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Attached primes and annihilators of top local cohomology modules defined by a pair of ideals

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Abstract. Assume that R is a complete Noetherian local ring and M is a non-zero finitely generated R-module of dimension $n = \dim(M) \geqslant 1$. It is shown that any non-empty subset T of $\mathrm{Assh}(M)$ can be expressed as the set of attached primes of the top local cohomology modules $H^n_{I,J}(M)$ for some proper ideals I, J of R. Moreover, for ideals $I, J = \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_I(M))} \mathfrak{p}$ and J' of R it is proved that $T = \mathrm{Att}_R(H^n_{I,J}(M)) = \mathrm{Att}_R(H^n_{I,J'}(M))$ if and only if $J' \subseteq J$. Let $H^n_{I,J}(M) \neq 0$. It is shown that there exists $Q \in \mathrm{Supp}(M)$ such that $\dim(R/Q) = 1$ and $H^n_Q(R/\mathfrak{p}) \neq 0$, for each $\mathfrak{p} \in \mathrm{Att}_R(H^n_{I,J}(M))$. In addition, we prove that if I and J are two proper ideals of a Noetherian local ring R, then $\mathrm{Ann}_R(H^n_{I,J}(M)) = \mathrm{Ann}_R(M/T_R(I, J, M))$, where $T_R(I, J, M)$ is the largest submodule of M with $\mathrm{cd}(I, J, T_R(I, J, M)) < \mathrm{cd}(I, J, M)$, here $\mathrm{cd}(I, J, M)$ is the cohomological dimension of M with respect to I and J. This result is a generalization of [1, Theorem 2.3] and [2, Theorem 2.6].

Introduction

Assume that R is a Noetherian ring and I, J are two ideals of R and M is an R-module. As a generalization of the usual local cohomology modules, the local cohomology modules with respect to a system of ideals was introduced, in [3]. As a special case of these extended modules, in [13],

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the local cohomology modules with respect to a pair of ideals is defined. To be more precise, let

$$W(I,J)=\{\mathfrak{p}\in \operatorname{Spec}(R)|I^n\subseteq \mathfrak{p}+J \text{ for some positive integer } n\}.$$

The (I,J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\operatorname{Supp}(Rx)\subseteq W(I,J)$, is considered. For an integer i, the local cohomology functor $H^i_{I,J}$ with respect to (I,J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$. The i-th local cohomology module of M with respect to (I,J) is denoted by $H^i_{I,J}(M)$. When $J=0$, then $H^i_{I,J}$ coincides with the usual local cohomology functor H^i_I with the support in the closed subset $V(I)$.

Recall that for an R-module K, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of K if $\mathfrak{p}=\operatorname{Ann}(K/N)$ for some submodule N of K. The set of attached prime ideals of K is denoted by $\operatorname{Att}_R(K)$. When K has a secondary representation, this definition agrees with the usual definition of attached primes in [12].

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R-module of dimension n. The main theorem in Section 2, shows that if R is complete with respect to $\mathfrak{m}$-adic topology, then for any non-empty subset T of $\operatorname{Assh}(M)$ there exist ideals I,J of R such that $T=\operatorname{Att}_R(H^n_I(M))=\operatorname{Att}_R(H^n_{I,J}(M))$ which is an another version of Theorem 2.8 in [8]. Moreover we show that for each $\mathfrak{p}\in\operatorname{Att}_R(H^n_{I,J}(M))$ there exists $Q\in\operatorname{Supp}(M)$ such that $\dim(R/Q)=1$ and $H^n_Q(R/\mathfrak{p})\neq 0$.

