcal{OP}_n$. Moreover, since an idempotent transformation may be characterized as a transformation that fixes each element of its image, for a non-idempotent α there exist distinct $u, v \in X_n$ such that $u\alpha = v$, $v\alpha \neq v$. Let C be the component of the digraph of α containing vertices u, v. Since C is a directed tree with all arcs directed towards the root, say, $z \in X_n$, there exists a unique directed path in C from u through v to z. Therefore there exist distinct vertices x, y distinct from z in this path such that $x\alpha = y$, $y\alpha = z$, and $z\alpha = z$. We may assume without loss of generality that $x < y$. Then since α is order-preserving we have that $y = x\alpha \leqslant y\alpha = z$, so that $x < y < z$ since $y \neq z$.

Since β is an inverse of α, $\beta\alpha$ is an idempotent transformation with image $X_n\beta\alpha = X_n\alpha$, so $y \in X_n\beta\alpha$ and $y\beta\alpha = y$. Let w denote $y\beta$. If $y \leqslant w$, then since α is order-preserving we have that $z = y\alpha \leqslant w\alpha = y\beta\alpha = y$, a contradiction to our earlier observation that $y < z$. Therefore we have $y\beta = w < y$.

Assume first that β is order-preserving, so an application of β to both sides of the inequality $y\beta < y$ yields $y\beta^2 \leqslant y\beta < y$, so $y\beta^2 < y < z$. By using a similar argument we obtain that $y\beta^3 < y < z$, and indeed

$$y\beta^m < y < z \text{ for any integer } m \geqslant 2. \tag{1}$$

Let $k \geqslant 2$ be chosen such that α^k is an idempotent, say ε. Put $m = k$ in Equation (1) above. On one hand by Corollary 2.2 we have that $y\alpha^k$ is the root of the common component of y under α and under ε, so that $y\alpha^k = z$. On the other hand we now obtain by Lemma 2.3 and Equation (1) that $y\alpha^k = y\beta^k < y < z$, a contradiction. It follows that if $\beta \in \mathcal{O}_n$ then α is an idempotent, and so the first statement is proved.

Finally assume that α has at least two fixed points and $\beta \in \mathcal{OP}_n$. Consider the (common) components $C(1)$ and $C(n)$ associated with digraphs of α and β containing 1 and n respectively. Since the components of α are intervals of the standard chain X_n (see Lemma 2.8 of [5]), it follows that if $C(1) = C(n)$ then α would have just one component and so just one fixed point, contrary to hypothesis. Hence $C(1) = \{1, 2, \ldots, i\}$ and $C(n) = \{j, j+1, \ldots, n\}$, for some $i < j$. But since these are also components of β, and β maps each of its components into itself, it follows that $1\,\beta$ lies in $C(1)$ and $n\,\beta$ lies in $C(n)$; in particular $1\,\beta < n\,\beta$, whence it follows from Proposition 2.3 of [5] that β lies in $\mathcal{O}_n$. But that contradicts the first part of our theorem. Therefore α does not have a strong inverse in $\mathcal{OP}_n$. □

An immediate consequence of the above is the result of A. Vernitski [18, Corollary 4].

Theorem 2.7. *Any inverse subsemigroup of $\mathcal{O}_n$ is a semilattice. The union of all inverse subsemigroups of $\mathcal{O}_n$ is just the set of idempotents of $\mathcal{O}_n$, or equivalently, the set of group elements of $\mathcal{O}_n$.*

Next we apply the above results to the semigroups $\mathcal{OP}_n$ of all orientation-preserving transformations of X_n and $\mathcal{P}_n$ of all orientation-preserving or orientation-reversing transformations of X_n. Let $\mathcal{OR}_n$ denote the set of all orientation-reversing transformations in $\mathcal{T}_n$. It was shown in [5] that $\mathcal{P}_n = \mathcal{OP}_n \cup \mathcal{OR}_n$,

$$\mathcal{OP}_n \cap \mathcal{OR}_n = \{\alpha \in \mathcal{T}_n : \mathrm{rank}(\alpha) \leqslant 2\},$$

$$\mathcal{OP}_n \cdot \mathcal{OR}_n = \mathcal{OR}_n = \mathcal{OR}_n \cdot \mathcal{OP}_n \text{ and } (\mathcal{OR}_n)^2 = \mathcal{OP}_n = (\mathcal{OP}_n)^2. \quad (2)$$

Note that for $n \leqslant 2$ we have $\mathcal{OP}_n = \mathcal{T}_n$ and so every element of $\mathcal{OP}_n$ has a strong inverse in $\mathcal{OP}_n$. Now $|\mathcal{OP}_3| = 24$ (see [5], Corollary 2.7), and $\mathcal{T}_3 \setminus \mathcal{OP}_3$ consists of the three transpositions, which reverse orientation. It is easily seen that each member of $\mathcal{OP}_3$ has a strong inverse: indeed, $\mathcal{P}_3 = \mathcal{T}_3$ (see [5]), and so $\mathcal{P}_3$ is covered by its inverse subsemigroups. Since the elements of $\mathcal{P}_3$ and $\mathcal{OP}_3$ of rank at most two coincide, and the ranks of a transformation and its inverse are the same, we only need to observe that the three permutations in $\mathcal{OP}_3$ each have strong inverses in $\mathcal{OP}_3$ as together they form a (cyclic) group.