Let R be a Noetherian ring, I,J be two ideals of R and M be a non-zero finitely generated R-module of dimension n. Let $\operatorname{cd}(I,J,M)$ denote the supremum of all integers r for which $H^r_{I,J}(M)\neq 0$. We call this integer the cohomological dimension of M with respect to ideals I,J, see [7]. In Section 3, first we define $T_R(I,J,M)$ the largest submodule of M such that $\operatorname{cd}(I,J,T_R(I,J,M))<\operatorname{cd}(I,J,M)$ and we show that

$$T_R(I,J,M)=\Gamma_{\mathfrak{a}}(M)=\bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=c} N_j,$$

where $0=\bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule of M, N_j is a $\mathfrak{p}_j$-primary submodule of M and $\mathfrak{a}=\prod_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)\neq c}\mathfrak{p}_j$. We show $\operatorname{Ann}_R(H^n_{I,J}(M))=\operatorname{Ann}_R(M/T_R(I,J,M))$, which is a generalization of [1, Theorem 2.3] and [2, Theorem 2.6] and some applications of this theorem are given.

1. Attached prime ideals of top local cohomology modules

In this section we assume that $(R, \mathfrak{m})$ is local and complete with respect to $\mathfrak{m}$-adic topology, M is a non-zero finitely generated R-module of dimension $n \geqslant 1$ and T is a non-empty subset of $\mathrm{Assh}(M)$.

Definition 1. Let K be an R-module, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of K if $\mathfrak{p} = \mathrm{Ann}(K/N)$ for some submodule N of K. The set of attached prime ideals of K is denoted by $\mathrm{Att}_R(K)$.

Lemma 1. [1, Lemma 3.2] *Let K be an R-module. Then the set of minimal elements of $V(\mathrm{Ann}_R(K))$ coincides with that of $\mathrm{Att}_R(K)$. In particular, $\sqrt{\mathrm{Ann}_R(K)} = \bigcap_{\mathfrak{p}\in\mathrm{Att}_R(K)} \mathfrak{p}$.*

Theorem 1. *Let M be a non-zero finitely generated R-module of dimension n and T be a non-empty subset of $\mathrm{Assh}(M)$. Then the following statements are true:*

(i) *If $T \subseteq \mathrm{Att}_R(H_I^n(M))$ for some ideal I, then $T = \mathrm{Att}_R(H_{I,J}^n(M))$, where $J = \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$.*

(ii) *$T = \mathrm{Att}_R(H_I^n(M)) = \mathrm{Att}_R(H_{I,J}^n(M))$, where I, J are ideals of R and $J = \sqrt{\mathrm{Ann}_R(H_I^n(M))}$.*

Proof. (i) By assumption T is a non-empty subset of $\mathrm{Assh}(M)$. Set $J := \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. We show that $T = \mathrm{Att}_R(H_{I,J}^n(M))$. Assume that $\mathfrak{q} \in \mathrm{Att}_R(H_{I,J}^n(M))$. Then by [6, Theorem 2.1] it follows that $J \subseteq \mathfrak{q}$. Thus $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{p} \in T$. Hence, this fact that $\mathfrak{p}, \mathfrak{q}$ are in $\mathrm{Assh}(M)$ shows that $\mathfrak{p} = \mathfrak{q}$ and so $\mathfrak{q} \in T$. Now, let $\mathfrak{q} \in T$. Then $J \subseteq \mathfrak{q}$ and also $\mathfrak{q} \in \mathrm{Att}_R(H_I^n(M))$. Therefore, $\mathfrak{q} \in \mathrm{Att}_R(H_{I,J}^n(M))$ by [6, Theorem 2.1].

(ii) In view of [8, Theorem 2.8] there exists an ideal I of R such that $T = \mathrm{Att}_R(H_I^n(M))$. Thus by (i) and Lemma 1 the result follows. □

Corollary 1. *Let M be a non-zero finitely generated R-module of dimension n and let I_1, I_2, J_1, J_2 be ideals of R. Then the following statements hold:*

(i) *If $T \subseteq \mathrm{Att}_R(H_{I_1,J_1}^n(M)) \cup \mathrm{Att}_R(H_{I_2,J_2}^n(M))$ is a non-empty set and $J = \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, then $\mathrm{Att}_R(H_{I_1+I_2,J}^n(M)) = T$.*

(ii) *If $T = \mathrm{Att}_R(H_{I_1,J_1}^n(M)) \cup \mathrm{Att}_R(H_{I_2,J_2}^n(M))$ is a non-empty set and $J = \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, then $\mathrm{Att}_R(H_{I_1+I_2,J}^n(M)) = \mathrm{Att}_R(H_{I_1,J_1}^n(M)) \cup \mathrm{Att}_R(H_{I_2,J_2}^n(M))$.*

(iii) *If $T = \mathrm{Att}_R(H_{I_1,J_1}^n(M)) \cap \mathrm{Att}_R(H_{I_2,J_2}^n(M))$ is a non-empty set and $J = \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, then $\mathrm{Att}_R(H_{I_1+I_2,J}^n(M)) = \mathrm{Att}_R(H_{I_1,J_1}^n(M)) \cap \mathrm{Att}_R(H_{I_2,J_2}^n(M))$.*

Proof. Let $\mathfrak{p} \in T$ and $\mathfrak{p} \in \mathrm{Att}_R(H^n_{I_1,J_1}(M))$. Then $\mathfrak{p} \in \mathrm{Att}_R(H^n_{I_1}(M))$, by [6, Theorem 2.1]. Thus $\dim R/\mathfrak{p} = n$ and by Lichtenbaum-Hartshorne Vanishing Theorem $\dim R/(I_1 + \mathfrak{p}) = 0$. Since $I_1 + \mathfrak{p} \subseteq I_1 + I_2 + \mathfrak{p}$, it follows that $\dim R/(I_1 + I_2 + \mathfrak{p}) = 0$ and so $H^n_{I_1+I_2}(R/\mathfrak{p}) \neq 0$. Thus $\mathfrak{p} \in \mathrm{Att}_R(H^n_{I_1+I_2}(M))$, by [9, Theorem A]. Therefore, $T \subseteq \mathrm{Att}_R(H^n_{I_1+I_2}(M))$ and the result follows by Theorem 1(i). □

Corollary 2. *Let I, J be ideals of R and let M be a non-zero finitely generated R-module of dimension n. If $T = \mathrm{Att}_R(H^n_I(M))$ and $J' = J + \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, then $\mathrm{Att}_R(H^n_{I,J}(M)) = \mathrm{Att}_R(H^n_{\mathfrak{m},J'}(M))$.*

Proof. In view of [6, Theorem 2.1] and Lichtenbaum-Hartshorne Vanishing Theorem, we have

$$\mathrm{Att}_R(H^n_{I,J}(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M) \cap V(J) : \sqrt{I + \mathfrak{p}} = \mathfrak{m}\}.$$

Let $\mathfrak{p} \in \mathrm{Att}_R(H^n_{I,J}(M))$. Then $\mathfrak{p} \in \mathrm{Att}_R(H^n_I(M))$ and $0 \neq H^n_I(R/\mathfrak{p}) \cong H^n_{I+\mathfrak{p}}(R/\mathfrak{p}) \cong H^n_{\mathfrak{m}}(R/\mathfrak{p})$. Hence, $\mathfrak{p} \in \mathrm{Att}_R(H^n_{\mathfrak{m},J'}(M))$. The proof of the opposite inclusion is similar. □

Theorem 2. *Let M be a non-zero finitely generated R-module of dimension n and let I, $J = \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_I(M))} \mathfrak{p}$ and J' be ideals of R. Then $\mathrm{Att}_R(H^n_{I,J}(M)) = \mathrm{Att}_R(H^n_{I,J'}(M))$ if and only if $J' \subseteq J$.*

Proof. Let $\mathrm{Att}_R(H^n_{I,J}(M)) = \mathrm{Att}_R(H^n_{I,J'}(M))$. Then Theorem 1 shows that $\mathrm{Att}_R(H^n_I(M)) = \mathrm{Att}_R(H^n_{I,J}(M))$. Hence, by [6, Theorem 2.1]

$$J' \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_{I,J'}(M))} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_{I,J}(M))} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_I(M))} \mathfrak{p} = J.$$