Let θ denote the n-cycle $(1, 2, 3, \ldots, n)$ in $\mathcal{OP}_n$. As a consequence of Theorem 2.7 we can prove the following result:

Lemma 2.8. *A non-idempotent transformation in $\mathcal{OP}_n$ with at least two fixed points does not have a strong inverse in $\mathcal{OP}_n$.*

Proof. Observe that if $n \leqslant 3$ then any transformation in $\mathcal{OP}_n$ with at least two fixed points is an idempotent. Hence assume that $n \geqslant 4$. By Theorem 4.9 in [5], the digraph of any member of $\mathcal{OP}_n$ cannot have two cycles of different length. It follows that all the cycles of α are fixed points. By Corollary 4.12 in [5], the mapping α can be written as $\theta^{-m}\delta\theta^m$ for some $\delta \in \mathcal{O}_n$ and a non-negative integer m.

Now assume by way of contradiction that $\beta \in \mathcal{OP}_n$ is a strong inverse of α. Take the mapping

$$\varphi : \mathcal{OP}_n \to \mathcal{OP}_n \text{ defined by } \kappa\varphi = \theta^m \kappa \theta^{-m}$$

for $\kappa \in \mathcal{OP}_n$. Since θ is a permutation in $\mathcal{OP}_n$, the mapping φ is an automorphism of $\mathcal{OP}_n$. Moreover, $\alpha\varphi = \delta$ and $\beta\varphi = \theta^m\beta\theta^{-m}$, so φ maps

$\langle \alpha, \beta \rangle$ isomorphically onto $\langle \delta, \theta^m \beta \theta^{-m} \rangle$. Since, by our assumption, β is a strong inverse of α, we have that $\langle \alpha, \beta \rangle$ and $\langle \delta, \theta^m \beta \theta^{-m} \rangle$ are isomorphic inverse subsemigroups of $\mathcal{OP}_n$ and $\theta^m \beta \theta^{-m}$ is a strong inverse of δ.

We now note that α and its conjugate δ have the same number of fixed points. Indeed for any $x \in X_n$ we have that $x\,\alpha = x$ if and only if $x\,\theta^{-m}\delta\theta^m = x$, that is $(x\,\theta^{-m})\delta = x\,\theta^{-m}$. Thus $\delta \in \mathcal{O}_n$ has at least two fixed points, and by Theorem 2.6(2), δ does not have a strong inverse in $\mathcal{OP}_n$, a contradiction. $\square$

Putting together the observations above that $\mathcal{OP}_n$ is covered by its inverse subsemigroups when $n \leqslant 3$, and that if $n \geqslant 4$ then $\mathcal{OP}_n$ contains non-idempotent transformations with at least two fixed points, an application of the above lemma yields the following result.

Theorem 2.9. *The semigroup $\mathcal{OP}_n$ is covered by its inverse subsemigroups if and only if $n \leqslant 3$.*

Example. In $\mathcal{OP}_3$ we have the pair of strong inverses $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}$. We note that neither α nor β are idempotents, and α is a member of $\mathcal{O}_3$, while β is a member of $\mathcal{OP}_3$. The semigroup $\langle \alpha, \beta \rangle$ is the five-element combinatorial Brandt (inverse) semigroup, yet neither of α nor β is a group element. Hence, although $\mathcal{OP}_n$ is not covered by its inverse subsemigroups, its set of strong inverses encompasses more than its group elements (so that Theorem 2.7 is not true if $\mathcal{O}_n$ is replaced by $\mathcal{OP}_n$). We note that α is a member of $\mathcal{O}_3$ and β is a member of the semigroup of order-preserving mappings on the chain $3 < 1 < 2$. This however does not contradict Lemma 2.8 as both α and β have just one fixed point.

If $n \leqslant 3$, it is observed in [5] that $\mathcal{P}_n = \mathcal{T}_n$, and so $\mathcal{P}_n$ is covered by its inverse semigroups. The result below demonstrates that these are the only instances when this is true.

Theorem 2.10. *The semigroup $\mathcal{P}_n$ of all orientation-preserving or orientation reversing mappings is covered by its inverse subsemigroups if and only if $n \leqslant 3$.*

Proof. Assume $n \geqslant 4$ and choose, using Theorem 2.6, a transformation $\alpha \in \mathcal{OP}_n$ of rank at least 3 that has no strong inverse in $\mathcal{OP}_n$. Assume $\beta \in \mathcal{P}_n$ is a strong inverse of α in $\mathcal{P}_n$. Now any inverse of α has the same rank as α, so $\beta \in \mathcal{OR}_n$ with rank at least 3. But then by [5, Corollary 5.2] $\alpha = \alpha\beta\alpha \in \mathcal{OP}_n \cdot \mathcal{OR}_n \cdot \mathcal{OP}_n = \mathcal{OR}_n$. Since the rank of α is at least 3, and, in accordance with [5, Lemma 5.4], $\mathcal{OR}_n \cap \mathcal{OP}_n$ consists of transformations of rank at most 2, $\alpha \in \mathcal{OR}_n \setminus \mathcal{OP}_n$, a contradiction to the assumption that $\alpha \in \mathcal{OP}_n$. This completes the proof. □

Acknowledgement

The first author acknowledges support by the Portuguese Government through the Portuguese Foundation FCT under the project PEst-OE/MAT/UI4080/2014.

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Contact information

Paula Catarino Departamento de Matemática
Universidade de Trás-os-Montes e Alto Douro
5001-801 Vila Real, Portugal
E-Mail(s): pcatarin@utad.pt

Peter M. Higgins Department of Mathematical Sciences
University of Essex
Colchester CO4 3SQ U.K.
E-Mail(s): peteh@essex.ac.uk

Inessa Levi Department of Mathematics
Columbus State University
Columbus, GA 31907, USA
E-Mail(s): levi_inessa@columbusstate.edu

Received by the editors: 04.06.2014
and in final form 04.08.2014.