Conversely, if $J' \subseteq J$, then by [6, Theorem 2.1] we have

$$\mathrm{Att}_R(H^n_I(M)) = \mathrm{Att}_R(H^n_{I,J}(M)) \subseteq \mathrm{Att}_R(H^n_{I,J'}(M)) \subseteq \mathrm{Att}_R(H^n_I(M)).$$

So the result follows. □

Theorem 3. *Let M be a non-zero finitely generated R-module of dimension n and let I, I' and $J = \bigcap_{\mathfrak{p}\in \mathrm{Att}_R(H^n_I(M))} \mathfrak{p}$ be ideals of R such that $I \subseteq I'$. Then $\mathrm{Att}(H^n_{I,J}(M)) = \mathrm{Att}(H^n_{I',J}(M))$*

Proof. Assume that $\mathfrak{p} \in \mathrm{Att}(H^n_{I,J}(M))$. Thus [6, Theorem 2.1] shows that $\mathfrak{p} \in \mathrm{Att}(H^n_I(M))$ and $J \subseteq \mathfrak{p}$. By assumption and [10, Proposition

1.6], $\operatorname{Att}_R(H^n_I(M)) \subseteq \operatorname{Att}_R(H^n_{I'}(M))$ so that $\mathfrak{p} \in \operatorname{Att}(H^n_{I'}(M))$ and $\mathfrak{p} \in \operatorname{Att}(H^n_{I',J}(M))$. Thus $\operatorname{Att}(H^n_{I,J}(M)) \subseteq \operatorname{Att}(H^n_{I',J}(M))$. Therefore,

$$J \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Att}(H^n_{I',J}(M))} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Att}(H^n_{I,J}(M))} \mathfrak{p} = J$$

which shows that $\operatorname{Att}(H^n_{I,J}(M)) = \operatorname{Att}(H^n_{I',J}(M))$. □

Theorem 4. *Let M be a finitely generated R-module of dimension n and I, J be ideals of R such that $H^n_{I,J}(M) \neq 0$. Then there exists $Q \in \operatorname{Supp}(M)$ such that $\dim(R/Q) = 1$ and $H^n_Q(R/\mathfrak{p}) \neq 0$, for each $\mathfrak{p} \in \operatorname{Att}_R(H^n_{I,J}(M))$.*

Proof. By assumption $T = \operatorname{Att}_R(H^n_{I,J}(M)) \neq \varnothing$. Then in view of [8, Theorem 2.8] we have $T = \operatorname{Att}_R H^n_{\mathfrak{a}}(M)$ for some ideal $\mathfrak{a}$ of R. Now, [8, Proposition 2.1] shows that there exists an integer r such that for all $1 \leqslant i \leqslant r$ there exists $Q_i \in \operatorname{Supp}(M)$ with $\dim(R/Q_i) = 1$ such that $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$. In addition, we may assume that $\mathfrak{a} = \bigcap_{i=1}^r Q_i$. Let $\mathfrak{p} \in \operatorname{Att}_R H^n_{I,J}(M)$. Then $\mathfrak{p} \in \operatorname{Att}_R H^n_{\mathfrak{a}}(M)$ and so $H^n_{\bigcap_{i=1}^r Q_i}(R/\mathfrak{p}) \neq 0$. Now, by setting $\mathfrak{b} = \bigcap_{i=1}^{r-1} Q_i$ and $\mathfrak{c} = Q_r$ we have the following long exact sequence

$$H^n_{\mathfrak{b}+\mathfrak{c}}(R/\mathfrak{p}) \to H^n_{\mathfrak{b}}(R/\mathfrak{p}) \oplus H^n_{\mathfrak{c}}(R/\mathfrak{p}) \to H^n_{\mathfrak{b}\cap\mathfrak{c}}(R/\mathfrak{p}) \to 0,$$

where $H^n_{\mathfrak{b}\cap\mathfrak{c}}(R/\mathfrak{p}) = H^n_{Q_1\cap\cdots\cap Q_r}(R/\mathfrak{p}) \neq 0$. So $H^n_{\mathfrak{b}}(R/\mathfrak{p}) \oplus H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$. Therefore $H^n_{\mathfrak{b}}(R/\mathfrak{p}) \neq 0$ or $H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$. If $H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$ we are done. Otherwise, one can set $\mathfrak{b} = \bigcap_{i=1}^{r-2} Q_i$ and $\mathfrak{c} = Q_{r-1}$ and with repeat this method, to get the result. □

Corollary 3. *Let M be a finitely generated R-module of dimension $n \geqslant 1$ and I, J be ideals of R such that $H^n_{I,J}(M) \neq 0$. Then there exists $Q \in \operatorname{Supp}(M)$ such that $\dim(R/Q) = 1$ and $\operatorname{Att}_R(H^n_{Q,J}(M)) \neq \varnothing$.*

2. Annihilators of top local cohomology modules

In this section $(R, \mathfrak{m})$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and I, J are two proper ideals of R.

Let M be a non-zero finitely generated R-module and let $\operatorname{cd}(I, J, M)$ denote the supremum of all integers r for which $H^r_{I,J}(M) \neq 0$. We call this integer the cohomological dimension of M with respect to ideals I, J. When $J = 0$, we have $\operatorname{cd}(I, 0, M) = \operatorname{cd}(I, M)$ which is just the supremum of all integers r for which $H^r_I(M) \neq 0$. In [7, Corollary 3.3] a characterization of $\operatorname{cd}(I, J, M)$ is provided.

Lemma 2. *[7, Proposition 3.2] Let M and N be two finitely generated R-modules such that* $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$. *Then* $\operatorname{cd}(I,J,N) \leqslant \operatorname{cd}(I,J,M)$.

Definition 2. Let M be a non-zero finitely generated R-module of cohomological dimension c. We denote by $T_R(I,J,M)$ the largest submodule of M such that $\operatorname{cd}(I,J,T_R(I,J,M)) < \operatorname{cd}(I,J,M)$.

It is easy to check that $T_R(I,J,M) = \cup\{N : N \leqslant M,\ \operatorname{cd}(I,J,N) < \operatorname{cd}(I,J,M)\}$. When $J = 0$, this definition coincides with that of [1, Definition 2.2].

Lemma 3. *Let M be a non-zero finitely generated R-module of dimension n such that $n = \operatorname{cd}(I,J,M)$. Then $T_R(\mathfrak{m},M) \subseteq T_R(I,M) \subseteq T_R(I,J,M)$.*

Proof. For the first inclusion let $x \notin T_R(I,M)$. Then $\operatorname{cd}(I,J,Rx) = n$ and so $H^n_I(Rx) \neq 0$. Thus $\dim(Rx) = n$. Hence, by Grothendieck's Vanishing Theorem $H^n_{\mathfrak{m}}(Rx) \neq 0$ and $x \notin T_R(\mathfrak{m},M)$. Let $x \notin T_R(I,J,M)$. Then $\operatorname{cd}(I,J,Rx) = n$ and so $H^n_{I,J}(Rx) \neq 0$. Thus $\operatorname{Att}_R(H^n_{I,J}(Rx)) \neq \varnothing$ and $\operatorname{Att}_R(H^n_I(Rx)) \neq \varnothing$ by [6, Theorem 2.1]. Hence, $H^n_I(Rx) \neq 0$ and $\operatorname{cd}(I,Rx) = n$. Therefore, $x \notin T_R(I,M)$. □

Theorem 5. *Let M be a non-zero finitely generated R-module with cohomological dimension $c = \operatorname{cd}(I,J,M)$. Then*

$$T_R(I,J,M) = \Gamma_{\mathfrak{a}}(M) = \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=c} N_j.$$

Here $0 = \bigcap_{j=1}^n N_j$ is a reduced primary decomposition of the zero submodule of M, N_j is a $\mathfrak{p}_j$-primary submodule of M and $\mathfrak{a} = \prod_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)\neq c} \mathfrak{p}_j$.

Proof. First we show the equality $\Gamma_{\mathfrak{a}}(M) = \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=c} N_j$. To do this, the inclusion $\bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=c} N_j \subseteq \Gamma_{\mathfrak{a}}(M)$ follows easily by the proof of [11, Theorem 6.8(ii)]. In order to prove the opposite inclusion, suppose, the contrary is true. Then there exists $x \in \Gamma_{\mathfrak{a}}(M)$ such that $x \notin \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=c} N_j$. Thus there exists an integer t such that $x \notin N_t$ and $\operatorname{cd}(I,J,R/\mathfrak{p}_t) = c$. Now, as $x \in \Gamma_{\mathfrak{a}}(M)$, it follows that there is an integer $s \geqslant 1$ such that $\mathfrak{a}^s x = 0$, and so $\mathfrak{a}^s x \subseteq N_t$. Because of $x \notin N_t$ and N_t is a $\mathfrak{p}_t$-primary submodule, it yields that $\mathfrak{a} \subseteq \mathfrak{p}_t$. Hence, there is an integer j such that $\mathfrak{p}_j \subseteq \mathfrak{p}_t$ and $\operatorname{cd}(I,J,R/\mathfrak{p}_j) \leqslant c-1$. Therefore, in view of Lemma 2, we have

$$\operatorname{cd}(I,J,R/\mathfrak{p}_t) \leqslant \operatorname{cd}(I,J,R/\mathfrak{p}_j) \leqslant c-1,$$

which is a contradiction. Now, we show that $T_R(I,J,M) = \Gamma_{\mathfrak{a}}(M)$. Let $x \in T_R(I,J,M)$. Then in view of Lemma 2, $\operatorname{cd}(I,J,Rx) \leqslant c-1$. Let $\mathfrak{p}$ be a minimal prime ideal of $\operatorname{Ann}_R(Rx)$, it follows that $\operatorname{cd}(I,J,R/\mathfrak{p}) \leqslant c-1$. So

$$\mathfrak{a} \subseteq \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)\leqslant c-1} \mathfrak{p}_j \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Ass}_R(Rx)} \mathfrak{p} = \sqrt{\operatorname{Ann}_R(Rx)}.$$

Thus there exists an integer $k \geqslant 1$ such that $\mathfrak{a}^k \subseteq \operatorname{Ann}_R(Rx)$. Hence, $\mathfrak{a}^k x = 0$ and $x \in \Gamma_{\mathfrak{a}}(T_R(I,J,M))$. Thus $T_R(I,J,M) = \Gamma_{\mathfrak{a}}(T_R(I,J,M))$. Now, we have $T_R(I,J,M) = \Gamma_{\mathfrak{a}}(T_R(I,J,M)) \subseteq \Gamma_{\mathfrak{a}}(M)$. We show that $\Gamma_{\mathfrak{a}}(M) \subseteq T_R(I,J,M)$, to do this, we show $\operatorname{cd}(I,J,\Gamma_{\mathfrak{a}}(M)) \leqslant c-1$. Let $\mathfrak{p} \in \operatorname{Supp}(\Gamma_{\mathfrak{a}}(M))$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ and there exists $\mathfrak{p}_j \subseteq \mathfrak{p}$ such that $\operatorname{cd}(I,J,R/\mathfrak{p}_j) \leqslant c-1$. Thus by Lemma 2, $\operatorname{cd}(I,J,R/\mathfrak{p}) \leqslant c-1$. Hence, $\operatorname{cd}(I,J,\Gamma_{\mathfrak{a}}(M)) \leqslant c-1$ by [7, Theorem 3.1]. Therefore, $T_R(I,J,M) = \Gamma_{\mathfrak{a}}(M)$. □

Corollary 4. *Let M be a non-zero finitely generated R-module of dimension n with cohomological dimension $c = \operatorname{cd}(I,J,M)$. Then the following statements are true:*

(i) $\operatorname{Ass}_R(T_R(I,J,M)) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) : \operatorname{cd}(I,J,R/\mathfrak{p}) \leqslant c-1\}$,

(ii) $\operatorname{Ass}_R(M/T_R(I,J,M)) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) : \operatorname{cd}(I,J,R/\mathfrak{p}) = c\}$. *If $n = c$, then* $\operatorname{Ass}_R(M/T_R(I,J,M)) = \operatorname{Att}_R(H^n_{I,J}(M))$.

Proof. By Theorem 5, $T_R(I,J,M) = \Gamma_{\mathfrak{a}}(M)$, where $\prod_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)\leqslant c-1} \mathfrak{p}_j = \mathfrak{a}$. So by [4, Section 2.1, Proposition 10] we have

$$\operatorname{Ass}_R(T_R(I,J,M)) = \operatorname{Ass}_R(M) \cap V(\mathfrak{a}).$$

Now (i) follows from Lemma 2.

In order to show (ii), use [5, Exercise 2.1.12] and [6, Theorem 2.1]. □

Corollary 5. *Let M be a non-zero finitely generated R-module of dimension n such that $n = \operatorname{cd}(I,J,M)$. Then there exists a positive integer t such that $J^t M \subseteq T_R(I,J,M)$.*

Proof. Let $0 = \bigcap_{j=1}^n N_j$ denote a reduced primary decomposition of the zero submodule of M, where N_j is a $\mathfrak{p}_j$-primary submodule of M. In view of Theorem 5, $T_R(I,J,M) = \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=n} N_j$. If $\operatorname{cd}(I,J,R/\mathfrak{p}_j) = n$, then $H^n_{I,J}(R/\mathfrak{p}_j) \neq 0$. Thus $J \subseteq \mathfrak{p}_j = \sqrt{\operatorname{Ann}_R(M/N_j)}$ by [13, Theorem 4.3]. Hence, there exists a positive integer t_j such that $J^{t_j} M \subseteq N_j$. Let $t = \max\{t_j : \operatorname{cd}(I,J,R/\mathfrak{p}_j) = n\}$. Then $J^t M \subseteq \bigcap_{\operatorname{cd}(I,J,R/\mathfrak{p}_j)=n} N_j = T_R(I,J,M)$. □

Theorem 6. *Let M be a non-zero finitely generated R-module of dimension $n = \operatorname{cd}(I, J, M)$. Then*

$$\operatorname{Ann}_R(H^n_{I,J}(M)) = \operatorname{Ann}_R(M/T_R(I, J, M)).$$

Proof. By Corollary 5, $J^tM \subseteq T_R(I, J, M)$ for some integer $t \geqslant 1$ and by [13, Proposition 1.4(8)], $H^i_{I,J}(M) \cong H^i_{I,J^t}(M)$ for all $i \geqslant 0$. Then we can assume that $JM \subseteq T_R(I, J, M)$. First we show that $T_R(I, M/JM) = T_R(I, J, M)/JM$. Let $x \in M$ and consider the exact sequence

$$0 \to Rx \cap JM \to Rx \to \frac{Rx}{Rx \cap JM} \to 0$$

that induce an exact sequence

$$\cdots \to H^n_{I,J}(Rx \cap JM) \to H^n_{I,J}(Rx) \to H^n_{I,J}(\frac{Rx}{Rx \cap JM}) \to 0. \qquad (*)$$

If $x + JM \in T_R(I, M/JM)$, then $H^n_{I,J}(Rx/Rx \cap JM) \cong H^n_I(R(x + JM)) = 0$, by [13, Corollary 2.5]. As $Rx \cap JM \subseteq T_R(I, J, M)$ it follows that $H^n_{I,J}(Rx \cap JM) = 0$. Hence, $H^n_{I,J}(Rx) = 0$. So that $x \in T_R(I, J, M)$. If $x \in T_R(I, J, M)$, then $H^n_{I,J}(Rx) = 0$. Thus $H^n_{I,J}(Rx/Rx \cap JM) = H^n_I(R(x + JM)) = 0$ by $(*)$. Therefore, $x + JM \in T_R(I, M/JM)$. Now, from the exact sequence

$$0 \to JM \to M \to \frac{M}{JM} \to 0$$

we have the exact sequence

$$\cdots \to H^n_{I,J}(JM) \to H^n_{I,J}(M) \to H^n_{I,J}(\frac{M}{JM}) \to 0.$$

Since $JM \subseteq T_R(I, J, M)$, it follows that $H^n_{I,J}(JM) = 0$ and so we have $H^n_{I,J}(M) \cong H^n_{I,J}(M/JM)$. Thus $H^n_{I,J}(M) \cong H^n_I(M/JM)$. Therefore,

$$\begin{aligned}\operatorname{Ann}_R(H^n_{I,J}(M)) = \operatorname{Ann}_R(H^n_I(M/JM)) &= \operatorname{Ann}_R(\frac{M/JM}{T_R(I, M/JM)}) \\ &= \operatorname{Ann}_R(\frac{M/JM}{T_R(I, J, M)/JM}) = \operatorname{Ann}_R(M/T_R(I, J, M)),\end{aligned}$$

see [1, Theorem 2.3]. □

Corollary 6. *Let M be a non-zero finitely generated R-module of dimension $n = \operatorname{cd}(I, J, M)$ and $JM \subseteq T_R(I, M)$. Then $\operatorname{Ann}_R(H^n_I(M)) = \operatorname{Ann}_R(H^n_{I,J}(M))$.*

Proof. By a similar argument to that of Theorem 6, one can show that $T_R(I, M/JM) = T_R(I, M)/JM$. Also, by Lemma 3 we have $T_R(I, M) \subseteq T_R(I, J, M)$. Thus $JM \subseteq T_R(I, J, M)$ and so $H^n_{I,J}(JM) = 0$. Hence, it follows by $(*)$ that $H^n_{I,J}(M) \cong H^n_I(M/JM)$. Therefore,

$$\begin{aligned}\operatorname{Ann}_R(H^n_{I,J}(M)) = \operatorname{Ann}_R(H^n_I(M/JM)) &= \operatorname{Ann}_R(\frac{M/JM}{T_R(I, M/JM)})\\ = \operatorname{Ann}_R(\frac{M/JM}{T_R(I, M)/JM}) &= \operatorname{Ann}_R(M/T_R(I, M)).\end{aligned}$$

Now, the result follows by [1, Theorem 2.3] and Theorem 6. □

Corollary 7. *Let M be a non-zero finitely generated R-module of dimension $n = \operatorname{cd}(I, J, M)$. Then the following statements hold:*

(i) $\sqrt{\operatorname{Ann}_R(H^n_{I,J}(M))} = \bigcap_{\mathfrak{p}\in \operatorname{Ass}_R M, \operatorname{cd}(I,J,R/\mathfrak{p})=n} \mathfrak{p}$,

(ii) $V(\operatorname{Ann}_R(H^n_{I,J}(M))) = \operatorname{Supp}(M/T_R(I, J, M))$,

(iii) *If* $T_R(I, J, M) = 0$, *then* $\operatorname{Supp}(M) = V(\operatorname{Ann}_R(H^n_{I,J}(M)))$.

Proof. (i) It follows by Lemma 1 and [6, Theorem 2.1].

To prove (ii) use Theorem 6.

(iii) It follows from (ii). □

Corollary 8. *Let $d = \dim R = \operatorname{cd}(I, J, R)$. Then the following statements hold:*

(i) $\operatorname{cd}(I, J, \operatorname{Ann}_R(H^d_{I,J}(R))) < \dim R$.

(ii) *If* $d \geqslant 1$, *then* $\dim R = \dim R/\operatorname{Ann}_R(H^d_{I,J}(R)) = \dim R/\Gamma_{I,J}(R)$.

Proof. (i) It follows from Theorem 6.

(ii) By [13, Corollary 1.13 (4)], $H^d_{I,J}(R) \cong H^d_{I,J}(R/\Gamma_{I,J}(R))$. So that $\Gamma_{I,J}(R) \subseteq \operatorname{Ann}_R(H^d_{I,J}(R))$. □

Corollary 9. *If R is a domain of $\dim R = d$ and $H^d_{I,J}(R) \neq 0$, then* $\operatorname{Ann}_R(H^d_{I,J}(R)) = 0$.

Proof. It follows by Theorems 5 and 6. □

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